Chapter 5

Frequency Response, Fourier Analysis and Filter Design

The Laplace transform analysis of the previous chapter provides a means whereby the response of a rather arbitrary linear (and time invariant) system to a rather arbitrary stimulus may be computed. However, it is very common in the process of engineering design to consider much more restrictive cases in which the response to only sinusoidal signals is of interest. This restricted aspect of system performance is known as ‘frequency response’. Associated with it is a process of splitting signals themselves into constituent sinusoidal components in much the same way as light can be decomposed into colour components. This signal-splitting is called ‘Fourier Analysis’ and describing it, together with its connections to the idea of system frequency response and methods for the synthesis of certain prescribed frequency responses, is the purpose of this chapter.

5.1 System Frequency Response and System Transfer function

Suppose we have a linear and time invariant system that is characterised by its impulse response \( h(t) \), or equivalently by its transfer function \( H(s) = \mathcal{L}\{h(t)\} \).

Then by definition, the frequency response at \( \omega \) radians per second (rad/s) of this system is its output \( y(t) \) in steady state (ie. after initial condition effects have decayed to zero) when its input \( u(t) \) is a unit amplitude and sinusoidally shaped signal of frequency \( \omega \) rad/s.

Now, as a convenient means for representing such a sinusoidally shaped signal, we can use the (by now) well known complex valued representation

\[
e^{j\omega t} = \cos \omega t + j \sin \omega t.
\] (5.1)

If, in fact, only a cosine component is of interest, then only the real part of (5.1) (and its consequences) is considered. If only a sinusoidal component is desired, then the imaginary part attracts sole attention. Furthermore, for any linear and time invariant system, then by (3.169) the relationship between input \( u(t) \) and output \( y(t) \) is one of convolution with the system impulse response and hence

\[
y(t) = [h \ast u](t) = \int_{-\infty}^{\infty} h(\sigma)u(t - \sigma)\,d\sigma.
\]

Therefore, for the input \( u(t) = e^{j\omega t} \) being the unit amplitude (complex) sinusoid (5.1), and recalling
the definition (4.1) for the Laplace transform

\[
y(t) = \int_{-\infty}^{\infty} h(\sigma) e^{j\omega(t-\sigma)} \, d\sigma
\]

\[
= e^{j\omega t} \int_{-\infty}^{\infty} h(\sigma)e^{-j\omega \sigma} \, d\sigma
\]

\[
= e^{j\omega t} H(j\omega).
\]  

(5.2)

There are several important points exposed by this calculation. First, (5.2) indicates that if the input \( u(t) \) is purely sinusoidal at \( \omega \) rad/s and hence of the form \( u(t) = e^{j\omega t} \), then \( y(t) \) is also purely sinusoidal at exactly that same frequency \( \omega \). No new frequency components are generated, nor is the frequency \( \omega \) altered.

Second, although the frequency \( \omega \) of the sinusoidal response \( y(t) \) is unaltered as compared to that of \( u(t) \), the magnitude and phase may be changed. That is, remembering that \( H(j\omega) \) is complex valued, and hence may be represented in polar form as

\[
R(\omega) e^{j\phi(\omega)}
\]

(5.3)

where

\[
R(\omega) \triangleq |H(j\omega)|, \quad \phi(\omega) \triangleq \angle H(j\omega),
\]  

(5.4)

then by substitution of (5.3) in (5.2)

\[
y(t) = e^{j\omega t} R(\omega)e^{j\phi(\omega)}
\]

\[
= R(\omega)e^{j(\omega t + \phi(\omega))}
\]

\[
= R(\omega) \cos (\omega t + \phi(\omega)) + j R(\omega) \sin (\omega t + \phi(\omega)).
\]

That is, \( y(t) \) will be scaled in magnitude by a factor \( R(\omega) \), and offset in phase by an amount \( \phi(\omega) \) relative to the magnitude and phase of \( u(t) = e^{j\omega t} \).

Finally, according to (5.4) these magnitude and scaling factors \( R(\omega) \) and \( \phi(\omega) \) are completely determined by the transfer function \( H(s) \) evaluated at the imaginary axis point \( s = j\omega \). In fact, these frequency dependent functions \( R(\omega) \) and \( \phi(\omega) \) are, by definition, called the ‘magnitude frequency response’ and ‘phase frequency response’ respectively.

| Magnitude Frequency Response: \( R(\omega) \triangleq |H(j\omega)| \) |
| Phase Frequency Response: \( \phi(\omega) \triangleq \angle H(j\omega) \) |

Collectively, these preceding principles are known as the system ‘Frequency Response’.
Consider a linear system characterised by an impulse response \( h(t) \), and hence also by a transfer function \( H(s) = \mathcal{L}\{h(t)\} \). Then the system response \( y(t) = [h \ast u](t) \) to a sinusoidal input \( u(t) = \sin \omega t \) of angular frequency \( \omega \) rad/s is of the form
\[
y(t) = R(\omega) \sin(\omega t + \phi(\omega)).
\]
That is, the response \( y(t) \) is also sinusoidal at the same frequency of \( \omega \) rad/s, but of a changed amplitude \( R(\omega) \) and with a phase shift \( \phi(\omega) \) which both depend upon \( \omega \) and are given by
\[
R(\omega) = |H(j\omega)|, \quad \phi(\omega) = \angle H(j\omega).
\]
In particular, since \( H(s) \) evaluated at \( s = j\omega \) is given as
\[
H(j\omega) = R(\omega)e^{j\phi(\omega)}
\]
then the complex valued function \( H(j\omega) \) of \( \omega \) is called the ‘Frequency Response’ of the linear system.

As an alternative (slightly specialised) derivation of the result (5.2), notice that in the particular case of the relationship between \( y(t) \) and \( u(t) \) satisfying a differential equation of the form
\[
\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} y(t) = \sum_{\ell=0}^{n} b_\ell \frac{d^\ell}{dt^\ell} u(t)
\]  
(5.5)
then putting \( u(t) = e^{j\omega t} \) and noting that
\[
\frac{d^\ell}{dt^\ell} e^{j\omega t} = (j\omega)^\ell e^{j\omega t}
\]
leads to
\[
\sum_{\ell=0}^{n} b_\ell (j\omega)^\ell e^{j\omega t} = e^{j\omega t} \sum_{\ell=0}^{n} b_\ell (j\omega)^\ell = e^{j\omega t} B(j\omega)
\]  
(5.6)
where
\[
B(s) \triangleq b_0 + b_1 s + b_2 s^2 + \cdots + b_n s^n.
\]
Similarly, if we assume that the form of the solution \( y(t) \) of (5.5) when \( u(t) = e^{j\omega t} \) is
\[
y(t) = Re^{j\omega t + \phi}\]
then
\[
\frac{d^k}{dt^k} y(t) = Re^{j\phi(j\omega)^k} e^{j\omega t}
\]
so that
\[
\sum_{k=0}^{n} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{n} a_k Re^{j\phi(j\omega)^k} e^{j\omega t} = e^{j\omega t} Re^{j\phi} \sum_{k=0}^{n} a_k (j\omega)^k = e^{j\omega t} Re^{j\phi} A(j\omega)
\]  
(5.7)
where
\[
A(s) \triangleq a_0 + a_1 s + a_2 s^2 + \cdots + a_n s^n.
\]
Therefore, substituting (5.6) and (5.7) into the differential equation (5.5) leads to
\[
e^{j\omega t} Re^{j\phi} A(j\omega) = e^{j\omega t} B(j\omega).
\]
Consequently, after dividing both sides of this equation by the common factor $e^{j\omega t}$

$$Re^{j\phi} = \frac{B(j\omega)}{A(j\omega)} = \frac{B(s)}{A(s)} \bigg|_{s=j\omega} = H(s)|_{s=j\omega} = H(j\omega).$$

Therefore, by an alternate route, we have again established that the magnitude scaling factor $R$ and phase offset factor $\phi$ are given (respectively) by the magnitude $|H(j\omega)|$ and argument $\angle H(j\omega)$ of $H(j\omega)$, the system Laplace transform Transfer function evaluated at $s = j\omega$.

### 5.2 The Fourier Transform

Having now presented the idea of the frequency response $H(s)|_{s=j\omega} = H(j\omega)$ of linear systems, it is appropriate to recognise that in the special case of this Laplace transform value $H(s)$ being evaluated at the purely imaginary value $s = j\omega$, it is known as the ‘Fourier transform’.

For a given signal $f(t)$, its Fourier transform $F(\omega) = \mathcal{F}\{f\}$ is defined as

$$F(\omega) = \mathcal{F}\{f(t)\} \triangleq \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt. \quad (5.8)$$

The physical interpretation of the Fourier transform follows from that of the Laplace transform given in section 4.2 after $s = j\omega$ is substituted. That is, the value $F(\omega)$ is a measure of the ‘strength’ of the dual components $\cos \omega t$ and $\sin \omega t$ in the signal $f(t)$.

To elaborate on this, since $e^{j\omega t} = \cos \omega t + j \sin \omega t$, then $(5.8)$ can be rewritten as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cos \omega t \, dt - j \int_{-\infty}^{\infty} f(t) \sin \omega t \, dt.$$ 

so that real part of $F(\omega)$ pertains to the strength of the $\cos \omega t$ component in $f(t)$ and the imaginary part to the $\sin \omega t$ strength. Because of this interpretation, the Fourier transform $F(\omega)$ is sometimes referred to as the ‘spectrum’ of the signal $f(t)$.

It is particularly important, especially in the context of filter design, to recognise the following role for the Fourier transform in regard to system frequency response.

For a linear and time invariant system with impulse response $h(t)$, its frequency response $H(\omega) = R(\omega)e^{j\phi(\omega)}$ is given as the Fourier transform of the impulse response:

$$H(\omega) = \mathcal{F}\{h(t)\}. \quad (5.9)$$

The Fourier transform is named after a French mathematician named Joseph Fourier who, at the time in the early 19th Century that he formulated his ideas that signals could be decomposed into sinusoids, was ridiculed for his opinions. Fourier’s original motivation was his interest in solving the partial differential equations that describe how heat propagates in a physical medium.

**Example 5.1 Rectangular Pulse** Consider the rectangular pulse signal $f(t)$ shown in the left diagram of figure 5.1:

$$f(t) = \Pi_\tau(t) \triangleq \begin{cases} 
1 & : |t| \leq \tau \\
0 & : |t| > \tau.
\end{cases} \quad (5.10)$$
Then by the definition (5.8)

\[ F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt \]

\[ = \int_{-\tau}^{\tau} e^{-j\omega t} \, dt \]

\[ = \left. \frac{1}{j\omega} e^{-j\omega t} \right|_{t=-\tau}^{t=\tau} \]

\[ = \frac{2j\sin\omega\tau}{j\omega} = 2\sin\omega\tau \omega. \] (5.11)

A plot of this spectrum \( F(\omega) \) is shown in the right diagram of figure 5.1. This sort of \( \sin x / x \) type function occurs so frequently in signal processing applications that it is given the special name of the ‘Cardinal Sine’ or ‘sinc’ function

\[ \text{sinc}(x) \triangleq \frac{\sin \pi x}{\pi x}. \] (5.12)

With this in mind, in the special case (5.11) considered here

\[ F(\omega) = 2\tau \text{sinc} \left( \frac{\omega\tau}{\pi} \right). \]

That is, a pulse of width \( 2\tau \) has a spectrum with nulls at integer multiples of \( 2\pi / \tau \). In particular, a unit area pulse made up of a height \( 1 / 2\pi \) and over an interval \( t \in [-\pi, \pi] \) has a spectrum of \( \text{sinc}(\omega) \), which has has spectral nulls at integer values of \( \omega \). As will be seen in the following chapter, a pulse of this sort, and hence the associated spectrum \( \text{sinc}(\omega) \), has particular relevance in the area of digital signal processing, and this underlies the special definition (5.12) of the cardinal sine function.

![Figure 5.1: Rectangular pulse \( f(t) \) and its Fourier transform \( F(\omega) = \mathcal{F}\{f(t)\} = 2\sin(\omega\tau/\pi) \). - see Example 5.1](image)

**Example 5.2** Two sided exponential Consider the signal \( f(t) \) shown in the left diagram of figure 5.2

\[ f(t) = e^{-\alpha|t|}; \text{Real} \{\alpha\} > 0. \] (5.13)
Then by the definition (5.8)

\[
F(\omega) = \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-j\omega t} \, dt
\]

\[
= \int_{-\infty}^{0} e^{\alpha t} e^{-j\omega t} \, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} \, dt
\]

\[
= \int_{-\infty}^{0} e^{(\alpha-j\omega)t} \, dt + \int_{0}^{\infty} e^{-(\alpha+j\omega)t} \, dt
\]

\[
= \frac{e^{(\alpha-j\omega)t}}{\alpha-j\omega} \bigg|_{t=-\infty}^{t=0} + \frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)} \bigg|_{t=0}^{t=\infty}
\]

\[
= \frac{1}{\alpha-j\omega} + \frac{1}{\alpha+j\omega}
\]

\[
= \frac{2\alpha}{\omega^2 + \alpha^2}
\]

(5.14)

A plot of \(f(t)\) given by (5.13) and its corresponding spectrum \(F(\omega)\) given by (5.14) are shown in the right-hand diagram of figure 5.2.

![Plot of f(t) and its Fourier transform F(\omega)](image)

Figure 5.2: Two-sided exponential \(f(t)e^{-\alpha|t|}\) and its Fourier transform \(F(\omega)\) = \(F\{f(t)\}\) = \(2\alpha/(\omega^2 + \alpha^2)\) - see Example 5.2.

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**Example 5.3** Pure Tone Consider the signal \(f(t)\) shown in the left diagram of figure 5.3

\[f(t) = \cos \omega_0 t\]

(5.15)

Then by the definition (5.8)

\[
F(\omega) = \int_{-\infty}^{\infty} \cos \omega_0 t e^{-j\omega t} \, dt
\]

\[
= \int_{-\infty}^{\infty} \left( \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} \right) e^{-j\omega t} \, dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t} \, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(\omega_0+\omega)t} \, dt
\]

\[
= \frac{1}{2} \int_{-\infty}^{\infty} e^{j(\omega_0-\omega)t} \, dt + \frac{1}{2} \int_{-\infty}^{\infty} e^{-j(\omega_0+\omega)t} \, dt.
\]

(5.16)
There are difficulties (discussed in the following paragraph) associated with evaluating these integrals, but first notice that
\[ \int_{-\infty}^{\infty} e^{j(\omega_0 - \omega)t} \, dt = \int_{-\infty}^{\infty} \cos(\omega_0 - \omega)t \, dt + j \int_{-\infty}^{\infty} \sin(\omega_0 - \omega)t \, dt. \] (5.17)

Now when \( \omega \neq \omega_0 \), the above integrals will clearly be zero, since they measure the area under sine and cosine functions which have positive area components exactly cancelling negative area ones. On the other hand, when \( \omega = \omega_0 \), then the integrands are equal to 1 and 0 respectively, and hence the total integral is equal to \( +\infty \). Via the discussion of section 2.1.5 (and elsewhere), it has been made clear that it is often important to use the Dirac delta function in order to be precise about what is meant by \( +\infty \), in which case (see Appendix 5.A.1) these intuitive ideas of the integrals in (5.16) being either zero or \( +\infty \) are more precisely expressed as
\[ \int_{-\infty}^{\infty} e^{j(\omega_0 - \omega)t} \, dt = 2\pi \delta(\omega - \omega_0). \]

Substituting this in (5.16) then provides
\[ F(\omega) = \pi \delta(\omega + \omega_0) + \pi \delta(\omega - \omega_0). \]

A plot of \( f(t) \) given by (5.15) together with this spectrum \( F(\omega) \) is shown in the right-hand diagram of figure 5.3.

![Diagram of pure tone and its Fourier transform](image)

Figure 5.3: Pure tone \( f(t) = \cos \omega_0 t \) and its Fourier transform \( F(\omega) = \pi \delta(\omega \pm \omega_0) \) - see Example 5.3

Notice that in all these examples, the Fourier transform is purely real valued, which indicates that the underlying signal \( f(t) \) is not composed of any \( \sin \omega t \) terms. This makes sense, since all the signals \( f(t) \) just presented are symmetric about the \( y \) axis just as the \( \cos \omega t \) signal is, and hence only \( \cos \omega t \) components should be present in their spectral decomposition via \( F(\omega) \). That is, signals \( f(t) \) that are symmetric about the \( y \)-axis are associated with purely real valued spectra \( F(\omega) \).

In general though, the Fourier transform will be a complex valued function of frequency \( \omega \), as reference to table 5.1 shows where a range of common Fourier transform results are presented. Note that many of the results in table 5.1 are obtained simply from the corresponding Laplace transform result presented previously in table 4.1 but with the substitution \( s = j\omega \) being made.
5.2.1 Inverse Fourier Transform

If the choice $\gamma = 0$ is made in the inverse Laplace transform definition (4.43) and the substitution $s = j\omega$ is made, then recognising that in this case $ds = j\,d\omega$ delivers the inverse Fourier transform as follows.

Given a Fourier transform $F(\omega)$, the signal $f(t)$ such that $F(\omega) = \mathcal{F}\{f\}$ may be found via the inverse Fourier transform.

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega. \quad (5.18)$$

**Example 5.4** Suppose that

$$F(\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

which is illustrated in figure 5.4. Then according to the inverse Fourier transform definition (5.18)

![Figure 5.4: Signal whose inverse Fourier transform is found in Example 5.4](image)

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0) \right] e^{j\omega t} \, d\omega$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} \, d\omega + \frac{1}{2} \int_{-\infty}^{\infty} \delta(\omega + \omega_0) e^{j\omega t} \, d\omega$$

$$= \frac{1}{2} \left[ e^{j\omega_0 t} + e^{-j\omega_0 t} \right]$$

$$= \cos \omega_0 t.$$

**Example 5.5** Suppose that $\alpha \in \mathbb{R}$ with $\alpha > 0$ and that

$$F(\omega) = \frac{2\alpha}{\omega^2 + \alpha^2}.$$
Then according to (5.18), the inverse transform \( f(t) \) of this function is given as

\[
f(t) = \mathcal{F}^{-1} \{ F(\omega) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\alpha}{\omega^2 + \alpha^2} e^{j\omega t} \, d\omega.
\]  

(5.19)

Initially, this may appear difficult to evaluate this integral. However it can be handled by using the Cauchy Residue Theorem techniques of the previous Laplace transform chapter. This is achieved by making the substitution \( s = j\omega \) in (5.19) which implies \( ds = j\, d\omega \) and hence

\[
f(t) = \frac{1}{2\pi j} \int_{\Lambda} \frac{2\alpha}{\alpha^2 - s^2} e^{st} \, ds = -\frac{1}{2\pi j} \int_{\Lambda} \frac{2\alpha}{(s - \alpha)(s + \alpha)} e^{st} \, ds
\]

(5.20)

where \( \Lambda = (-j\infty, j\infty) \) is the imaginary axis as illustrated in figure 5.5. Now, if \( t > 0 \), then \( |e^{st}| \to 0 \)

Figure 5.5: Integration paths used in Example 5.5 – they are to be taken as extending out infinitely.

along the infinite left half plane semi-circle contour \( \Gamma_1 \) shown in figure 5.5 since REAL \( \{ s \} \to -\infty \) along \( \Gamma_1 \). Therefore, for \( t > 0 \)

\[
f(t) = -\frac{1}{2\pi j} \int_{\Lambda+\Gamma_1} \frac{2\alpha}{(s - \alpha)(s + \alpha)} e^{st} \, ds = -\frac{1}{2\pi j} \int_{\Gamma_1} \left. \frac{2\alpha}{(s - \alpha)(s + \alpha)} e^{st} \right|_{s=\alpha} \, ds.
\]

However, this is now an integral around the closed contour \( \Lambda + \Gamma_1 \) whose value, according to Cauchy’s Residue Theorem, is governed by the pole location at \( s = -\alpha \) within that contour according to

\[
f(t) = -\frac{1}{2\pi j} \text{Res}_{s=-\alpha} \left\{ \frac{2\alpha}{(s - \alpha)(s + \alpha)} e^{st} \right\} = \frac{2\alpha}{(s - \alpha)} e^{st} \bigg|_{s=-\alpha} = e^{-\alpha t}.
\]
Similarly, if $t < 0$, then $|e^{st}| \to 0$ along the infinite right half plane semi-circle contour $\Gamma_2$ shown in figure 5.5 and hence for $t < 0$

$$f(t) = -\frac{1}{2\pi j} \int_{\Lambda+\Gamma_2} \frac{2\alpha}{(s - \alpha)(s + \alpha)} e^{st} ds$$

$$= \frac{1}{2\pi j} \text{Res}_{s=\alpha} \left\{ \frac{2\alpha}{(s - \alpha)(s + \alpha)} e^{st} \right\}$$

$$= \frac{2\alpha}{(s + \alpha)} e^{st} \bigg|_{s=\alpha} = e^{\alpha t}$$

where in progressing to the second line, a sign change occurred because of an implied clockwise (as opposed to counter-clockwise) integration path. In summary then, the $t < 0$ and $t \geq 0$ cases can be amalgamated as

$$\mathcal{F}^{-1} \left\{ \frac{2\alpha}{\omega^2 + \alpha^2} \right\} = e^{-\alpha|t|}$$

which is consistent with Example 5.2.
<table>
<thead>
<tr>
<th>Time Domain Signal $f(t)$</th>
<th>Fourier Transform $F(\omega) = \mathcal{F}{f(t)}$</th>
<th>Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(t) = \delta(t)$</td>
<td>$F(\omega) = 1$</td>
<td></td>
</tr>
<tr>
<td>$f(t) = \Pi_\tau(t) \triangleq \begin{cases} 1 &amp;</td>
<td>t</td>
<td>\leq \tau \ 0 &amp;</td>
</tr>
<tr>
<td>$f(t) = \frac{\sin T t}{t}$</td>
<td>$F(\omega) = \pi \Pi_T(\omega)$</td>
<td></td>
</tr>
<tr>
<td>$f(t) = e^{-\alpha</td>
<td>t</td>
<td>}$</td>
</tr>
<tr>
<td>$f(t) = e^{j\omega_0 t}$</td>
<td>$F(\omega) = 2\pi \delta(\omega - \omega_0)$</td>
<td></td>
</tr>
<tr>
<td>$f(t) = \cos \omega_0 t$</td>
<td>$F(\omega) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)$</td>
<td></td>
</tr>
<tr>
<td>$f(t) = \sin \omega_0 t$</td>
<td>$F(\omega) = -j\pi \delta(\omega - \omega_0) + j\pi \delta(\omega + \omega_0)$</td>
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</tr>
<tr>
<td>$f(t) = e^{-\alpha</td>
<td>t</td>
<td>} \cos \omega_0 t$</td>
</tr>
<tr>
<td>$f(t) = e^{-\alpha</td>
<td>t</td>
<td>} \sin \omega_0 t$</td>
</tr>
<tr>
<td>$f(t) = 1(t)$</td>
<td>$F(\omega) = \frac{1}{j\omega}$</td>
<td></td>
</tr>
<tr>
<td>$f(t) = 1(t) \cdot e^{-\alpha t}$</td>
<td>$F(\omega) = \frac{1}{j\omega + \alpha}$</td>
<td>$\text{Real } {\alpha} &gt; 0$</td>
</tr>
<tr>
<td>$f(t) = 1(t) \cdot t^n e^{-\alpha t}$</td>
<td>$F(\omega) = \frac{n!}{(j\omega + \alpha)^{n+1}}$</td>
<td>$\text{Real } {\alpha} &gt; 0$</td>
</tr>
<tr>
<td>$f(t) = 1(t) \cdot e^{-\alpha t} \cos \omega_0 t$</td>
<td>$F(\omega) = \frac{(j\omega - \alpha)}{(j\omega - \alpha)^2 + \omega_0^2}$</td>
<td>$\text{Real } {\alpha} &gt; 0$</td>
</tr>
<tr>
<td>$f(t) = 1(t) \cdot e^{-\alpha t} \sin \omega_0 t$</td>
<td>$F(\omega) = \frac{\omega_0}{(j\omega - \alpha)^2 + \omega_0^2}$</td>
<td>$\text{Real } {\alpha} &gt; 0$</td>
</tr>
</tbody>
</table>

Table 5.1: Fourier transforms of some common signals
5.2.2 Properties of the Fourier Transform

Because the Fourier transform is nothing more than a special case of the Laplace transform, it inherits all the properties of the Laplace transform that were presented in section 4.4 after the substitution $s = j\omega$ has also been made. Additionally, precisely because of the special case nature of $s = j\omega$, the Fourier transform possesses some important special properties of its own that are discussed in this section.

Properties inherited from Laplace transform Properties

These properties have already been established and discussed in detail in Chapter 4 and hence they are simply listed here. In what follows, $f(t)$, $g(t)$, $h(t)$, $u(t)$ and $y(t)$ are arbitrary signals while $\alpha, \beta \in \mathbb{C}$ are arbitrary constant scalars.

**Linearity**

$$\mathcal{F}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{F}\{f(t)\} + \beta \mathcal{F}\{g(t)\}.$$  

**Time shift**

$$\mathcal{F}\{f(t-T)\} = e^{-j\omega T} \mathcal{F}\{f(t)\}. \quad (5.21)$$

**Time Scale** If $\mathcal{F}\{f(t)\} = F(\omega)$ then

$$\mathcal{F}\{f(\alpha t)\} = \frac{1}{\alpha} F\left(\frac{\omega}{\alpha}\right).$$

**Modulation Property** This is obtained by setting $\alpha = j\omega_0$ in (4.35) to establish that if $\mathcal{F}\{f(t)\} = F(\omega)$ then

$$\mathcal{F}\{e^{j\omega_0 t} \cdot f(t)\} = F(\omega - \omega_0). \quad (5.22)$$

**Example 5.6 Radio Transmission** The modulation property is fundamental to analysing the transmission of information via propagation at radio frequencies. For instance, consider a signal $f(t)$ with spectrum $F(\omega)$ that is to be transmitted. This might, for instance, be the voltage developed by the microphone in a mobile telephone in response to the user speaking into it.

Suppose that spectrum $F(\omega)$ consists only of components at frequencies $f = \omega/(2\pi)$ below 10kHz. Again, this would be typical for a voice signal $f(t)$. Due to the electromagnetic properties of free space (as described by Maxwell’s equations), signals in this frequency range are not well suited to long distance transmission.

In recognition of this, and the fact that signals in frequency ranges of 1MHz and higher are well suited to transmission, the key idea of radio frequency transmission is to shift the spectral content of the signal $f(t)$ into a higher, and more appropriate range.

For this purpose, a carrier frequency $\omega_c$, at an appropriately high value is specified, and then used to generate a carrier wave signal $s(t) = \cos \omega_c t$. In turn this is used to generate a modulated signal $m(t)$ according to the multiplication of the signal $f(t)$ to be transmitted and the carrier signal $\cos \omega_c t$

$$m(t) = f(t) s(t) = f(t) \cos \omega_c t. \quad (5.23)$$
The spectrum $M(\omega)$ of this modulated signal can then be found using the modulation property (5.22) as follows

\[
M(\omega) = \mathcal{F}\{f(t) \cos \omega_c t\} \\
= \mathcal{F}\left\{f(t) \cdot \frac{1}{2} [e^{j\omega_c t} + e^{-j\omega_c t}]\right\} \\
= \frac{1}{2} \mathcal{F}\{f(t) \cdot e^{j\omega_c t}\} + \frac{1}{2} \mathcal{F}\{f(t) \cdot e^{-j\omega_c t}\} \\
= \frac{1}{2} F(\omega - \omega_c) + \frac{1}{2} F(\omega + \omega_c).
\]

That is, the modulated signal $m(t)$ can be considered as two copies of the original signal $f(t)$, but with spectral components shifted to the carrier frequencies $\omega_c$ and $-\omega_c$. This is illustrated graphically in figure 5.6.

![Figure 5.6: Radio Transmission via the modulation property. The signal $f(t)$ with spectrum $F(\omega)$ is, via multiplication with the carrier signal $\cos \omega_c t$, changed to the modulated signal $m(t)$ with spectrum $M(\omega) = F(\omega \pm \omega_c)$.](image)

**Convolution** Suppose that $y(t) = [h \circledast u](t)$. Then

\[
\mathcal{F}\{y(t)\} = \mathcal{F}\{[h \circledast u](t)\} = \mathcal{F}\{h(t)\} \mathcal{F}\{u(t)\}.
\]

In the context of system frequency response, this principle is particularly important. It states that for a linear and time invariant system with impulse response $h(t)$, then the spectrum $U(\omega) = \mathcal{F}\{u(t)\}$ of an input to that system is related to the spectrum $Y(\omega) = \mathcal{F}\{y(t)\}$ of the output of that system according to

\[
Y(\omega) = H(\omega)U(\omega)
\]

where $H(\omega) = \mathcal{F}\{h(t)\}$ is the frequency response of the system.
Differentiation Suppose that \( \lim_{|t| \to \infty} f(t) = 0 \). Then
\[
\mathcal{F} \left\{ \frac{d^n}{dt^n} f(t) \right\} = (j \omega)^n \mathcal{F} \{ f(t) \}.
\]

Integration
\[
\mathcal{F} \left\{ \int \cdots \int f(t_1) \, dt_1 \cdots dt_n \right\}_{n \text{ times}} = \frac{1}{(j \omega)^n} \mathcal{F} \{ f(t) \}.
\]

Transform of a product Suppose that \( F(\omega) = \mathcal{F} \{ f(t) \} \) and \( G(\omega) = \mathcal{F} \{ g(t) \} \) where \( f(t) \) and \( g(t) \) are both real valued signals. Then
\[
\mathcal{F} \{ f(t)g(t) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega - \sigma)G(\sigma) \, d\sigma = \frac{1}{2\pi} [F \odot G](\omega). \tag{5.24}
\]

Symmetry and Realness

In practical situations, any signals of interest are real valued because they represent physical quantities. For example, the voltage signal coming out of a microphone in a mobile telephone handset cannot take on imaginary number values. It is therefore important to examine the consequences on the spectrum \( F(\omega) = \mathcal{F} \{ f(t) \} \) of \( f(t) \) being real valued; this topic has already been raised on page 183 in relation to the Fourier transform examples presented there.

To investigate this, notice that if \( f(t) \) is real valued, then it is equal to its complex conjugate. That is, for any complex number \( z = x + jy \) its conjugate is \( \overline{z} = x - jy \). Therefore, if \( z \) is purely real, then \( y = 0 \) and hence \( z = \overline{z} \).

Consequently, if \( f(t) \) is purely real valued, then we can characterise this property by the equality \( f(t) = \overline{f(t)} \) in which case the complex conjugate of the Fourier transform of \( f(t) \) is given by
\[
\overline{F(\omega)} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt = \int_{-\infty}^{\infty} \overline{f(t)}e^{j\omega t} \, dt = \int_{-\infty}^{\infty} f(t)e^{j\omega t} \, dt ; \text{Since } f \text{ is purely real } \Rightarrow \ f = \overline{f}
\]
\[
= F(-\omega).
\]

If \( f(t) \) is real valued then
\[
F(\omega) = \overline{F(-\omega)} \tag{5.25}
\]
In this case, \( F(\omega) \) is called an ‘Hermitian’ function.

This principle has further ramifications. Firstly, (5.25) indicates that since the real part of \( F(\omega) \) will be unaffected by conjugation, then
\[
\text{Re}\{F(\omega)\} = \text{Re}\{F(-\omega)\}.
\]
That is, if \( f(t) \) is a real valued signal then the real part of \( F(\omega) \) is an even function of frequency \( \omega \), and hence symmetrical about the \( y \) axis when plotted.

Similarly, the effect of conjugation of \( F(\omega) \) is to flip the sign of its imaginary part, in which case by (5.25)

\[
\text{Im}\{F(\omega)\} = -\text{Im}\{F(-\omega)\}
\]

so that \( \text{Im}\{F(\omega)\} \) is an odd function of frequency and hence symmetrical about the line \( y = x \) when plotted. Finally, again via (5.25)

\[
|F(\omega)|^2 = F(\omega) \overline{F(\omega)} = F(\omega)F(-\omega)
\]

and by the same reasoning

\[
|F(-\omega)|^2 = F(-\omega) \overline{F(-\omega)} = F(-\omega)F(\omega)
\]

and hence

\[
|F(\omega)| = |F(-\omega)|
\]

so that \( |F(\omega)| \) is an even function of \( \omega \). Summarising these derivations:

\[
f(t) \text{ purely real } \iff \begin{cases} 
\text{Re}\{F(\omega)\} & \text{is an even function of } \omega \\
\text{Im}\{F(\omega)\} & \text{is an odd function of } \omega \\
|F(\omega)| & \text{is an even function of } \omega
\end{cases} \tag{5.26}
\]

There is an important practical aspect of (5.26) which depends upon recalling from (5.9) that, viewed as a function of \( \omega \), the frequency response \( H(j\omega) \) of a linear system \( H(s) \) is the Fourier transform of its impulse response \( h(t) \).

Since any physically realisable system must have a real-valued impulse response, the relationship (5.26) implies that such systems must also have frequency responses that satisfy \( F(\omega) = \overline{F(-\omega)} \), and this will be important in the subsequent study of filter design.

**Energy Equivalence**

If a signal \( f(t) \) is imagined to represent the voltage across (or current through) a 1\( \Omega \) resistor, then the energy dissipated in that resistor over an infinite amount of time would be

\[
\text{Energy} = \int_{-\infty}^{\infty} |f(t)|^2 \, dt.
\]

At the same time, the signal \( f(t) \) has a spectrum \( F(\omega) = \mathcal{F}\{f(t)\} \), and one would hope that the amount of energy in \( f(t) \) was somehow reflected in the ‘size’ of \( F(\omega) \). That is, a ‘large’ signal should have a ‘large’ spectrum and vice-versa.

Parseval’s Theorem guarantees that this is the case by asserting that the energy in the time and frequency domains is the same.

Parseval’s Theorem states that

\[
\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega. \tag{5.27}
\]
To see how this principle arises, note that via (5.18) a signal $g(t)$ can be written in terms of its inverse Fourier transform to yield, upon swapping orders of integration
\[
\int_{-\infty}^{\infty} f(t)g(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \overline{\mathcal{F}\{G(\omega)\}} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[ \int_{-\infty}^{\infty} G(\omega)e^{j\omega t} \, d\omega \right] \, dt
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) \left[ \int_{-\infty}^{\infty} f(t) G(\omega) \, d\omega \right] \, dt
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G(\omega) \, d\omega.
\]

The generalised Parseval’s (or Plancherel’s) Theorem is that for two signals $f(t)$ and $g(t)$.

\[
\int_{-\infty}^{\infty} f(t)g(t) \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega)F(-\omega) \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(-\omega)F(\omega) \, d\omega.
\]  
\[\text{(5.28)}\]

If $f(t) = g(t) \in \mathbb{R}$, then this provides (5.27) as a special case.

**The Paley–Wiener Theorem**

As already pointed out, a key practical application of Fourier transform properties arises by noting that the Frequency response $H(j\omega)$ of a linear system is the Fourier transform of the impulse response $h(t)$. Because of the need that $h(t) = 0$ for $t < 0$ so that a causal (and hence realistic) system is considered, this places some constraints on the nature of $H(j\omega) = \mathcal{F}\{h(t)\}$. In particular, the Paley–Wiener Theorem asserts the following.

A signal $f(t) = 0$ for $t < 0$ if, and only if

\[
\int_{-\infty}^{\infty} \log |F(\omega)| \, d\omega > -\infty.
\]  
\[\text{(5.29)}\]

For the purposes of intuition, the $1 + \omega^2$ term in the denominator can be ignored to conclude that the condition (5.29) is equivalent to a restriction that $|H(j\omega)| \neq 0$ over any band of frequencies. That is, it is impossible to construct a linear system that provides perfect attenuation over a band of sinusoidal component frequencies.

The theorem is named after Norbert Wiener, one of the most brilliant mathematicians of the 20th Century and his colleague Paley, who was killed in a skiing accident at the age of 26.

**Impossibility of Perfect Band-Limiting**

There is an immediate and important consequence of the Paley–Wiener theorem (5.29). Namely, for a signal to be perfectly bandlimited in that its Fourier transform is zero for all $\omega$ beyond some finite ‘bandwidth’ $B$, then that same signal must be non-zero over an infinite duration in the time domain.
More specifically, first recall that according to (5.21), a fundamental property of the Fourier transform is that \( \mathcal{F}\{u(t + T)\} = e^{-j\omega T} \mathcal{F}\{u(t)\} \). Furthermore, since \( |e^{-j\omega T}| = 1 \) for all \( \omega \), then the integral in the Paley–Wiener condition (5.29) can be written as
\[
\int_{-\infty}^{\infty} \log \frac{|U(\omega)|}{1 + \omega^2} \, d\omega = \int_{-\infty}^{\infty} \log \frac{|e^{-j\omega T} U(\omega)|}{1 + \omega^2} \, d\omega.
\]
Now if \( U(\omega) \) is perfectly band-limited so that \( |U(\omega)| = 0 \) for \( |\omega| > B \), then the above left hand, and hence also right hand integrals will equal \(-\infty\). Therefore, the Paley–Wiener condition on \( e^{-j\omega T} U(\omega) \) will not be satisfied, and hence by the if-and-only-if nature of the Paley–Wiener theore, the time domain signal \( \mathcal{F}^{-1}\{e^{-j\omega T} U(\omega)\} = u(t + T) \) must be non-zero for some \( t + T < 0 \), and hence for some \( t < -T \).

However, in this argument, the time offset \( T \) is arbitrary, and it can therefore be made as large, both positively \textit{and} negatively, as we like. Therefore:

A consequence of the Paley–Wiener theorem is that a signal \( u(t) \) has Fourier transform \( U(\omega) \) that is perfectly band-limited as
\[
|U(\omega)| = 0 \quad \text{for} \quad |\omega| > B \tag{5.30}
\]
if, and only if for any \( T \in (-\infty, \infty) \), there exists a \( t_* < T \) such that \( u(t_*) \neq 0 \).

That is, a signal has perfectly bandlimited spectrum if and only if that signal has persisted for all time. Clearly, this is a physical impossibility, and hence any physically realisable signal can never be perfectly bandlimited. In spite of this, in these same practical applications it is always clear that despite the absence of any exact band-limiting frequency, there is a point \( B \) rad/s beyond which the spectrum \( U(\omega) \) of the signal of interest \( u(t) \) is so small and hence negligible, so that for all practical purposes \( u(t) \) can be safely assumed to be band-limited at \( B \) rad/s.

Duality

The intuitive understanding of the duality property is that if a particular principle holds in the Fourier transformation mapping from time to frequency domain, then the domains can be swapped and the same principle still holds.

To develop this in detail, consider a signal \( f(t) \) that under a Fourier transform operation maps to \( F(\omega) = \mathcal{F}\{f(t)\} \), which will be denoted as
\[
f(t) \overset{\mathcal{F}}{\longrightarrow} F(\omega) \tag{5.31}
\]
where, as should now be clear from the definition (5.8)
\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt. \tag{5.32}
\]
Now consider the idea of taking the Fourier transform of \( F(\omega) \) itself; ie. a second Fourier transform. For this purpose, consider \( F \) as a function of \( t \) (any symbol can be used for the argument of \( F \)) so that from (5.32)
\[
F(t) = \int_{-\infty}^{\infty} f(\tau) e^{-j\tau t} \, d\tau. \tag{5.33}
\]
In this case, using the definition (5.32)

\[ \mathcal{F}\{F(t)\} = \int_{-\infty}^{\infty} F(t)e^{-j\omega t} \, dt \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(\tau)e^{-j\tau t} \, d\tau \right] e^{-j\omega t} \, dt \]

\[ = \int_{-\infty}^{\infty} f(\tau) \int_{-\infty}^{\infty} e^{-j(\tau+\omega)t} \, dt \, d\tau \]

\[ = 2\pi f(-\omega) \]

where the results of Appendix 5.A.1 are used to assert the Dirac delta equivalence in the above computation. Furthermore, via an almost identical argument

\[ \mathcal{F}^{-1}\{f(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega)e^{j\omega t} \, d\omega \]

\[ = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} F(\sigma)e^{j\sigma t} \, d\sigma \right] e^{j\omega t} \, d\omega \]

\[ = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} F(\sigma) \int_{-\infty}^{\infty} e^{j(\tau+\sigma)t} \, dt \, d\sigma \]

\[ = \frac{1}{2\pi} F(-t). \]

These developments then provide a formal statement of the duality property of Fourier transforms.

Suppose that

\[ f(t) \xrightarrow{\mathcal{F}} F(\omega). \quad (5.34) \]

Then

\[ F(t) \xrightarrow{\mathcal{F}} 2\pi f(-\omega). \quad (5.35) \]

\[ f(\omega) \xrightarrow{\mathcal{F}^{-1}} \frac{1}{2\pi} F(-t). \quad (5.36) \]

In intuitive terms, what this means is that once a Fourier transform pair or a Fourier transform principle has been established, then the time and frequency domains \( t \) and \( \omega \) can be exchanged, and (modulo changes of sign and \( 2\pi \) scaling factors), the pair or principle still holds.

Put another way, and speaking somewhat loosely, the duality property says that if a certain ‘shape’ signal (for example a pulse shape) has a transform of a certain shape (a sinc shape), then the reverse applies, and a sinc shaped signal will have a transform that is a pulse shape.

The important and useful consequences of the duality principle are perhaps best conveyed by example. In the following, the preceding intuitive explanation is made more rigorous.

**Example 5.7** Computing the spectrum of sinc-type signals. Consider the signal

\[ f(t) = \frac{\sin Tt}{t} \]
which is shown in the left diagram of figure 5.7 and suppose that we wish to compute its spectrum as

\[ F(\omega) = \mathcal{F}\{ f(t) \} = \int_{-\infty}^{\infty} \left( \frac{\sin Tt}{t} \right) e^{-j\omega t} \, dt. \]

Now this integral is quite difficult to evaluate. However, this problem can be side-stepped using the duality property by noting that according to Example 5.1 it has already been established that with \( \Pi_\tau \) representing the width 2\( \tau \) pulse-function defined in that example.

\[ \Pi_\tau(t) \xrightarrow{\mathcal{F}} 2 \left( \frac{\sin \omega \tau}{\omega} \right). \]

Therefore, the duality principle asserts that

\[ 2 \left( \frac{\sin t\tau}{t} \right) \xrightarrow{\mathcal{F}} 2\pi \Pi_\tau(-\omega) \]

and hence, by changing \( \tau \mapsto T \)

\[ \frac{\sin tT}{t} \xrightarrow{\mathcal{F}} \frac{1}{2} \cdot 2\pi \Pi_\tau(-\omega) |_{\tau=T} = \pi \Pi_T(\omega) \]

where we note that \( \Pi_T(-\omega) = \Pi_T(\omega) \). This is shown in the right-hand diagram of figure 5.7 and illustrates how the duality principle permits the avoidance of a difficult computation by drawing on an already known, and far simpler to derive result.

Comparing with the result of Example 5.1 in broad terms the duality principle asserts that if a rectangular pulse signal has a sinc-type spectrum, then in a dual fashion a sinc signal has a rectangular pulse spectrum.

This is of great practical importance, since it illustrates the shape a signal should have if it is to have its spectral ‘bandwidth’ limited. Namely, if a sinc shaped pulse is used to transmit a binary bit of information, then this will result in the most efficient use of an available spectral bandwidth. This principle is widely exploited in the field of digital communications.

\[ f(t) = \frac{\sin Tt}{T} \]

\[ F(\omega) \]

\[ -5\pi/T \quad -3\pi/T \quad -\pi/T \quad 0 \quad \pi/T \quad 3\pi/T \quad 5\pi/T \]

\[ \pi \]

\[ T \]

\[ \omega \]

**Figure 5.7:** Sinc-type function \( f(t) \) and its rectangular pulse Fourier Transform. see Example 5.7
Example 5.8  Duality of product and convolution  Take two signals $f(t)$ and $g(t)$. Then a fundamental Fourier transform principle introduced on page 189 is that

$$[f \circledast g](t) \xrightarrow{F} F(\omega)G(\omega)$$  \hspace{1cm} (5.37)

where $F(\omega) = F\{f(t)\}$, $G(\omega) = F\{g(t)\}$. Therefore, according to the duality principle

$$F(t)G(t) \xrightarrow{F} 2\pi[f \circledast g](-\omega).$$  \hspace{1cm} (5.38)

Of course, we are free to take the signals $f(t)$ and $g(t)$ as we like, so in the above discussion we could substitute $f(t) \mapsto F^{-1}\{f\}$, $g(t) \mapsto F^{-1}\{g\}$ in which case (5.37) becomes

$$[F^{-1}\{f\} \circledast F^{-1}\{g\}](t) \xrightarrow{F} F\{F^{-1}\{f\}\}F\{F^{-1}\{g\}\} = f(\omega)g(\omega).$$  \hspace{1cm} (5.39)

According to (5.38), this then implies the dual relationship

$$f(t)g(t) \xrightarrow{F} 2\pi [F^{-1}\{f\} \circledast F^{-1}\{g\}](\omega)$$  \hspace{1cm} (5.40)

However, using the second duality property (5.36) implies (here we assume that $f(t)$ and $g(t)$ are real valued so that $F(-\omega) = \overline{F(\omega)}$ and likewise for $G(\omega)$),

$$f(t)g(t) \xrightarrow{F} \frac{1}{2\pi} \mathcal{H} \{f \circledast \overline{g}\} (\omega)$$

That is, starting from the principle that convolution in the time domain corresponds to multiplication in the frequency domain, we obtain a dual result by simply ‘flipping’ the domains. That is, multiplication in the time domain corresponds to convolution in the frequency domain, although a factor of $2\pi$ is required in this dual result.

Suppose that $f(t)$ and $g(t)$ are purely real valued. Then

$$H(\omega) \overset{\triangle}{=} F\{f(t) \cdot g(t)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\sigma)G(\omega - \sigma) \, d\sigma = \frac{1}{2\pi} [F \circledast \overline{G}] (\omega).$$  \hspace{1cm} (5.41)

Example 5.9  Duality of realness/Hermitian relationship  It has already been established that if $f(t) \in \mathbb{R}$, then $F(\omega) = F\{f(t)\}$ is Hermitian in that $F(\omega) = \overline{F(-\omega)}$. However, by the duality relationship

$$F(t) \xrightarrow{F} 2\pi f(-\omega),$$

the Hermitian signal $F(t)$ maps to a real valued signal. Since, again we are free to use whatever signal we like for $f(t)$, this again implies that time and frequency domains can be exchanged without altering the validity of a Fourier transform property.
F(\omega) \text{ purely real } \iff \begin{cases} \Re\{f(t)\} \text{ is an even function of } t \\ \Im\{f(t)\} \text{ is an odd function of } t \\ |f(t)| \text{ is an even function of } t \end{cases} \quad (5.42)

This dual result has an important practical ramification, that again depends on recognising that system frequency response is the Fourier transform of the system impulse response. In particular, suppose that a linear system is required to have no phase shift in its frequency response - such a system would be extremely useful in automatic control system design. Since the phase shift imposed by this system with an impulse response \( h(t) \) is the complex argument of \( \phi = \angle F\{h\} \), then zero phase shift requires that \( F\{h\} \) be purely real valued.

However, via the dual result developed in this example, if \( F\{h(t)\} \) is to be real valued, then \( jh(t) \) must be an even function of \( t \), and hence symmetrical about the \( y \)-axis. Therefore, the causality requirement \( h(t) = 0 \) for \( t < 0 \) (see section 3.12.2) is only achievable if \( h(t) = 0 \) for all \( t \). A consequence of the dual result of this example is that it is impossible to build a non-trivial (ie non-zero response) linear system \( H(s) \) that has both zero phase shift, and is also causal.

Example 5.10 Duality of Band-limiting and Time-limiting of signals

Previously, on page 193 it was established that an important consequence of the Paley–Wiener theorem was that if a signal \( f(t) \) was perfectly band-limited in the frequency domain in that \( |F\{f(t)\}| = |F(\omega)| = 0 \) for \( |\omega| > B \), then necessarily \( f(t) \) was not time limited; no finite \( T \) existed for which \( f(t) = 0 \) for all \( t < T \).

More specifically, it was established that for any \( B < \infty \), there exists a \( T \) with \( |T| < \infty \) such that

\[
f(t) : f(t) \neq 0, \forall t \in (-\infty, T) \quad \overset{\mathcal{F}}{\rightarrow} \quad F(\omega) : |F(\omega)| = 0, |\omega| \in (B, \infty).
\]

An immediate consequence of the duality principle (5.34), (5.35) therefore is that

\[
f(t) : |f(t)| = 0, \forall t \in [T, \infty) \quad \overset{\mathcal{F}}{\rightarrow} \quad F(\omega) : |F(\omega)| \neq 0 \in (-\infty, B]
\]

and furthermore, again by the duality principle result (5.36), for any \( B < \infty \), there exists some \( T < \infty \) such that

\[
F(\omega) : F(\omega) \neq 0, \omega \in (-\infty, B] \quad \overset{\mathcal{F}^{-1}}{\rightarrow} \quad f(t) : |f(t)| = 0, \forall t \in [T, \infty).
\]

Consequently, while the Paley–Wiener theorem immediately implies that a frequency domain band-limited signal must, in the time domain, be of infinite duration, the duality principle further asserts that this is an if-and-only-if principle.

That is a signal can be limited (i.e. equal to zero beyond some finite point) in one domain, if and only if it is of infinite duration in the other domain. Time limited signals occur only when they are of infinite bandwidth in the frequency domain. Band limited signal in the frequency domain are only possible if they exist for infinite time.

Some readers may recognise this principle as being reminiscent of the Heisenberg uncertainty principle in the theory of Quantum Mechanics. This asserts that any attempt to provide hard limits on the position of a quantum particle (eg. an electron) necessarily implies lack of any hard limits on the momentum of that same particle.
This similarity is, in fact, no co-incidence since there is a Fourier transform relationship between position and momentum in the Schrödinger, Dirac, Von–Neumann and other formulations of quantum theory. Therefore the Heisenberg uncertainty principle can be seen as a direct consequence of the Paley–Wiener theorem and its duality principle analogues.

Summary of Properties

The Fourier transform properties developed and illustrated in this section are summarised in table 5.2 following. Notice that in that table, and unlike the previous Laplace transform property table 4.2 for the case of function differentiation, it has not been assumed that \( f(t) = 0 \) for \( t < 0 \). Rather, it is assumed that \( f(-\infty) = 0 \). This is reasonable on the grounds that we only wish to consider finite energy signals.
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<tr>
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<td>$f(t) = g(t - T)$</td>
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<tr>
<td>Integration</td>
<td>$y(t) = \int \cdots \int f(t_1) , dt_1 \cdots , dt_n$ $n$ times</td>
<td>$Y(s) = \frac{1}{(j\omega)^n} F(\omega)$</td>
</tr>
<tr>
<td>Multiply by $t$</td>
<td>$f(t) = t^n g(t)$</td>
<td>$F(\omega) = (j)^n \frac{d^n}{d\omega^n} G(\omega)$</td>
</tr>
<tr>
<td>Real valued signal</td>
<td>$f(t) \in \mathbb{R}$</td>
<td>$F(\omega) = \overline{F(-\omega)}$</td>
</tr>
<tr>
<td>Hermitian signal</td>
<td>$f(t) = \overline{f(-t)}$</td>
<td>$F(\omega) \in \mathbb{R}$</td>
</tr>
<tr>
<td>Energy Relationship</td>
<td>Energy $= \int_{-\infty}^{\infty}</td>
<td>f(t)</td>
</tr>
</tbody>
</table>

Table 5.2: Fourier Transform Properties; it is assumed that $f(-\infty) = f(\infty) = 0.$
5.3 Fourier Series

The Fourier transform, as just presented, is an analysis tool that historically was developed much later than a technique known as ‘Fourier series expansion’. The reason behind this is, that as just illustrated, Fourier analysis can be conducted on rather arbitrary signals. However, the Fourier series expansion is a method only applicable to periodic signals, and hence in some sense considers a simpler, and certainly more restricted case.

To expose the relationships between these ideas of Fourier transforms and Fourier series, recall that a key principle underlying the Fourier transform is one of decomposing a signal into its constituent spectral parts. In the rectangular pulse case of example 5.1, the sinc-function spectrum illustrated in figure 5.1 indicated a continuous distribution of spectral components. However, it happens to be the case that for the important class of periodic signals which satisfy

\[ f(t) = f(t + T) \]

(where \( T \) is the period - see section 2.1.6) then the spectrum is no longer continuous. It is in fact made up only of discrete points. Therefore, a periodic signal can be decomposed into a number of discrete sinusoidal components; see section 5.3.1 following for more justification on this point. This is known as the Fourier series expansion of the signal.

A periodic function \( f(t) = f(t + T) \) of period \( T \) possesses an exponential Fourier series representation

\[
f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}; \quad \omega_s \triangleq \frac{2\pi}{T} \tag{5.43}
\]

where the co-efficients \( \{c_k\} \) are given by the formula

\[
c_k = \frac{1}{T} \int_0^T f(t) e^{-jk\omega_s t} \, dt. \tag{5.44}
\]

An intuitive interpretation of the Fourier expansion (5.43) is that a periodic signal \( f(t) \) of period \( T \) can be decomposed into an infinite sum of sinusoidally shaped signals at the distinct frequencies \( 0, \omega_s, 2\omega_s, 3\omega_s, \ldots \). Each of these frequencies is a multiple (also called a ‘harmonic’) of the ‘fundamental frequency’ \( \omega_2 = 2\pi/T \), which is the rate in radians per second at which the signal \( f(t) = f(t + T) \) repeats.

Furthermore, it is important to recognise that in the case where \( f(t) \) is real valued, then the expansion (5.43) in terms of the exponential terms \( e^{j\omega_s k t} \) can be re-written in terms of \( \sin \omega_s k t \) and \( \cos \omega_s k t \) signals, in which case the resultant expansion is known as the ‘trigonometric Fourier series’.
A real valued periodic function \( f(t) = f(t + T) \) of period \( T \) possesses a trigonometric Fourier series representation

\[
f(t) = c_0 + 2 \sum_{k=1}^{\infty} \alpha_k \cos \omega_k t + 2 \sum_{k=1}^{\infty} \beta_k \sin \omega_k t
\]  

(5.45)

where the co-efficients \( \{\alpha_k\} \), \( \{\beta_k\} \) and \( c_0 \) are given by

\[
\alpha_k = 2 \text{Real}\{c_k\} = \frac{2}{T} \int_0^T f(t) \cos \omega_k t \, dt,
\]  

(5.46)

\[
\beta_k = -2 \text{Imag}\{c_k\} = \frac{2}{T} \int_0^T f(t) \sin \omega_k t \, dt,
\]  

(5.47)

\[
c_0 = \frac{1}{T} \int_0^T f(t) \, dt = \text{‘d.c.’ value of the signal}.
\]  

(5.48)

The trigonometric series representation (5.45) follows directly from the exponential series (5.43) by simply rewriting (5.43) as:

\[
f(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{j\omega_k t} + \sum_{k=1}^{\infty} c_{-k} e^{-j\omega_k t}.
\]  

(5.49)

Now, from (5.56), and since \( f(t) \) is assumed to be real valued

\[
c_{-k} = \frac{1}{T} \int_0^T f(t) e^{j\omega_k t} \, dt = \frac{1}{T} \int_0^T f(t) e^{-j\omega_k t} \, dt = c_k
\]  

(5.50)

Therefore, since \( e^{j\omega_k t} = e^{-j\omega_k t} \) then the substitution of (5.50) in (5.49) provides

\[
f(t) = c_0 + \sum_{k=1}^{\infty} c_k e^{j\omega_k t} + \sum_{k=1}^{\infty} c_k e^{j\omega_k t} = c_0 + \sum_{k=1}^{\infty} 2 \text{Real}\{c_k e^{j\omega_k t}\}
\]  

(5.51)

Now put \( c_k = a_k + j b_k \) and notice that \( e^{j\omega_k t} = \cos \omega_k t + j \sin \omega_k t \) so that (5.51) becomes

\[
f(t) = c_0 + 2 \sum_{k=1}^{\infty} a_k \cos \omega_k t + 2 \sum_{k=1}^{\infty} b_k \sin \omega_k t
\]

\[
c_0 + 2 \sum_{k=1}^{\infty} a_k \cos \omega_k t + 2 \sum_{k=1}^{\infty} b_k \sin \omega_k t
\]

\[
\begin{align*}
\alpha_k &= 2a_k, \quad \beta_k = -2b_k
\end{align*}
\]

which is the trigonometric series (5.45).

The trigonometric Fourier series is perhaps more intuitive, since it is more reasonable that a real valued function \( f(t) \) can be expressed as a sum of other real valued functions \( \sin \) and \( \cos \) as opposed to the complex valued function \( e^{jx} \). However, the trigonometric Fourier series is then also restricted to representing only real valued signals, while the exponential Fourier series (5.43) can accommodate complex valued signals.

A further advantage of the exponential Fourier series is that in general it is easier to calculate \( c_k \) defined by (5.56) than to calculate \( \alpha_k \) and \( \beta_k \) from the \( \sin \) and \( \cos \) integrals. If necessary, the trigonometric Fourier series components \( \{\alpha_k\} \) and \( \{\beta_k\} \) can be obtained by taking real and imaginary parts of the exponential Fourier series components \( \{c_k\} \) and then using the relationships (5.46)–(5.48).
Example 5.11 Cosine Signal Example

Consider the simple case of the signal \( f(t) \) being a cosine signal

\[
f(t) = \cos \omega_0 t
\]

which is periodic with period

\[
T = \frac{2\pi}{\omega_0}
\]

and hence repeats itself at the rate of

\[
\omega_s = \frac{2\pi}{T} = \omega_0 \text{ rad/s}.
\]

Now, this book has frequently employed the technique of expressing such a cosine signal via using De-Moivre’s theorem (see page 13) to write

\[
\cos \omega_0 t = \frac{1}{2} \left( e^{j\omega_0 t} + e^{-j\omega_0 t} \right).
\]

In fact, this is the exponential Fourier series representation of \( \cos \omega_0 t \) since with \( \omega_s = \omega_0 \) it is (5.43) with the substitutions

\[
c_1 = c_{-1} = \frac{1}{2}, \quad c_k = 0 \text{ for } |k| \neq 1.
\]

To verify this, note that via (5.44) and using \( T = 2\pi/\omega_0, \omega_s = \omega_0 \)

\[
c_k = \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega_0} \cos(\omega_0 t) e^{-jkt} dt
\]

\[
= \frac{\omega_0}{4\pi} \int_0^{2\pi/\omega_0} \left[ e^{j\omega_0 t} + e^{-j\omega_0 t} \right] e^{-jkt} dt
\]

\[
= \frac{\omega_0}{4\pi} \int_0^{2\pi/\omega_0} \left[ e^{j(1-k)\omega_0 t} + e^{-j(1+k)\omega_0 t} \right] dt
\]

\[
\begin{align*}
&= \frac{\omega_0}{4\pi} \left[ \frac{e^{j(1-k)\omega_0 t}}{j(1-k)\omega_0} \bigg|_{t=0}^{t=2\pi/\omega_0} + \frac{e^{-j(1+k)\omega_0 t}}{-j(1+k)\omega_0} \bigg|_{t=0}^{t=2\pi/\omega_0} \right] \\
&= \frac{\omega_0}{4\pi} \left[ \frac{e^{j(1-k)2\pi} - 1}{j(1-k)\omega_0} + \frac{e^{-j(1+k)2\pi} - 1}{-j(1+k)\omega_0} \right].
\end{align*}
\]

(5.52)

(5.53)

(5.54)

Now, for any integer \( k \)

\[
e^{j(1-k)2\pi} = e^{-j(1+k)2\pi} = 1
\]

and hence the above expression is zero for all \( k \), except \( k = \pm 1 \) where the denominator terms above are also zero. When \( k = 1 \), then returning to (5.52) indicates that the first integrand is equal to 1, and hence

\[
c_1 = \frac{\omega_0}{4\pi} \int_0^{2\pi/\omega_0} dt = \frac{\omega_0}{4\pi} \cdot \frac{2\pi}{\omega_0} = \frac{1}{2}.
\]

and similarly when \( k = -1 \) equation (5.52) yields \( c_{-1} = 1/2 \).
Example 5.12 Square Wave Example Consider the problem of finding the Fourier series expansion for a periodic function which is a square wave of unit amplitude and period \( T = 0.5 \) seconds. This signal is illustrated as a solid line in figure 5.8. Then by (5.43), the required expansion is given as

\[
f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_k t}
\]

(5.55)

where

\[
c_k = \frac{1}{T} \int_{0}^{T} f(t) e^{-j\omega_k t} \, dt.
\]

(5.56)

Since the period \( T \) is equal to 0.5s then \( \omega_k = 2\pi/0.5 = 4\pi \) and hence by equation (5.56)

\[
c_k = \frac{2}{2\pi} \left\{ \int_{0}^{0.25} e^{-j4\pi kt} \, dt - \int_{0.25}^{0.5} e^{-j4\pi kt} \, dt \right\}
\]

\[
= \frac{j}{2\pi k} \left\{ e^{-j4\pi k0.25} - e^{-j4\pi k0.5} \right\}
\]

\[
= \frac{j}{2\pi k} \left\{ e^{-j2\pi k} - 1 \right\}
\]

\[
= \left\{ \begin{array}{ll}
-\frac{2j}{\pi k} & ; k \text{ odd} \\
0 & ; k \text{ even}
\end{array} \right.
\]

Therefore, the exponential and trigonometric Fourier series representations for \( f(t) \) are (respectively)

\[
f(t) = -\frac{2j}{\pi} \sum_{k=-\infty}^{\infty} \frac{1}{k} e^{j4\pi kt} = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \sin(4\pi kt).
\]

(5.57)

Note that the square wave function is an odd function \( f(t) = -f(-t) \) so that the trigonometric Fourier expansion contains only odd functions; the \( \sin \) functions.

The relationship between the square wave signal \( f(t) \) and a truncated at \( N \)-terms version \( f_N(t) \) of its Fourier expansion (5.57) defined as

\[
f_N(t) \triangleq \frac{4}{\pi} \sum_{k=1}^{N} \frac{1}{k} \sin(4\pi kt)
\]

(5.58)

is shown in figure 5.8 for the two cases of truncation length \( N = 3 \) and \( N = 20 \). Clearly, as shown there, if more terms \( N \) are included in the Fourier expansion, the discrepancy between the expansion and the underlying signal becomes smaller. In the limit as \( N \to \infty \) the discrepancy drops to zero so that the representation (5.57) is exact.

Example 5.13 Triangle Wave Example Consider the problem of finding the Fourier series expansion for the periodic function which is a triangle wave of unit amplitude and period \( T = 5 \) seconds.
Figure 5.8: Square wave signal $f(t)$ together with its partial Fourier expansions $f_N(t)$ for the cases of (left diagram) $N = 3$ Fourier coefficients being included in the expansion and (right diagram) $N = 20$ co-efficients - See Example 5.12.

This signal is illustrated as a dashed line in figure 5.9. In this case with the period $T = 2.5s$ then $\omega_s = 2\pi/2.5 = 4\pi/5$. Therefore, according to (5.44)

$$c_k = \frac{2}{5} \left\{ \int_0^{1.25} \frac{2}{1.25} \left( t - \frac{1.25}{2} \right) e^{-j\frac{4\pi k t}{5}} dt + \int_{1.25}^{2.5} \frac{2}{1.25} (1.875 - t) e^{-j\frac{4\pi k t}{5}} dt \right\}.$$

Now, using integration by parts and with $\phi$ being arbitrary

$$\int t e^{j\phi t} dt = \int t \frac{d}{dt} \left( \frac{e^{j\phi t}}{j\phi} \right) dt = \frac{te^{j\phi t}}{j\phi} - \frac{1}{j\phi} \int e^{j\phi t} dt = \frac{t}{j\phi} e^{j\phi t} + \frac{e^{j\phi t}}{\phi^2}$$

so that

$$c_k = \frac{0.8}{1.25} \left\{ \frac{-5}{j4\pi k} \left( t + \frac{5}{j4\pi k} \right) e^{-\frac{4\pi k t}{j} } + \frac{1.25}{2} \frac{5}{j4\pi k} e^{-\frac{4\pi k t}{j} } \right\}_{0}^{1.25} + $$$$
\left\{ \frac{5}{j4\pi k} \left( t + \frac{5}{j4\pi k} \right) e^{-\frac{4\pi k t}{j} } - 1.875 \frac{5}{j4\pi k} e^{-\frac{4\pi k t}{j} } \right\}_{1.25}^{2.5} \right\}$$

$$= -\frac{4}{\pi^2 k^2} \sin \frac{k\pi}{2}$$

$$= \left\{ \begin{array}{ll}
\frac{4}{\pi^2 k^2} (-1)^{\frac{k-1}{2}} & ; k \text{ odd} \\
0 & ; k \text{ even}
\end{array} \right.$$
The relationship between the triangle wave signal \( f(t) \) and a truncated at \( N \)-terms version \( f_N(t) \) of its Fourier expansion (5.59) defined as

\[
f_N(t) \triangleq \frac{8}{\pi^2} \sum_{k \text{ odd}}^{N} \frac{(-1)^{k+1}}{k^2} \sin \left( \frac{4\pi kt}{5} \right)
\]  

is shown in figure 5.8 for the two cases of truncation length \( N = 3 \) and \( N = 20 \). Clearly, as shown there, if more terms \( N \) are included in the Fourier expansion, the discrepancy between the expansion and the underlying signal becomes smaller. In the limit as \( N \to \infty \) the discrepancy drops to zero so that the representation (5.59) is exact.

Notice how much quicker (compared to the square wave case) the truncated Fourier series \( f_N(t) \) converges to the function \( f(t) \) as \( N \) grows. Without plotting, we could have predicted this would happen because for the triangle wave case, the Fourier co-efficients decay as \( 1/k^2 \). This is much faster than for the square wave case where they decay like \( 1/k^2 \). This means that the 'high frequency components' of the triangle wave signal are much smaller than for square wave signal, so it doesn’t matter so much when they are left out in the situation where expansion length \( N \) is small.

We also knew that this would be the case before we worked out the Fourier series. This is because the triangle wave signal is a much smoother signal than the square wave signal (the square wave signal has points where the derivative is \( \infty \)). Since frequency is a measure of rate of change, this means the presence of stronger high frequency components for the square wave signal.

![Figure 5.9: Triangle wave signal \( f(t) \) together with its partial Fourier expansions \( f_N(t) \) for the cases of (left diagram) \( N = 3 \) Fourier coefficients being included in the expansion on and (right diagram) \( N = 20 \) co-efficients - See Example 5.13](image)

### 5.3.1 Relationship between Fourier Series and the Fourier Transform

The exponential Fourier series (5.43) which decomposes a time domain signal \( f(t) \) into a an infinite sum of further time domain functions \( e^{j\omega_st} = \cos k\omega_st - j \sin k\omega_st \) is, superficially, different to the
Fourier transform $F(\omega) = \mathcal{F}\{f(t)\}$ which takes the same time domain signal $f(t)$, but returns a frequency domain function $F(\omega)$. However, in a more fundamental sense, Fourier series and Fourier transforms are intimately related in that they both decompose signals into spectral components.

To be more explicit on this point, note that since a periodic function of period $T$ satisfies $f(t) = f(t + T)$, then the time shift property for Fourier transforms (see table 5.2) implies that

$$F(\omega) = e^{j\omega T} F(\omega).$$

Now, if $\omega T$ is not a multiple of $2\pi$ (i.e., $\omega \neq k2\pi/T$) then $e^{j\omega T}$ is not equal to 1, and hence $F(\omega) = 0$ is the only solution to the above equation (5.61). At the same time, by Parseval’s Theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

Since the energy in a periodic signal is not zero (it is, in fact, infinite since $f(t) \neq 0$ as $|t| \to \infty$), then this further implies that even though (5.61) implies that $|F(\omega)| = 0$ almost everywhere, it still must have non-zero area underneath it.

Therefore, for a periodic signal $f(t) = f(t + T)$, its Fourier transform must be of the form

$$F(\omega) = \sum_{k=-\infty}^{\infty} b_k \delta(\omega - k\omega_s) \quad ; \omega_s \triangleq \frac{2\pi}{T}. \tag{5.62}$$

That is, the Fourier transform of a periodic signal indicates that its spectral decomposition involves only discrete frequencies at multiples of the ‘harmonic spacing’ $\omega_s = 2\pi/T$. However, the exponential Fourier series (5.43) also asserts this fact by showing how $f(t)$ may be decomposed into $e^{jk\omega_st}$ components which also are discrete frequency terms at the harmonic spacing $\omega_s$. Therefore, in this qualitative sense of decomposing a periodic signal into discrete spectral components that are harmonically related to a fundamental frequency $\omega_s = 2\pi/T$, both the Fourier transform and a Fourier series decomposition are identical.

The two methods are also related in a quantitative sense, which depends on how the Dirac delta weights $\{b_k\}$ in the Fourier transform (5.62) are related to the Fourier series co-efficients $\{c_k\}$ in (5.43).

To evaluate this, the Fourier series representation (5.43) of $f(t)$ may be substituted into the definition of the Fourier transform $F(\omega) = \mathcal{F}\{f(t)\}$

$$F(\omega) = \int_{-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} c_k e^{j\omega_st} \right] e^{-j\omega t} dt.$$

Swapping the order of the summation and integration operations (i.e. exploiting the linearity of the integration operation) then leads to

$$F(\omega) = \sum_{k=-\infty}^{\infty} c_k \int_{-\infty}^{\infty} e^{j(\omega - k\omega_s)t} dt.$$

It is easy to see that the inner integral is zero for $\omega \neq k\omega_s$ (in this case the integrand, being composed of sin and cos terms, has equal area above and below the $t$ axis), and is infinite for $\omega = k\omega_s$ (since the integral if over the whole range $t \in (-\infty, \infty)$), and hence is like a Dirac delta. However, as already mentioned several times now (see Appendix 5.A.1) there is a factor of $2\pi$ involved so that in fact

$$\int_{-\infty}^{\infty} e^{j(\omega - k\omega_s)t} dt = 2\pi \delta(\omega - k\omega_s) \tag{5.63}$$
and therefore

\[ F(\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k \delta(\omega - k\omega_s). \]  \hspace{1cm} (5.64)

Comparing this with (5.62) indicates that the weights \( b_{-2}, b_{-1}, b_0, b_1, b_2, \cdots \) can be simply computed from the Fourier co-efficients \( c_{-2}, c_{-1}, c_0, c_1, c_2, \cdots \) by a straightforward method of multiplying by \( 2\pi \)

\[ b_k = 2\pi c_k. \]  \hspace{1cm} (5.65)

A periodic function \( f(t) = f(t + T) \) has a Fourier transform \( F(\omega) \) consisting only of Dirac delta impulses of the form

\[ F(\omega) = \sum_{k=-\infty}^{\infty} b_k \delta(\omega - k\omega_s) \quad \omega_s = \frac{2\pi}{T} \]

where

\[ b_k = 2\pi c_k = \frac{2\pi}{T} \int_0^T f(t)e^{-jk\omega_s t} dt. \]

**Example 5.14 Transform of cosine and sine** Consider the periodic signal

\[ f(t) = \cos \omega_s t. \]

Then \( f(t) \) is periodic with period \( T = 2\pi/\omega_0 \) so that

\[ \omega_s = \frac{2\pi}{T} = \omega_0 \]

and, according to Example 5.11 has a Fourier series representation of

\[ \cos \omega_s t = \frac{1}{2}e^{-j\omega_0 t} + \frac{1}{2}e^{j\omega_0 t}. \]

That is, the Fourier co-efficients for \( \cos \omega_s t \) are

\[ c_k = \begin{cases} 
1/2 & ; k = \pm 1 \\
0 & ; \text{Otherwise.} 
\end{cases} \]

Therefore, according to (5.64), (5.65) the Fourier transform of \( f(t) = \cos \omega_s t \) has \( b_{\pm 1} = 2\pi c_{\pm 1} = \pi \) and hence

\[ F(\omega) = \pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0). \]

Note that this is identical to the result previously obtained in Example 5.3. As well, by the same arguments, since

\[ \sin \omega_0 t = \frac{j}{2}e^{-j\omega_0 t} - \frac{j}{2}e^{j\omega_0 t} \]

then for \( g(t) = \sin \omega_0 t \)

\[ G(\omega) = \mathcal{F} \{g(t)\} = -j\pi \delta(\omega - \omega_0) + j\pi \delta(\omega + \omega_0). \]
5.3.2 Properties of Fourier Series Co-efficients

As just highlighted, there is a very close connection between Fourier Series and Fourier transform. Namely, The Fourier transform of a periodic signal \( f(t) = f(t + T) \) consists of Direct delta impulses, centred at points \( \omega = k\omega_s \) where \( \omega_s = 2\pi / T \), and with areas \( b_k \) underneath these impulses that are equal to \( b_k = 2\pi c_k \) where \( c_k \) is the \( k \)’th Fourier co-efficient in the Fourier series expansion of \( f(t) \).

As such, Fourier series co-efficients inherit all the properties of the Fourier transform, after recog- nising the \( b_k = 2\pi c_k \) relationship. In particular, assuming in what follows that all signal are periodic of period \( t = T \) and that \( \omega_s = 2\pi / T \) is the rate in rad/s at which these signals repeat, then the following results follow directly from the corresponding Fourier transform result.

**Linearity**

\[
f(t) + g(t) = \sum_{k=-\infty}^{\infty} (c_k + d_k)e^{jk\omega_s t}
\]

where

\[
c_k = \frac{1}{T} \int_{0}^{T} f(t)e^{-jk\omega_s t}, \quad d_k = \frac{1}{T} \int_{0}^{T} g(t)e^{-jk\omega_s t}.
\]

**Time shift** If \( f(t) = g(t - \tau) \) and \( g(t) \) has Fourier series representation

\[
g(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_s t}, \quad d_k = \frac{1}{T} \int_{0}^{T} g(t)e^{-jk\omega_s t}.
\]

then \( f(t) \) has Fourier series representation

\[
f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}, \quad c_k = e^{-jk\omega_s \tau}d_k.
\]

**Time Scale** If \( f(t) = f(t + T) \) has Fourier series representation

\[
f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t},
\]

and \( g(t) = f(\alpha t) \), then \( g(t) = g(t + T/\alpha) \) has Fourier series representation

\[
g(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\alpha\omega_s t},
\]

where \( c_k = d_k / \alpha \).

**Differentiation** Suppose that

\[
f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}
\]

Then

\[
\frac{d^n}{dt^n} f(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_s t}, \quad d_k = (jk\omega_s)^n \cdot c_k.
\]
Integration Suppose that
\[ f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t} \]

Then
\[ \int \cdots \int f(t_1) \, dt_1 \cdots dt_n = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_s t}, \quad d_k = \frac{1}{(jk\omega_s)^n} c_k. \]

Symmetry and Realness Suppose that \( f(t) = f(t + T) \in \mathbb{R} \). Then \( f(t) \) has exponential Fourier series expansion
\[ f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}, \quad c_k = \overline{c_{-k}}. \quad (5.66) \]
Vice-versa, suppose that \( c_k \in \mathbb{R} \) for all \( k \). Then \( f(t) \) defined by \( (5.66) \) satisfies
\[ f(t) = \overline{f(-t)}. \]

Convolution
There are several further properties of Fourier series which are important direct analogues of those existing for Fourier transforms, however they do not follow directly from those Fourier transform results. The first of these cases concerns convolution. Suppose, as throughout this section, that \( f(t) \) and \( g(t) \) are periodic of period \( T \). Then in general the ‘infinite horizon’ convolution
\[ [f \circledast g](t) = \int_{-\infty}^{\infty} f(\sigma)g(t - \sigma) \, d\sigma \quad (5.67) \]
will be infinite for all \( t \), and hence not well defined. This follows directly from the fact that since \( f(t) \) and \( g(t) \) are periodic, then they cannot decay towards zero as \( |t| \to \infty \), and hence the integral \( (5.67) \) is infinite.

In recognition of this difficulty, when dealing with such periodic signals, the ‘horizon’ for convolution is taken as \([0, T]\), the region in which one period occurs, which leads to the following result

Suppose that
\[ f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}, \quad g(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_s t}. \]
Then the signal
\[ h(t) \triangleq \frac{1}{T} \int_0^T f(\sigma)g(t - \sigma) \, d\sigma = \frac{1}{T} \int_0^T g(\sigma)f(t - \sigma) \, d\sigma \]
has an exponential Fourier series representation
\[ h(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t} \quad (5.68) \]
where
\[ a_k = c_k \cdot d_k. \quad (5.69) \]
This can be established by noting that since \( g(t) \) is periodic and hence \( g(t - \sigma + T) = g(t - \sigma) \) then

\[
h(t + T) = \frac{1}{T} \int_{0}^{T} f(\sigma) g(t - \sigma + T) \, d\sigma = \frac{1}{T} \int_{0}^{T} f(\sigma) g(t - \sigma) \, d\sigma = h(t)
\]

and hence \( h(t) \) is periodic. Therefore, it has a Fourier series expansion (5.68) with co-efficients \( \{a_k\} \) given by (5.44)

\[
a_k = \frac{1}{T} \int_{0}^{T} h(t)e^{-j\omega_k t} \, dt
\]

\[
= \frac{1}{T^2} \int_{0}^{T} \left[ \int_{0}^{T} f(\sigma)g(t - \sigma) \, d\sigma \right] e^{-j\omega_k t} \, dt
\]

\[
= \frac{1}{T^2} \int_{0}^{T} \left[ \int_{0}^{T} f(\sigma)g(t - \sigma) \, d\sigma \right] e^{-j\omega_k \sigma} e^{-j\omega_k (t - \sigma)} \, dt
\]

\[
= \frac{1}{T} \int_{0}^{T} f(\sigma)e^{-j\omega_k \sigma} \, d\sigma \left( \frac{1}{T} \int_{0}^{T} g(t - \sigma)e^{-j\omega_k (t - \sigma)} \, dt \right)
\]

\[
= c_k \cdot d_k.
\]

which is (5.69).

**Transform of a product**

The result that is dual to the previous one, and concerns the Fourier co-efficients of the product of two functions, also does not follow from the corresponding Fourier transform result (5.24). This time the problem arises in the convolution of the two spectra (5.24), which in the case of periodic signals, will both consist of Dirac-delta functions. Hence the convolution of these on the right hand side of (5.24) will be infinite, and hence not well defined. Instead, the following result will need to be established separately.

Suppose that

\[
f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_k t}, \quad g(t) = \sum_{k=-\infty}^{\infty} d_k e^{j\omega_k t}.
\] (5.70)

Then the signal

\[
h(t) \triangleq f(t)g(t)
\]

has an exponential Fourier series representation

\[
h(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_k t}
\] (5.71)

where

\[
a_k = \sum_{m=-\infty}^{\infty} c_m d_{k-m}.
\] (5.72)
This can be established by noting that since \( g(t) \) and \( f(t) \) are both periodic, then clearly
\[
h(t + T) = f(t + T)g(t + T) = f(t)g(t) = h(t)
\]
and hence \( h(t) \) is periodic. Therefore, it has a Fourier series expansion \((5.71)\) with co-efficients \( \{a_k\} \) given by \((5.44)\), which on substituting in the expansions \((5.70)\) for \( f(t) \) and \( g(t) \) evaluates to
\[
a_k = \frac{1}{T} \int_0^T f(t)g(t)e^{-jk\omega_st} \, dt
\]
\[
= \frac{1}{T} \int_0^T \left( \sum_{m=-\infty}^\infty c_m e^{jm\omega_st} \right) \left( \sum_{n=-\infty}^\infty d_n e^{jn\omega_st} \right) e^{-jk\omega_st} \, dt
\]
\[
= \sum_{m=-\infty}^\infty c_m \sum_{n=-\infty}^\infty d_n \left( \frac{1}{T} \int_0^T e^{j(m+n-k)\omega_st} \, dt \right).
\]
(5.73)

However, since \( \omega_s T = (2\pi/T) \cdot T = 2\pi \) then
\[
\frac{1}{T} \int_0^T e^{j(m+n-k)\omega_st} \, dt = \frac{1}{T} \frac{e^{j(m+n-k)\omega_s T} - 1}{j(m+n-k)\omega_s} \bigg|_{t=0}^{t=T}
\]
\[
= \frac{e^{j(m+n-k)2\pi} - 1}{j(m+n-k)\omega_s}
\]
(5.74)
and for any integer \( m, n \) and \( k \)
\[
e^{j(m+n-k)2\pi} = 1
\]
so that \((5.75)\) is zero, provided the denominator is not also zero. This latter situation occurs when \( m + n - k = 0 \), in which case the integral \((5.74)\) is simply
\[
\frac{1}{T} \int_0^T e^{j(m+n-k)\omega_st} \, dt = 1
\]
\[
\text{when } m + n - k = 0
\]
and
\[
\int_0^T e^{j(m+n-k)\omega_st} \, dt = 1.
\]
Therefore
\[
\frac{1}{T} \int_0^T e^{j(m+n-k)\omega_st} \, dt = \begin{cases} 
1 & n = k - m \\
0 & n \neq k - m
\end{cases}
\]
(5.76)
Substituting this into \((5.73)\) then leads to
\[
a_k = \sum_{m=-\infty}^\infty c_m \sum_{n=-\infty}^\infty d_n \left( \frac{1}{T} \int_0^T e^{j(m+n-k)\omega_st} \, dt \right) = \sum_{m=-\infty}^\infty c_m d_{k-m}
\]
which is the result \((5.72)\).

**Energy Equivalence**

A further important Fourier series result that does not follow from the corresponding Fourier transform result is the equivalent of the Parseval’s Theorem relationship \((5.28)\). Again, this is because in the case of periodic signals \( f(t) \), \( g(t) \), their energy is infinite since \( f(t), g(t) \) \( \neq 0 \) as \( |t| \to \infty \). To circumvent this difficulty, energy can be measured just over one period of these periodic signals, to provide the following result.
Suppose that
\[ f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}, \quad g(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_s t}. \] (5.77)

Then Parseval’s Theorem states that
\[ \frac{1}{T} \int_0^T f(t) \overline{g(t)} \, dt = \sum_{k=-\infty}^{\infty} c_k \overline{d_k} \] (5.78)
for which an important special case of \( g(t) = f(t) \) implies the power relationship
\[ \frac{1}{T} \int_0^T |f(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2. \] (5.79)

To establish this result, substitute the Fourier series representations (5.77) into (5.78) and swap the order of integration and summation to obtain
\[
\frac{1}{T} \int_0^T f(t) \overline{g(t)} \, dt = \frac{1}{T} \int_0^T \left( \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t} \right) \left( \sum_{m=-\infty}^{\infty} \overline{d_m} e^{jm\omega_s t} \right) \, dt \\
= \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_k \overline{d_m} \left( \frac{1}{T} \int_0^T e^{j(k-m)\omega_s t} \, dt \right). \] (5.80)

However, as explained in the previous section
\[
\frac{1}{T} \int_0^T e^{j(k-m)\omega_s t} \, dt = \begin{cases} 
1 & : k = m \\
0 & : k \neq m.
\end{cases}
\]
Substituting this into the expression (5.80) then gives the result (5.78). An important application of this result is to assess the approximation accuracy of a partial Fourier series. To be more specific, consider a periodic function \( f(t) = f(t + T) \) with Fourier series expansion
\[ f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_s kt} \] (5.81)
together with a version of this expansion that is truncated at \( N \) terms in the summation
\[ f_N(t) \triangleq \sum_{k=-N}^{N} c_k e^{j\omega_s kt}. \] (5.82)

Such truncated expansions occur commonly in signal processing applications, particularly in the area of filter design. They were also considered in Examples 5.12 and 5.13. In all these cases, an important question is how closely \( f_N(t) \) approximates \( f(t) \) for a given \( N \). This can be addressed by considering the discrepancy
\[ f(t) - f_N(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_s kt} - \sum_{k=-N}^{N} c_k e^{j\omega_s kt} = \sum_{|k| > N} c_k e^{j\omega_s kt}. \]
However, by the above Parseval’s theorem, this discrepancy, squared and averaged over a period $T$, is given as

\[ \frac{1}{T} \int_0^T |f(t) - f_N(t)|^2 \, dt = \sum_{|k|>N} |c_k|^2. \]  

(5.83)

Therefore, the rate of decay of the magnitude $|c_k|$ of the Fourier co-efficients of a periodic signal $f(t)$ is a measure of how well a truncated expansion $f_N(t)$ will approximate $f(t)$.

Suppose that $f(t) = f(t + T)$ has exponential Fourier series co-efficients given as

\[ c_k = \frac{1}{T} \int_0^T f(t)e^{-jk\omega_s t} \, dt, \quad \omega_s \triangleq \frac{2\pi}{T} \]

which are used to define the $N$-term truncated Fourier expansion

\[ f_N(t) \triangleq \sum_{k=-N}^{N} c_k e^{jk\omega_s t}. \]

Then the average squared error between $f(t)$ and $f_N(t)$ is given as

\[ \frac{1}{T} \int_0^T |f(t) - f_N(t)|^2 \, dt = \sum_{|k|>N} |c_k|^2. \]

For instance, notice that in Example 5.13, the Fourier co-efficients for the triangle wave decay like $1/k^2$, which is faster than the $1/k$ rate for the square wave considered in Example 5.12. Likewise, the $N = 3$ and $N = 20$ truncated Fourier reconstructions $f_N(t)$ for the triangle wave shown in figure 5.9 are much closer to the original signal $f(t)$ than for the square wave case shown in figure 5.8.

**Summary of Properties**

The Fourier series properties developed and illustrated in this section are summarised in table 5.3 following. All three signals $f(t)$, $g(t)$ and $h(t)$ considered there are assumed periodic of period $T$ so that $\omega_s = 2\pi/T$ and they have Fourier series decompositions

\[ f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_s t}, \quad g(t) = \sum_{k=-\infty}^{\infty} d_k e^{jk\omega_s t}, \quad h(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}. \]  

(5.84)
<table>
<thead>
<tr>
<th>Property Name</th>
<th>Operation</th>
<th>Fourier Co-efficient Property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>$h(t) = \alpha f(t) + \beta g(t)$</td>
<td>$a_k = c_k + d_k$</td>
</tr>
<tr>
<td>Time Shift</td>
<td>$f(t) = g(t - \tau)$</td>
<td>$d_k = e^{j\omega_s k\tau} c_k$</td>
</tr>
<tr>
<td>Time Scale</td>
<td>$f(t) = g(\alpha t)$</td>
<td>$c_k = \frac{d_k}{\alpha}$</td>
</tr>
<tr>
<td>Convolution</td>
<td>$h(t) = \frac{1}{T} \int_0^T f(\sigma) g(t-\sigma) , d\sigma$</td>
<td>$a_k = c_k \cdot d_k$</td>
</tr>
<tr>
<td>Transform of Product</td>
<td>$h(t) = f(t) g(t)$</td>
<td>$a_k = \sum_{m=-\infty}^{\infty} c_m d_{k-m}$</td>
</tr>
<tr>
<td>Differentiation</td>
<td>$g(t) = \frac{d^n}{dt^n} f(t)$</td>
<td>$d_k = (j\omega_s)^n c_k$</td>
</tr>
<tr>
<td>Integration</td>
<td>$g(t) = \underbrace{\int \cdots \int}<em>{n \text{ times}} f(\sigma) , d\sigma \cdots dt</em>{n-1}$</td>
<td>$d_k = \frac{c_k}{(j\omega_s)^n}$</td>
</tr>
<tr>
<td>Real valued signal</td>
<td>$f(t) \in \mathbb{R}$</td>
<td>$c_k = e^{-j\omega_s t}$</td>
</tr>
<tr>
<td>Hermitian signal</td>
<td>$f(t) = \overline{f(-t)}$</td>
<td>$c_k \in \mathbb{R}$</td>
</tr>
<tr>
<td>Energy Relationship</td>
<td>Energy $= \frac{1}{T} \int_0^T</td>
<td>f(t)</td>
</tr>
</tbody>
</table>

Table 5.3: *Fourier Series Properties*: it is assumed that $f(t) = \sum_k c_k e^{j\omega_s t}$, $g(t) = \sum_k d_k e^{j\omega_s t}$, $h(t) = \sum_k a_k e^{j\omega_s t}$. .
5.4 Bode Frequency Response Diagrams

To delve more deeply into the idea of frequency response, it is necessary to investigate details of how graphical presentations of the functions \(|H(j\omega)|\) and \(\angle H(j\omega)\) are affected by certain identifiable properties of \(H(s)\) itself; namely the location of the poles and zeros of \(H(s)\).

In relation to this, it is most common that a wide range of values of frequency \(\omega\) need to be considered, and over this wide range, the magnitude \(|H(j\omega)|\) may also vary through a very wide range. In order to compress these variations, a logarithmic scale can be used for \(\omega\) and \(|H(j\omega)|\), and if so the resultant plot of

\[
20 \log_{10} |H(j\omega)| \quad \text{versus} \quad \log_{10} \omega
\]

is called a ‘Bode magnitude plot’, and a graph of

\[
\angle H(j\omega) \quad \text{versus} \quad \log_{10} \omega
\]

is called a ‘Bode phase plot’. The name is chosen in honour of Henrik Bode, an engineer who pioneered the use and analysis of such plots. The reason for the inclusion of the factor of 20 in (5.85) is that if \(|H(j\omega)|\) represents a current or voltage gain (which, in practice, it very commonly does), then \(20 \log_{10} |H(j\omega)|\) represents a power gain in units called ‘decibels’ (dB). Therefore, the factor of 20 in (5.85) allows the \(y\) axis of a Bode magnitude plot to be conveniently labelled in units of dB.

5.4.1 Straight line approximation of Bode magnitude

Aside from the advantage of axes compression, a second key feature of Bode frequency response diagrams is that their logarithmic scales permit a piece-wise linear (i.e. straight-line) approximation of them to be obtained in a fashion which is both simple, and in most instances accurate enough to be highly informative.

To explain it, first suppose that as is most commonly the case (see section 4.7), the system transfer function \(H(s)\) has the following rational form

\[
H(s) = \frac{B(s)}{A(s)} = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}
\]

(5.87)

In this case

\[
20 \log_{10} |H(j\omega)| = 20 \log_{10} \left| \frac{K(j\omega-z_1)(j\omega-z_2)\cdots(j\omega-z_m)}{(j\omega-p_1)(j\omega-p_2)\cdots(j\omega-p_n)} \right|
\]

\[
= 20 \log_{10} K + 20 \sum_{l=1}^{m} \log_{10} \left| j\omega - z_l \right| - 20 \sum_{k=1}^{n} \log_{10} \left| j\omega - p_k \right|.
\]

(5.88)

Therefore, the total log magnitude \(\log_{10} |H(j\omega)|\) may be decomposed into individual components, each of which measures the distance from a point \(j\omega\) on the imaginary axis to a point \(p_k\) which is a pole of \(H(s)\) or a point \(z_l\) which is a zero of \(H(s)\). The manner in which these distances vary, and hence (5.88) varies as \(\omega\) changes, then depends on the location of these poles \(\{p_k\}\) and zeros \(\{z_l\}\).

Real Valued Poles and Zeros

Via (5.88) it is clear that fundamental features of the Bode magnitude response \(20 \log_{10} |H(j\omega)|\) are governed by the actual location of the poles and zeros of \(H(s)\). To provide more detail on this point,
consider the case of $H(s)$ possessing a single pole

$$H(s) = \frac{1}{s - p} \quad (5.89)$$

so that according to (5.88)

$$20 \log_{10} |H(j\omega)| = -20 \log_{10} |j\omega - p|. \quad (5.90)$$

Then, assuming that $p$ is real and negative, the distance $|j\omega - p|$ is as shown in figure 5.10. Clearly, as

![Figure 5.10: Graphical representation of the distance $|j\omega - p|$ between the vectors $j\omega$ and $p$ for the case of $p$ being real and negative.](image)

$\omega$ is increased from zero and as the vector $j\omega$ climbs up the imaginary axis, then the distance $|j\omega - p|$ grows monotonically (i.e. in a non-decreasing fashion). In this case, the contribution $-20 \log_{10} |j\omega - p|$ to $20 \log_{10} |H(j\omega)|$ is a decreasing one (decreasing because of the minus sign in (5.90)), and hence the presence of a real valued pole $p$ in $H(s)$ is such as to make $20 \log_{10} |H(j\omega)|$ ‘roll off’ with increasing frequency.

To be more precise on this point, suppose that $\omega$ is much less than $|p|$; we write this as $|\omega| \ll |p|$. Then $|j\omega - p|$ will be dominated by $|p|$ ($|j\omega - p| \approx |p|$), and hence

$$-20 \log_{10} |j\omega - p| \approx -20 \log_{10} |p| \quad : |\omega| \ll |p|. \quad (5.91)$$

On the other hand, if $|\omega| \gg |p|$, then $|j\omega - p|$ will be dominated by $|j\omega| = |\omega|$ and hence

$$-20 \log_{10} |j\omega - p| \approx -20 \log_{10} |\omega| \quad : |\omega| \gg |p|. \quad (5.91)$$

If $20 \log_{10} |H(j\omega)|$ is then plotted on a logarithmic $x$ axis in which the fundamental unit is $\log_{10} |\omega|$, then (5.91) represents a contribution which is a straight line of slope $-20$dB/decade change in $|\omega|$, which is a one unit change in $\log_{10} |\omega|$.
Where is the point of cross-over between $|\omega| \ll |p|$ and $|\omega| \gg |p|$? There can be no unique answer to a subjective question such as this, but a convenient choice for the cross-over point, which is also called a ‘break-point’, is $|\omega| = |p|$. This leads to the so-called ‘straight line’ approximation

$$20 \log_{10} \left| \frac{1}{j\omega - p} \right| \approx \begin{cases} -20 \log_{10} |p| & : |\omega| \leq |p| \\ -20 \log_{10} |\omega| & : |\omega| > |p|. \end{cases} \tag{5.92}$$

See figure 5.11 for an illustration of this approximation.

The error in this approximation clearly depends on the value of $\omega$, but at the ‘break-point’ of $|\omega| = |p|

$$20 \log_{10} \left| \frac{1}{j\omega - p} \right|_{\omega=|p|} = 20 \log_{10} \frac{1}{\sqrt{2p^2}} = -10 \log_{10} 2 - 20 \log_{10} |p|$$

so that the error between this and the straight line approximation (5.92) is

$$-20 \log_{10} |p| + 20 \log_{10} |p| + 10 \log_{10} 2 = 10 \log_{10} 2 = 3.01 \text{dB}.$$ 

Therefore, for the case (5.89) of a single real pole in $H(s)$, the relationship between the Bode Magnitude response $20 \log_{10} |H(j\omega)|$ and the straight line approximation (5.92) is as shown in figure 5.11. Note in particular that both the Bode plot and its straight line approximation depend only on $|p|$, and not on the sign of $p$; a pole at $p = 2$ and a pole at $p = -2$ both lead to the same Bode magnitude plot.

This is because (see figure 5.10) the distance $|j\omega - p|$ does not depend on which side of the imaginary axis $p$ lies on.

By starting with the simple single-pole case (5.89) we have considered a situation in which there is only one term $-20 \log_{10} |j\omega - p|$ in the expansion (5.88). To be more general than this, suppose

![Figure 5.11: Relationship between (thick line) Bode Magnitude $20 \log_{10} |H(j\omega)|$ and straight line approximation (5.92) for the case (5.89) of a single real pole.](image)
that $H(s)$ contains a real valued zero as well

$$H(s) = \frac{s - z}{s - p}$$  \hspace{1cm} (5.93)

so that according to (5.88)

$$20 \log_{10} |H(j\omega)| = 20 \log_{10} \left| \frac{j\omega - z}{j\omega - p} \right| = 20 \log_{10} |j\omega - z| - 20 \log_{10} |j\omega - p|. \hspace{1cm} (5.94)$$

The effect of the zero is thus to add a term $20 \log_{10} |j\omega - z|$ which is identical to the component $-20 \log_{10} |j\omega - p|$ due to the pole, except for a change of sign. Drawing on the previous argument that led to (5.92) then implies the following straight line approximation for the component due to the zero.

$$20 \log_{10} |j\omega - z| \approx \begin{cases} 
20 \log_{10} |z| & : |\omega| \leq |z| \\
20 \log_{10} |\omega| & : |\omega| > |z| 
\end{cases} \hspace{1cm} (5.95)$$

Figure 5.12: Relationship between (thick line) Bode Magnitude $20 \log_{10} |H(j\omega)|$ and straight line approximation (5.92) for the case (5.89) of a single real pole.

If we combine this straight line approximation with the one (5.92) used for the effect of a pole, then assuming that $z < p$, the combined straight line approximation to (5.94) together with its actual response are both shown in figure 5.12. Here, the effect $20 \log_{10} |j\omega - z|$ of the zero initially causes the response $20 \log_{10} |H(j\omega)|$ to increase at 20dB/decade once the breakpoint at $|\omega| = |z|$ is reached,
but then the $-20\text{dB/decade} \, 20\log_{10} |j\omega - p|$ of the pole, which begins at the breakpoint $|\omega| = |p|$ exactly cancels the effect of the zero so that the total response $20\log_{10} |H(j\omega)|$ ‘flattens out’. The clear point to notice is the following.

The presence of a pole in $H(s)$ generically causes the Bode magnitude frequency response $20\log_{10} |H(j\omega)|$ to decrease, while the presence of zeros in $H(s)$ generically cause the Bode magnitude frequency response $20\log_{10} |H(j\omega)|$ to increase.

Another clear point is that since according to (5.88) the Bode magnitude response can be decomposed into a sum of components $20\log_{10} |j\omega - z| \, 20\log_{10} |j\omega - p|$, each of which have straight line approximations that depend on the values of $|z|$ and $|p|$, then the total response $20\log_{10} |H(j\omega)|$ can be approximated by simply summing the individual straight line approximations.

**Example 5.15** Summing of straight line approximations to produce total approximation

Suppose that the Bode magnitude frequency response of the system with transfer function

$$H(s) = \frac{250(s + 2)}{(s + 10)(s + 50)}.$$ 

is required. Then according to (5.88) this response is

$$20\log_{10} |H(j\omega)| = 20\log_{10} 250 + 20\log_{10} |j\omega + 2| - 20\log_{10} |j\omega + 10| - 20\log_{10} |j\omega + 50|. $$

Now, according to the previous analysis and example, the term $20\log_{10} |j\omega + 2|$ due to the zero has a straight line approximation which starts increasing at $20\text{dB/decade}$ once $\omega > 2$; this is shown labelled on figure 5.13 as a dash-dot line.

Similarly, the term $-20\log_{10} |j\omega + 10|$ due to the pole at $10 \text{ rad/s}$ has a straight line approximation which starts decreasing at $20\text{dB/decade}$ once $\omega > 10$; this approximating component is also labelled on figure 5.13. Likewise, the term $-20\log_{10} |j\omega + 50|$ produces a final decreasing component that starts decreasing once $\omega > 50$.

Adding all these straight line approximation components together vertically then gives the total straight line approximation shown as the dashed line in figure 5.13. The true Bode magnitude response $20\log_{10} |H(j\omega)|$ is shown as the solid line in that figure.

**Example 5.16** Incorporation of DC gain The previous example was somewhat simplified in that all that was required was to add together straight line approximations that began as horizontal lines at 0dB. The reason this approach worked, was that the overall D.C. gain $H(j0)$ was unity, which is 0dB.

If the D.C. gain is not unity, then a trivial modification of the straight line approximation process is necessary. First, one proceeds as before simply adding straight line approximations together. Then a second and final step occurs in which the D.C. gain is calculated in dB as $20\log_{10} |H(j0)|$, and this is used to fix the $y$ axis labelling.

For example, suppose that the Bode magnitude frequency response of the system with transfer function

$$H(s) = \frac{(s + 5)^2}{(s + 1)(s + 100)}.$$ 

is required.
Then the straight line approximation components for the 2 zeros at $s = -5$ and 2 poles at $s = -1, -100$ are as shown as dash-dot line in figure 5.14, and their sum which is the total straight line approximation to $20 \log_{10} |H(j\omega)|$ is shown as the dashed line in that figure.

This establishes the general shape of the Bode Magnitude plot. To establish the $y$ axis labelling we calculate the D.C. gain in dB as

$$20 \log_{10} |H(j0)| = 20 \log_{10} \left| \frac{5 \cdot 5}{1 \cdot 100} \right| \approx -12 \text{dB}$$

Therefore, ‘sliding’ the straight line approximation vertically until the magnitude at the lowest frequency is $-12$ dB ensures that the $y$ axis labelling is correct, and the straight line approximation process is complete. To assess its accuracy, the true Bode magnitude response is shown as the solid line in figure 5.14.

**Complex valued poles and zeros**

The previous analysis addresses the case of real valued poles and zeros. Unfortunately, the complex valued case has to be considered separately and is somewhat more complicated. In order to deal with this, consider first the situation in which $H(s)$ contains only complex valued poles as follows.

$$H(s) = \frac{|p|^2}{(s - p)(s - \bar{p})} \quad p = re^{j\theta}.$$  \hspace{1cm} (5.96)
Figure 5.14: Figure generated via example 5.16. The dash-dot lines are the straight line approximation for the individual pole and zero contributions. Once summed, they generate the total straight line approximation shown as the dashed line.

Note that we have to consider two complex poles, since one complex pole cannot exist in isolation. The reason for this is simple. If $H(s)$ in (5.96) were to possess only the first complex pole, then the zero frequency (D.C.) gain of the system would be

$$\text{D.C. gain} = H(j0) = \frac{|p|^2}{-p} = \frac{p\overline{p}}{-p} = -\overline{p}$$

which is complex valued if $p$ is complex valued. It is impossible for any physical system to have such a D.C. gain; a complex valued quantity can never be the physical response of a system when fed a constant input of unit size. Therefore, if a system $H(s)$ possesses a complex pole at $s = p$, it must also possess a second complex pole at $s = \overline{p}$ in order to ensure that the D.C. gain is real valued.

To continue, the Bode magnitude frequency response of (5.96) is

$$20\log_{10}|H(j\omega)| = 20\log_{10}|p|^2 - 20\log_{10}|j\omega - p| - 20\log_{10}|j\omega - \overline{p}|$$ (5.97)

so that, as before, it is decomposed into individual elements which are the distances from the poles $p$ and $\overline{p}$ to a point $j\omega$ on the imaginary axis. However, with reference to figure 5.15 and in contradiction to the real-valued pole case, the distance $|j\omega - p|$ will not monotonically increase with increasing $|\omega|$. Instead, starting from $\omega = 0$, the distance $|j\omega - p|$ will decrease as $\omega$ increases until it hits a minimum at $\omega = \text{Imag} \{p\}$. Only then does the distance $|j\omega - p|$ start to monotonically increase as $\omega$ is increased further past $\omega = \text{Imag} \{p\}$. This implies that the effect of the $|j\omega - p|$ term in $20\log_{10}|H(j\omega)|$ will be to produce an initial increase of $20\log_{10}|H(j\omega)|$ up to some peak value before it starts decreasing; see figure 5.16. Actually, what is shown there is a range of scenarios for how close $p$ is to the imaginary axis. The closer it is, the smaller $|j\omega - p|$ is at the minimising point $\omega = \text{Imag} \{p\}$, and therefore the sharper the peak produced in $20\log_{10}|H(j\omega)|$. 
The way that this closeness of \( p \) to the imaginary axis is measured is via what is known as the ‘damping factor’ \( \xi \) which is defined by writing the denominator of (5.96) in a non-factored form as

\[
(s - p)(s - \overline{p}) = (s - re^{j\theta})(s - re^{-j\theta}) \\
= s^2 - (2r \cos \theta)s + r^2 \\
= s^2 + 2r \zeta s + r^2 \\
\quad; \theta \geq \frac{\pi}{2}
\]

where the damping ratio \( \zeta \) has been defined as the absolute value of the cosine of the angle that \( p \) makes with the positive real axis:

\[
\zeta \triangleq |\cos \theta| \quad; p = re^{j\theta}.
\]

Note that in progressing to (5.98) it has been assumed that the system involved is stable so that \( p \) is in the left half plane and hence \( \theta \geq \pi/2 \). Given this, there are then three regimes that \( \zeta \) can then fall into.

**Underdamped:** \( (\zeta < 1) \) In this case \( p = re^{j\theta} \) with \( \theta \in [\pi/2, \pi) \), and hence \( p \) is complex valued. As \( \theta \to \pi/2 \) and hence \( p \) gets closer to the imaginary axis, then \( \zeta \to 0 \).
Figure 5.16: Bode Magnitude responses for the case of a complex valued pole-pair and various choices of distances of poles to imaginary axis - the latter measured via the damping factor $\zeta$.

**Critically damped:** ($\zeta = 1$) Here $p = re^{j\theta}$ with $\theta = \pi$ so that $p$ is real valued with $p = \overline{p}$.

**Overdamped:** ($\zeta > 1$) Note that (5.98) may be factored as

$$s^2 + 2r\zeta s + r^2 = (s - p_1)(s - p_2)$$

where

$$p_{1,2} = r \left( \zeta \pm \sqrt{\zeta^2 - 1} \right).$$

so that in the overdamped case with $\zeta > 1$, then both poles are real valued and different to one another. In this case, for $\zeta > 1$ the damping factor loses its interpretation as $\zeta = |\cos \theta|$.

Therefore, in order to examine how complex valued poles affect the Bode Magnitude plot, it is only necessary to consider the case $\zeta < 1$, since for other values the poles are real valued, and this situation has already been dealt with.

Returning then to (5.97), to proceed further with this complex valued poled case, when $|\omega| \ll |p|$ and as previously argued (see figure 5.16 as well)

$$|j\omega - p| \approx |j\omega - \overline{p}| \approx |p| \quad ; |\omega| \ll |p|$$

and therefore for $H(s)$ given by (5.96) and using (5.97)

$$20 \log_{10} |H(j\omega)| \approx 40 \log_{10} |p| - 2 \cdot 20 \log_{10} |p| \approx 0 \quad ; |\omega| \ll |p|.$$

On the other hand, when $|\omega| \gg |p|$ then

$$|j\omega - p| \approx |j\omega - \overline{p}| \approx |\omega| \quad ; |\omega| \gg |p|$$
and hence
\[ 20 \log_{10} |H(j\omega)| \approx -40 \log_{10} |\omega| \quad ; |\omega| \gg |p|. \]

which, when plotted with respect to an x-axis labelled in units of \(\log_{10} \omega\), is a straight line of slope \(-40 \text{dB/decade}\).

These straight-line approximation values are shown as the labelled dash-dot line on figure 5.16 and as shown there, the transition between the straight line approximators can be smooth if \(p\) is well away from the imaginary axis or (\(\zeta > 1\)). However, as already discussed, the transition between the straight line approximator can also involve a sharp peak if \(\zeta = |\cos \theta|\) is small, in which case \(\theta\) is near \(\pi/2\) so that \(p\) is near the imaginary axis and \(|j\omega - p|\) becomes close to zero for \(\omega \approx p\).

To test for this latter case of a peak occurring, we check to see if the derivative of \(10 \log_{10} |H(j\omega)|^2\) with respect to \(\omega\) ever becomes zero, which indicates the presence of a maximising value. For this purpose, using (5.96) with the formulation (5.98) for the denominator leads to
\[
10 \log_{10} |H(j\omega)|^2 = 10 \log_{10} \left| \frac{r^2}{(r^2 - \omega^2) + j2\zeta r\omega} \right|^2 = 10 \log_{10} r^4 - 10 \log_{10}(\omega^4 + 2r^2(2\zeta^2 - 1)\omega^2 + r^4). \quad (5.100)
\]

Therefore, for a peak in \(\log_{10} |H(j\omega)|\) to occur, a value of \(\omega^2\) must exist such that
\[
\frac{d}{d\omega} \log_{10} |H(j\omega)|^2 = \frac{2\omega^2 + 2r^2(2\zeta^2 - 1)}{\omega^4 + 2r^2(2\zeta^2 - 1)\omega^2 + r^4} = 0.
\]

This can occur when the numerator of the expression is zero. That is, for
\[
\omega^2 = r^2(1 - 2\zeta^2) = r^2(1 - 2 \cos^2 \theta) = r^2((1 - \cos^2 \theta) - \cos^2 \theta) = r^2(\sin^2 \theta - \cos^2 \theta) = (\text{Imag} \{p\})^2 - (\text{Real} \{p\})^2.
\]

Clearly, this is only possible for real valued \(\omega\) if the right hand side of the above is positive, and this only occurs for \(|\text{Imag} \{p\}| > |\text{Real} \{p\}|\) which implies \(\theta \in [\pi/2, 3\pi/4]\) and hence
\[
\zeta = |\cos \theta| < \cos \left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}} \approx 0.707.
\]

That is, for values of \(\zeta > 0.707\), there is no definable peak in \(\log_{10} |H(j\omega)|\), while for values of \(\zeta\) less than 0.707 there is a peak. Via (5.101), when \(\zeta\) and \(\text{Real} \{p\} \approx 0\), the location of the peak is approximately at \(\omega = |\text{Imag} \{p\}| \approx |p|\), while when \(\zeta\) is larger, the peak will be at a somewhat smaller frequency.

The reason why a peak does not occur for \(\zeta > 0.707\) is that although the term \(-20 \log_{10} |j\omega - p|\) in (5.97) will, for any \(\zeta < 1\), always contribute a peaking effect around \(\omega \approx \text{Imag} \{p\}\), at this same point the other term \(-20 \log_{10} |j\omega - \bar{p}|\) in (5.97) will be in its region of monotonic decrease with increasing \(\omega\) (see figure 5.15), and this may ‘swamp’ the total response so that no peak occurs.
If it exists, the height of the peak increases as the imaginary part of \( p \) tends to zero, which can be seen by substituting (5.101) into (5.100) to obtain

\[
10 \log_{10} |H(j\omega)|^2 \big|_{\omega^2=r^2(1-2\zeta^2)} = 10 \log_{10} \frac{r^4}{4r^4\zeta^2(1-\zeta^2)} \approx 15 \log_{10} \left( \frac{0.707}{\zeta} \right).
\]

Therefore, if \( \zeta = 0.707 \) then \( 15 \log_{10} \frac{0.707}{\zeta} = 0 \) and there is no peak, while for \( \zeta = 0.707/10 = 0.07 \), then \( 15 \log_{10} \frac{0.707}{\zeta} = 15 \), and hence the peak can be expected to be 15dB higher than the straight-line approximation that holds for \( |\omega| \ll |p| \).

**Example 5.17 Complex Valued Poles** Suppose that the Bode magnitude frequency response of the system with transfer function

\[
H(s) = \frac{250(s + 2)}{s^2 + 4s + 400}
\]

is required. Note that this is essentially the previous example 5.15 with a modified denominator which is of the form \( s^2 + 2\zeta rs + r^2 \) with \( r = 20 \) and \( \zeta = 0.1 \).

Since \( \zeta < 0.707 \), this denominator will produce a peaking component in the final response, and the peak will be at a height of approximately

\[
15 \log_{10} \frac{0.707}{\zeta} = 15 \log_{10} \frac{0.707}{0.1} \approx 13\text{dB}
\]

above the straight-line approximation.

As reference to figure 5.17 shows, this is indeed the case. There the straight-line approximation is the sum of two parts, one which is increasing at 20dB/decade and is due to the zero at 2 rad/s, and another which is decreasing at -40dB/decade and is due to the two complex valued poles at \( |p| = 20 \) rad/s.

**Example 5.18 Complex Valued Zeros** The discussion so far has concentrated on the fact that a complex valued pole introduces a component \(-20 \log_{10} |j\omega - p|\) in the Bode magnitude response \(20 \log_{10} |H(j\omega)|\) that may produce a peaking behavior. Of course, it is also possible to have a complex values zero in \( H(s) \), and this will introduce a component of \( 20 \log_{10} |j\omega - z| \) into \( 20 \log_{10} |H(j\omega)| \).

The only difference between the contributions \(-20 \log_{10} |j\omega - p|\) and \( 20 \log_{10} |j\omega - z| \) is one of sign. Therefore, all the arguments previously made regarding complex poles and peaking behaviour apply equally well to complex zeros. However, with this in mind, complex zeros with damping ratio \( \zeta < 0.707 \) produce a ‘dipping behaviour’ in contrast to the peaking behaviour just illustrated with underdamped poles.

For instance, consider the previous example 5.16 revisited so that the numerator is modified such that

\[
H(s) = \frac{(s^2 + s + 25)}{(s + 1)(s + 100)}.
\]

Clearly, the numerator is of the form \( s^2 + 2\zeta rs + r^2 \) with \( r = 5 \) and \( \zeta = 0.1 \). Since \( \zeta < 0.707 \), this numerator will produce a dipping component in the final response, and the dip will be at a distance of approximately

\[
15 \log_{10} \frac{0.707}{\zeta} = 15 \log_{10} \frac{0.707}{0.1} \approx 13\text{dB}
\]

below the straight-line approximation.
Figure 5.17: *Figure generated via example 5.17.* The dash-dot lines are the straight line approximation for the individual pole and zero contributions. Once summed, they generate the total straight line approximation shown as the dashed line. The actual Bode magnitude response is the solid line.

As reference to figure 5.18 shows, this is indeed the case. There the straight line approximation is the sum of three parts, one which is increasing at 40dB/decade and is due to the complex zeros at \(|z| = r = 5\) rad/s, another which is decreasing at -20dB/decade due to the pole at 1 rad/s, and a final one which is also decreasing at -20dB/decade due to the pole at 10 rad/s.

### Poles and Zeros at the origin

There is a final special case to consider before closing this section on the approximation of Bode magnitude plots. To be more specific, this section has established the key principle that a useful straight-line approximation of the magnitude \(20 \log_{10} |H(j\omega)|\) is one that, with respect to a \(\log_{10} \omega\) \(x\)-axis, possesses ‘breakpoints’ at the poles and zeros of \(H(s)\). However, if these poles or zeros are at the origin \(s = 0\), then the associated breakpoint will lie to the left of the left-most point on the \(x\)-axis; remember that \(\log_{10} 0 = -\infty\).

The consequence of this is that the straight line approximation of \(20 \log_{10} |H(j\omega)|\) will not begin as a ‘level’ line before it hits further breakpoints at \(\omega > 0\). Rather, it will begin with a slope of \(20\ell\) dB/decade where \(\ell\) is the net number of zeros at the origin. Specifically, suppose that the zeros \(z_1, \cdots, z_\ell\) in (5.87) are all at \(s = 0\) so that

\[
H(s) = \frac{B(s)}{A(s)} = \frac{K s^{\ell}(s - z_{\ell+1})(s - z_{\ell+2}) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}.
\]

(5.102)
The 20 log\(10\) term in the above expression then establishes a baseline slope of 20 \(\ell\) dB/decade before the further effect of non-origin poles and zeros. Of course, the complementary situation of an \(s^\ell\) term in the denominator of \(H(s)\) producing \(\ell\) poles at the origin would lead to a \(-20\,\log\!_{10}\,|\omega|\) term in the Bode magnitude response and hence a baseline slope of \(-20\,\ell\) dB/decade.

An example makes these ideas concrete.

**Example 5.19 Poles and zeros at the origin** Suppose that the Bode magnitude frequency response of the system with transfer function

\[
H(s) = \frac{10s}{(s + 10)(s + 50)}
\]

is required. Then according to (5.104) the magnitude frequency response of this transfer function may be decomposed as

\[
20 \log_{10} |H(j\omega)| = 20 \log_{10} 10 + 20 \log_{10} |\omega| - 20 \log_{10} |j\omega + 10| - 20 \log_{10} |j\omega + 50|. \tag{5.104}
\]

The straight line approximation components that are commensurate with this decomposition are shown as the dash-dot lines in figure 5.19. In particular, notice that the first straight line approxi-
Figure 5.19: Figure generated via example 5.19 with zero at the origin. The dash-dot lines are the straight line approximation for the individual pole and zero contributions. Once summed, they generate the total straight line approximation shown as the dashed line. The actual Bode magnitude response is the solid line.

The pole at 50 rad/s contributes a downward slope of 20 dB/decade due to the $20 \log_{10} |s - 50|$ term that is a consequence of the zero at $s = 0$.

The pole at 10 rad/s introduces a counteracting component of an upward slope of 20 dB/decade is introduced by the $20 \log_{10} |s - 10|$ term that is a consequence of the zero at $s = 0$.

Note also that the baseline gain for the approximating components is determined by the lowest frequency of 0.1 rad/s used on the chosen x-axis in figure 5.19 as follows:

$$20 \log_{10} |H(j0.1)| = 20 \log_{10} \left| \frac{10(0.1)}{|0.1 - 10||j0.1 - 50|} \right| = -54 \text{dB}.$$  

To illustrate the complementary case of a pole at the origin, consider the situation of

$H(s) = \frac{10(s + 10)}{s(s + 50)}$.

Then according to (5.103) the associated frequency response has decomposition

$$20 \log_{10} |H(j\omega)| = 20 \log_{10} 10 + 20 \log_{10} |j\omega + 10| - 20 \log_{10} \omega - 20 \log_{10} |j\omega + 50|.$$  (5.105)

This response, together with its straight line approximation is shown in figure 5.20. Note that, as opposed to the previous case of a zero at the origin, this situation of a pole at the origin involves an initial downward trend of $20 \text{dB/decade}$ due to the $-20 \log_{10} |\omega|$ in the decomposition (5.105).

5.4.2 Straight Line Approximation of Bode Phase

The previous discussion has established how the Bode magnitude response $20 \log_{10} |H(j\omega)|$ depends on the poles and zeros of $H(s)$, but what of the phase response $\angle H(j\omega)$? To address this question, return to the pole-zero decomposition of $H(s)$ in (5.87). This indicates that the phase response $\angle H(j\omega)$
Figure 5.20: Figure generated via example 5.19 with pole at the origin. The dash-dot lines are the straight line approximation for the individual pole and zero contributions. Once summed, they generate the total straight line approximation shown as the dashed line. The actual Bode magnitude response is the solid line.

may be written as

\[ \angle H(j\omega) = \angle K \left(\frac{(j\omega - z_1)(j\omega - z_2) \cdots (j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2) \cdots (j\omega - p_n)} \right) \]

\[ = \angle K + \sum_{\ell=1}^{m} \angle(j\omega - z_\ell) - \sum_{k=1}^{n} \angle(j\omega - p_k). \] (5.106)

Note, in particular, the term \( \angle K \), which since \( K \) is a real number, will be equal to zero if \( K \geq 0 \) and equal to \( \pi \) if \( K < 0 \).

**Real valued poles and zeros**

Taking a real valued pole \( p \) now as an example, with reference to figure 5.21, its angle contribution will be of the form

\[ \angle(j\omega - p) = \begin{cases} \tan^{-1}\frac{\omega}{p} : p < 0 \\ \pi - \tan^{-1}\frac{\omega}{p} : p \geq 0 \end{cases} \] (5.107)

An important point raised here is that, in contrast to the previous case where the magnitude \( |H(j\omega)| \) was considered, the phase contribution \( \angle(j\omega - p) \) of a pole at \( s = p \) depends on which half plane a pole lies in.

That is, referring back to figure 5.10, the distance contribution \( |j\omega - p| \) to the total magnitude \( |H(j\omega)| \) increases with increasing \( \omega \), regardless of sign of \( p \). However, now referring to figure 5.21
the angle contribution $\angle(j\omega - p)$ to the phase $\angle H(j\omega)$ increases with increasing $\omega$ when $p < 0$, but decreases with increasing $\omega$ when $p > 0$.

![Graphical representation of the phase contribution $\angle(j\omega - p)$ between the vectors $j\omega$ and $p$ for the case of $p$ being real and for the two situations of it being in either the left or right half plane.](image)

**Figure 5.21**: Graphical representation of the phase contribution $\angle(j\omega - p)$ between the vectors $j\omega$ and $p$ for the case of $p$ being real and for the two situations of it being in either the left or right half plane.

Furthermore, via an argument involving Taylor series expansions, for $x$ near 1

$$\tan^{-1} x \approx \frac{\pi}{4} (\log_{10} x + 1)$$

so that for $|\omega| \approx |p|

$$\angle(j\omega - p) \approx \begin{cases} \frac{\pi}{4} \left( \log_{10} \left| \frac{\omega}{p} \right| + 1 \right) : p < 0 \\ \frac{\pi}{4} \left( \log_{10} \left| \frac{\omega}{p} \right| + 1 \right) : p \geq 0 \end{cases} \quad (5.108)$$

Again, with respect to a logarithmic $x$-axis, this is a straight line approximation, whose quality in the 2 decade range

$$\frac{|p|}{10} \leq \omega \leq 10|p|$$

is illustrated in figure 5.22. As clearly shown there, the approximation (5.108) is quite accurate, since the maximum error illustrated in figure 5.22 is only 6 degrees. Therefore, just as was the case for Bode magnitude plots, Bode phase plots can be approximated by straight line graphs which are dictated by the location of the poles and zeros in $H(s)$. 
Suppose that a pole $p$ of $H(s)$ is real valued. Then a straight-line approximation of its contribution $\angle(j\omega - p)$ to the total phase $\angle H(j\omega)$ is

$$\angle(j\omega - p) \approx \begin{cases} 0 & ; \omega \leq |p|/10 \\ \frac{\pi}{4} \left( \log_{10} \left| \frac{\omega}{p} \right| + 1 \right) & ; \frac{|p|}{10} \leq \omega \leq 10|p| \\ \frac{\pi}{2} & ; \omega > 10|p|, \end{cases}$$

$$; p < 0$$

$$\begin{cases} \pi & ; \omega \leq |p|/10 \\ \frac{\pi}{2} - \frac{\pi}{4} \left( \log_{10} \left| \frac{\omega}{p} \right| - 1 \right) & ; \frac{|p|}{10} \leq \omega \leq 10|p| \\ \frac{\pi}{2} & ; \omega > 10|p|, \end{cases}$$

$$; p > 0$$

(5.109)

Of course, in the preceding discussion a zero at $s = z$ could also have been addressed, which would lead to an identical approximation (5.109) save that in the above, $p$ would be replaced by $z$.

With this in mind, and referring to (5.106), since the contribution $\angle(j\omega - p)$ has a minus sign in front of it, while the contribution $\angle(j\omega - p)$ does not, then the straight-line approximations given...
in table 5.4 can be distilled. According to (5.106), the total phase response \( \angle H(j\omega) \) is then the

<table>
<thead>
<tr>
<th>Pole/Zero</th>
<th>Slope</th>
<th>Range</th>
<th>Start Phase</th>
<th>Stop Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>RHP pole</td>
<td>( \frac{\pi}{4} ) rad/s (+45°) per decade</td>
<td>( \frac{</td>
<td>p</td>
<td>}{10} \leq \omega \leq 10</td>
</tr>
<tr>
<td>LHP pole</td>
<td>( -\frac{\pi}{4} ) rad/s (-45°) per decade</td>
<td>( \frac{</td>
<td>p</td>
<td>}{10} \leq \omega \leq 10</td>
</tr>
<tr>
<td>RHP zero</td>
<td>( -\frac{\pi}{4} ) rad/s (-45°) per decade</td>
<td>( \frac{</td>
<td>z</td>
<td>}{10} \leq \omega \leq 10</td>
</tr>
<tr>
<td>LHP zero</td>
<td>( \frac{\pi}{4} ) rad/s (+45°) per decade</td>
<td>( \frac{</td>
<td>z</td>
<td>}{10} \leq \omega \leq 10</td>
</tr>
</tbody>
</table>

Table 5.4: Summary of straight line approximation of phase contributions for real-valued poles and zeros

summation of these straight line approximations. This procedure is in fact best conveyed by example.

Example 5.20 Summing of straight line approximations to produce total approximation

Suppose that the Bode phase frequency response of the system with transfer function

\[
H(s) = \frac{250(s + 2)}{(s + 10)(s + 50)}
\]

is required; this same \( H(s) \) was introduced in Example 5.15 where its magnitude response was considered. According to (5.106) the phase response of the system represented by \( H(s) \) is

\[
\angle H(j\omega) = \angle (j\omega + 2) - \angle (j\omega + 10) - \angle (j\omega + 50).
\]

Now, according to the previous analysis, the term \( \angle (j\omega + 2) \) due to the left half plane zero has a straight line approximation which starts increasing at 45 degrees/decade once \( \omega \) is a decade below this zero (\( \omega = 2/10 = 0.2 \)) and finishes increasing once \( \omega \) is a decade above the zero location (\( \omega = 2 \cdot 10 = 20 \)). This is shown labelled on figure 5.23 as a dash-dot line.

Similarly, the term \( -\angle (j\omega + 10) \) due to the pole at 10 rad/s has a straight line approximation which starts decreasing at −45 degrees/decade once \( \omega > 10/10 = 1 \) and stops decreasing once \( \omega = 10 \cdot 10 = 100 \); this approximating component is also labelled on figure 5.23. Finally, the term \( -\angle (j\omega + 50) \) produces a decreasing component of −45 degrees/decade over the two decade range \( \omega \in [5, 500] \).

Adding all these straight line approximation components together vertically then gives the total straight line approximation shown as the dashed line in figure 5.23. The true Bode phase response \( \angle H(j\omega) \) is shown as the solid line in that figure.
Example 5.21  Phase effect of right half plane poles

Suppose that the Bode phase frequency response of the system with transfer function

$$H(s) = \frac{(s + 5)}{(s - 1)(s + 100)}.$$ 

is required.

Then the straight line approximation components for the left half plane zero at $s = 5$ and the 2 poles at $s = 1, -100$ are as shown as dash-dot line in figure 5.24 and their sum which is the total straight line approximation to $\angle H(j\omega)$ is shown as the dashed line in that figure.

Note that, as opposed to the previous example, the pole-induced phase components are *not* all decreasing with increasing frequency - only the component due to the 100 rad/s pole is decreasing, and this is because that pole lies in the left half plane. In contrast, since the pole at 1 rad/s lies in the right half plane, its phase contribution is an increasing one.

Finally, again as a consequence of right half plane pole, the D.C. gain $H(0)$ is given as

$$H(0) = \frac{5}{(-1)(100)} = -0.05.$$ 

so that the initial phase at $\omega = 0$ is $-180^\circ$. For this reason, the various approximating components are initialised at $-180^\circ$, and not $0^\circ$ as in the previous example.
Complex valued poles and zeros

The case of complex valued poles is illustrated in figure 5.25 where as in the case where magnitude response was considered, we have been careful to note that a complex valued pole \( p \) in \( H(s) \) must be accompanied by its conjugate \( \overline{p} \) in order to ensure that the d.c. gain \( H(0) \) is real valued. Therefore, the relevant angle contribution to the total phase \( \angle H(j\omega) \) will be

\[
\angle(j\omega - p) + \angle(j\omega - \overline{p}).
\]

(5.110)

Clearly when \( \omega = 0 \) then

\[
\angle(j\omega - p) = -\angle(j\omega - p)
\]

so that (5.110) is equal to zero. Furthermore, as \( \omega \to \infty \)

\[
\angle(j\omega - p) + \angle(j\omega - \overline{p}) \to \frac{\pi}{2} + \frac{\pi}{2} = \pi.
\]

Therefore, for low frequencies (\( \omega < |p|/10 \)) and high frequencies (\( \omega > 10|p| \)), the phase contribution of a complex pole \( p \) and its conjugate \( \overline{p} \) is the same as for two real poles at \( |p| \). It is either \( 2 \cdot 0 = 0 \) for low frequencies or \( 2 \cdot \pi/2 = \pi \) for high frequencies.

However, in contrast to the real pole case, on a \( \log_{10} \omega \) axis, the transition from \( 0 \to \pi \) in the angle of (5.110) is not well approximated by a straight-line of slope \( \pi/4 \) rad/s/decade. Instead, as illustrated in figure 5.25 the rate at which the phase changes, particularly around the point \( \omega = |p| \), will depend on how close \( p \) is to the imaginary axis.
To investigate this, notice that by the law of sines (see Appendix 5.A.3), when \( \omega \approx \text{Imag} \{p\} \), then
\[
\frac{\omega}{\sin \angle (j\omega - p)} \approx \frac{|p|}{\sin(\pi/2)} = |p|
\]
and hence
\[
\angle (j\omega - p) \approx \sin^{-1} \left( \frac{\omega}{|p|} \right). \tag{5.112}
\]
The slope of this angular change with changes in \( \log_{10}(\omega) \) is then
\[
\frac{d}{d \log_{10} \omega} \angle (j\omega - p) = \left( \frac{d}{d \omega} \angle (j\omega - p) \right) \left( \frac{d}{d \log \omega} \omega \right) \left( \frac{d}{d \log_{10} \omega} \log \omega \right)
\]
\[
\approx \frac{d}{d \omega} \sin^{-1} \left( \frac{\omega}{|p|} \right) \cdot \log 10 \quad \text{[5.113]}
\]
\[
= \frac{\omega}{\sqrt{|p|^2 - \omega^2}} \cdot \log 10 = \frac{\log 10}{\zeta} \approx \frac{2.3}{\zeta}. \tag{5.114}
\]
Here, we have recalled from page 222 that according to (5.99) the angle \( \theta \) that a pole \( p \) makes with the positive real axis is used to define the damping ratio \( \zeta = |\cos \theta| \) of that pole. However, with reference to figure 5.25 when \( \omega = \text{Imag} \{p\} \) then the vector \( p \) is the hypotenuse of a right triangle making angle \( \pi - \theta \) with the other side \( j\omega - p \), which itself has length \( \sqrt{|p|^2 - \omega^2} \). Therefore, \( |\cos \theta| = \omega^{-1} \cdot \sqrt{|p|^2 - \omega^2} \), and hence the appearance of \( \zeta \) in (5.114).

As a result, the straight line approximation for the phase contribution of a complex pole \( p \) and its conjugate pair \( \overline{p} \) should have a slope \( 2.3/\zeta \) that depends on how close the pole \( p \) is to the imaginary axis. The closer it is, and hence the smaller that \( \zeta = |\cos \theta| \) is (since \( \theta \) is close to \( \pi/2 \)), then the greater the slope of the phase approximation. This makes sense according to figure 5.25 which shows that when \( \omega \approx \text{Imag} \{p\} \), then the phase \( \angle (j\omega - p) \) becomes more sensitive to changes in \( \omega \) as \( p \) approaches the imaginary axis. The quality of this straight line approximation is illustrated in figure 5.26 where it is profiled against the true phase contribution for two damping ratio cases.

This discussion of phase approximation for complex poles and zeros is summarised in table 5.5, and is illustrated in the following examples.

**Example 5.22 Complex Valued Poles** Suppose that the Bode phase frequency response of the system with transfer function
\[
H(s) = \frac{250(s + 2)}{s^2 + 4s + 400}
\]
is required; the magnitude response of this system was considered in Example 5.17. The denominator here is of the form \( s^2 + 2\zeta|p|s + |p|^2 \) with \( |p| = 20 \) and damping ratio \( \zeta = 0.1 \). Since the poles produced by this denominator are at
\[
p, \overline{p} = 20e^{j\pi \pm \cos^{-1} 0.1}
\]
which are in the left half plane, then according to the summary in table 5.5 this denominator contributes a phase response which is decreasing at a rate of
\[
m = -\frac{2.3}{0.1} = -23 \text{ rad/s/decade} = -23 \cdot \frac{180^\circ}{\pi} = -1318^\circ/\text{decade}.
\]
Clearly, this is a very sharp roll-off in phase, as is illustrated in figure 5.27. This decreasing phase response, needs to be added to a \(+45^\circ/\text{decade}\) increasing phase component due to the left half plane zero at \( s = -2 \). This total result together with the true phase response is shown in figure 5.27.
Figure 5.25: Graphical representation of the angles $\angle(j\omega - p)$, $\angle(j\omega - \overline{p})$ between the vectors $j\omega$ and $p, \overline{p}$ for the case of $p$ being complex valued.

Example 5.23 Complex Valued Zeros

Consider now the case of

$$H(s) = \frac{s^2 + 4s + 25}{s^2 + 101s + 100} = \frac{s^2 + 2 \cdot 0.4 \cdot 5s + 5^2}{(s + 1)(s + 100)}$$

so that the transfer function has two complex valued zeros $z$ and $\overline{z}$ at $|z| = 5$ rad/s and with damping ratio $\zeta = 0.4$. Since these zeros are in the left half plane, then according to the summary in table 5.5 they contribute a phase response which is increasing at a rate of

$$m = \frac{2.3}{0.4} = 5.75 \text{ rad/s/decade} = \left(5.75 \cdot \frac{180}{\pi}\right)^\circ = 330^\circ/\text{decade}.$$ 

This increasing phase response, needs to be added the two $-45^\circ/\text{decade}$ decreasing phase components due to the left half plane poles at $s = -1, -100$. The result is shown in figure 5.28.
Figure 5.26: True phase contribution $\angle(j\omega - p) = \tan^{-1}(\omega/p)$ together with the straight line approximations.

<table>
<thead>
<tr>
<th>Pole</th>
<th>Slope $m$</th>
<th>Range</th>
<th>Start Phase</th>
<th>Stop Phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>RHP</td>
<td>$\frac{2.3}{\zeta}$ rad/s $\left(\pm \frac{132^\circ}{\zeta}\right)$ per decade</td>
<td>$-\frac{\pi}{2m} \leq \log</td>
<td>\frac{\omega}{p}</td>
<td>\leq \frac{\pi}{2m}$</td>
</tr>
<tr>
<td>LHP</td>
<td>$-\frac{2.3}{\zeta}$ rad/s $\left(-\frac{132^\circ}{\zeta}\right)$ per decade</td>
<td>$-\frac{\pi}{2</td>
<td>m</td>
<td>} \leq \log</td>
</tr>
<tr>
<td>RHP</td>
<td>$\frac{2.3}{\zeta}$ rad/s $\left(-\frac{132^\circ}{\zeta}\right)$ per decade</td>
<td>$-\frac{\pi}{2</td>
<td>m</td>
<td>} \leq \log</td>
</tr>
<tr>
<td>LHP</td>
<td>$\frac{2.3}{\zeta}$ rad/s $\left(\pm \frac{132^\circ}{\zeta}\right)$ per decade</td>
<td>$-\frac{\pi}{2m} \leq \log</td>
<td>\frac{\omega}{z}</td>
<td>\leq \frac{\pi}{2m}$</td>
</tr>
</tbody>
</table>

Table 5.5: Summary of straight line approximation of phase contributions for complex-valued poles and zeros arising from a term $s^2 + 2\zeta|p|s + |p|^2$ or $s^2 + 2\zeta|z|s + |z|^2$. 
Figure 5.27: Figure generated via example 5.22. The dash-dot lines are the straight line approximation for the individual pole and zero contributions. Once summed, they generate the total straight line approximation shown as the dashed line. The actual Bode phase response is the solid line.

Figure 5.28: Figure generated via example 5.18. The dash-dot lines are the straight line approximation for the individual pole and zero contributions. Once summed, they generate the total straight line approximation shown as the dashed line. The actual Bode phase response is the solid line.
Poles and Zeros at the origin

There is a final scenario to address before closing this section on the approximation of Bode phase plots, and it concerns the special case of when there are poles or zeros of a transfer function \(H(s)\) at the origin.

That is, suppose that the zeros \(z_1, \cdots, z_\ell\) in (5.87) are all at \(s = 0\) so that (in what follows, it is assumed that \(K > 0\)) which as phase contribution zero, otherwise add a phase contribution of \(-\pi\) rad/s to all subsequent conclusions

\[
H(s) = \frac{B(s)}{A(s)} = \frac{K s^\ell(s-z_{\ell+1})(s-z_{\ell+2}) \cdots (s-z_m)}{(s-p_1)(s-p_2) \cdots (s-p_n)}.
\]

Then

\[
\angle H(j\omega) = K (j\omega)^\ell \frac{(j\omega - z_{\ell+1})(j\omega - z_{\ell+2}) \cdots (j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2) \cdots (j\omega - p_n)}
\]

\[
= \angle K + \angle (j\omega)^\ell + \angle (j\omega - z_{\ell+1}) + \angle (j\omega - z_{\ell+2}) \cdots + \angle (j\omega - z_m)
\]

\[
- \angle (j\omega - p_1) - \angle (j\omega - p_2) \cdots - \angle (j\omega - p_n)
\]

\[
= \angle K + \ell \cdot \left(\frac{\pi}{2}\right) + \angle (j\omega - z_{\ell+1}) + \angle (j\omega - z_{\ell+2}) \cdots + \angle (j\omega - z_m)
\]

\[
- \angle (j\omega - p_1) - \angle (j\omega - p_2) \cdots - \angle (j\omega - p_n).
\]

The \(\ell \cdot (\pi/2)\) term in the above expression which is due to the \(\ell\) zeros in \(H(s)\) at the origin then establishes a constant offset of \(\ell \cdot (\pi/2)\) radians, or \(\ell \cdot (90^\circ)\) degrees in the total phase response \(\angle H(j\omega)\).

Of course, the complementary situation of an \(s^\ell\) term in the denominator of \(H(s)\) producing \(\ell\) poles at the origin would lead to a \(-\ell \cdot (\pi/2)\) constant phase offset in \(\angle H(j\omega)\).

Therefore, poles or zeros at the origin are quite simple to deal with, simply by adding an appropriate constant phase offset to the straight line approximation obtained by the methods already presented in this chapter.

Finally, the term \(\angle K\) in (5.116) will (since \(K\) is a real number) contribute a constant phase offset of zero if \(K \geq 0\) and \(\pi\) if \(K < 0\).

**Example 5.24 Zero at the origin** Suppose that the Bode phase frequency response of the system with transfer function

\[
H(s) = \frac{10s}{(s+10)(s+50)}
\]

is required. Then according to (5.116) the phase frequency response of this transfer function may be decomposed as

\[
\angle H(j\omega) = \angle (j\omega) - \angle (j\omega + 10) - \angle (j\omega + 50).
\]

The straight line approximation components that result from the poles in this decomposition are shown as the dash-dot lines in figure 5.29. Furthermore, since

\[
\angle (j\omega) = \frac{\pi}{2} \text{ rad/s}
\]

then the component due to the zero at \(s = 0\) is a constant phase offset of \(\pi/2\) rad/s which is also shown as a dash-dot line in figure 5.29. Summing these approximating components yields the dashed straight line approximation in that figure, which is profiled against the true phase response \(\angle H(j\omega)\).
Figure 5.29: Figure generated via example 5.24 with zero at the origin. The dash-dot lines are the straight line approximation for the individual pole and zero contributions. Once summed, they generate the total straight line approximation shown as the dashed line. The actual Bode phase response is the solid line.

5.5 Filter Design

The final section of this chapter concentrates not on system analysis (as we have done so far). Instead, this section examines system synthesis. That is, given a target magnitude frequency response specification $S(\omega)$, how do we design a linear system $H(s)$ such that its frequency response approximates this target? That is, how do we design a transfer function $H(s)$ such that $|H(j\omega)| \approx S(\omega)$?

Such a process is often called ‘filter design’, since it commonly arises in cases where one wishes to ‘filter’ certain frequency components out of a signal while retaining others.

5.5.1 Constraints on Achievable Performance

In attacking this filter design problem, there are several critical constraints on what is achievable, all of them arising from the requirement that it be feasible to physically implement the designed $H(s)$.

Constraint of Rational $H(s)$

Virtually the only way that a designed transfer function $H(s)$ can be implemented is via electronic circuitry. This circuitry can either be active (i.e. incorporate energy from an external power supply) and involve the use of operational amplifiers (op-amps), or it may be passive. In either case, as section 4.10 of the previous chapter exposed, the $s$ terms in the achieved $H(s)$ transfer function will arise from the use of capacitors and inductors in the circuit. This is by virtue of their complex impedances which are, for capacitor and inductor respectively,

$$Z_C = \frac{1}{sC}, \quad Z_L = sL.$$
As the examples in section 4.10 demonstrated, the overall \( H(s) \) of any circuit containing these elements is then governed by a generalised Ohm’s law, together with the linear combination relationships imposed by Kirchoff’s voltage and current laws.

The net result of this is that any \( H(s) \) achieved by these means **must** be of the following form

\[
H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}.
\]  

(5.118)

This form, which consists of a polynomial in \( s \) divided by a polynomial in \( s \) is called a ‘rational’ form.

Since it is only transfer functions of this form that can be implemented, then it is important to ensure that the \( H(s) \) designed to meet the specification \(|H(j\omega)| \approx S(\omega)|\) is also of a rational form.

The designed \( H(s) \) must be rational. That is, it must be a polynomial in \( s \) divided by a polynomial in \( s \).

**Constraints on magnitude response \(|H(j\omega)||**

When considering the design of \( H(s) \) with a prescribed \(|H(j\omega)| \approx S(\omega)\) it is crucial to recognise the relationship established in (5.9) of section 5.1 between frequency response \( H(j\omega) \) and impulse response \( h(t) \). Namely

\[
H(j\omega) = \mathcal{F} \{ h(t) \}.
\]  

(5.119)

This implies that the fundamental properties of Fourier transforms that were presented in section 5.2.2 will impose important restrictions on what is achievable for \( H(j\omega) \).

In particular, whatever design \( H(j\omega) \) we seek to achieve, it must be associated with an impulse response \( h(t) = \mathcal{F}^{-1} \{ H(j\omega) \} \) that is causal, and hence satisfies \( h(t) = 0 \) for \( t < 0 \); see section 3.12.2. If this constraint is not satisfied then, although the filter \( H(j\omega) \) can be specified ‘on paper’, it will be impossible to construct since the output of the filter \( y(t) \) at time \( t = t_* \) will depend upon *future* inputs \( u(t), t > t_* \). The following example is designed to clarify this important point.

**Example 5.25** Design of a Perfect Low-pass filter Consider the design of a filter \( H(s) \) which has a perfect ‘low-pass’ response that completely attenuates all spectral components above \( \omega_c \) rad/s, and passes all those below \( \omega_c \) rad/s with zero phase shift or magnitude alteration. That is, consider a design specification of

\[
H(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_c, \\ 0 & |\omega| > \omega_c. \end{cases}
\]  

(5.120)

Since \( H(j\omega) \) is the Fourier transform \( H(\omega) = \mathcal{F} \{ h(t) \} \) of the impulse response \( h(t) \), then \( h(t) \) must be the inverse Fourier transform \( H(j\omega) \). Therefore, using the expression (5.13) for this inverse transform, the impulse response \( h(t) \) associated with the specification (5.120) is given as

\[
h(t) = \mathcal{F}^{-1} \{ H(j\omega) \} = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega = \frac{1}{2\pi j t} e^{j\omega_c t} \bigg|_{\omega = -\omega_c} - e^{-j\omega_c t} \bigg|_{\omega = \omega_c} = \frac{\sin \omega_c t}{\pi t} = \frac{\omega_c}{\pi} \text{sinc} \left( \frac{\omega_c t}{\pi} \right).
\]
At first glance, this may appear to be a perfectly reasonable impulse response, but remember that the output \( y(t) \) of the filter in question will depend on \( h(t) \) and the input \( u(t) \) to the filter by the convolution equation

\[
y(t) = \int_{-\infty}^{\infty} h(t-\sigma)u(\sigma)\,d\sigma
\]

so that for this perfect low pass filter design

\[
y(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega_c(t-\sigma)}{(t-\sigma)}u(\sigma)\,d\sigma.
\]

This illustrates a fundamental problem, since when \( \sigma > t \) then

\[
h(t-\sigma) = \frac{\sin \omega_c(t-\sigma)}{(t-\sigma)} \neq 0
\]

and hence the filter output \( y(t) \) will depend upon the input \( u(\sigma) \) with \( \sigma > t \). That is, the output of the filter at some time \( t = t_* \) depends on the input to the filter at a time \( t > t_* \) in the future. This non-causal nature implies that the filter cannot be physically constructed.

In fact, the result of non-causal \( h(t) \) presented in this preceding example could have been predicted via the Paley–Wiener theorem (5.29) which asserts that \( h(t) \) will be causal if, and only if, the area under

\[
\frac{\log |H(\omega)|}{1 + \omega^2}; H(\omega) = \mathcal{F}\{h(t)\}
\]

is finite. However, since the above function is, in fact, infinite whenever \( |H(\omega)| = 0 \), then the area under that function will certainly also be infinite if \( |H(\omega)| = 0 \) across any band of frequencies, and hence the associated impulse response \( h(t) \) will not be causal.

The magnitude response \( |H(j\omega)| \) cannot satisfy

\[
|H(j\omega)| = 0, \quad \omega \in [\omega_1, \omega_2], \quad \omega_1 \neq \omega_2
\]

if the associated impulse response \( h(t) \) is to be causal.

**Constraints on phase response** \( \angle H(j\omega) \)

In very many applications, particularly in control system design, it would be desirable to achieve a certain magnitude response \( |H(j\omega)| \) while at the same time not altering the phase of sine-wave components so that \( \angle H(j\omega) = 0 \) for all \( \omega \). Furthermore,

\[
\angle H(j\omega) = \tan^{-1}\left( \frac{\text{Imag}\{H(j\omega)\}}{\text{Real}\{H(j\omega)\}} \right).
\]

Therefore, in order for \( \angle H(j\omega) = 0 \), \( \text{Imag}\{H(j\omega)\} = 0 \) is necessary, which implies that \( H(j\omega) \) is purely real valued

However, again in recognition of the fact that \( H(j\omega) = \mathcal{F}\{h(t)\} \), the fundamental symmetry/realness properties of Fourier transform discussed in section 5.2.2 indicated that \( H(j\omega) \in \mathbb{R} \) if, and only if, \( h(t) = h(-t) \) and hence its amplitude is symmetric about the vertical y axis. In this case, if \( h(t) \) is non-trivial so that \( h(t) \) is not zero for all \( t > 0 \), then to be symmetric it must by non-zero for \( t < 0 \) as well, and hence also be non-causal.
The achieved non-trivial $H(j\omega)$ must have a non-zero phase response $\angle H(j\omega)$.

**Example 5.26** Counter-examples Some filter responses may initially appear as counter examples to the above principle. Consider for example the impulse response $h(t) = \delta(t)$ which is clearly causal, and yet has associated frequency response $H(\omega) = \mathcal{F}\{\delta(t)\} = 1$ which is purely real valued, and hence exhibits non-zero phase shift.

However, since this same frequency response $H(\omega) = 1$ also implements no magnitude response alteration of signals, it is not of any interest in the context of filter design, and hence for the purposes of the discussion here is considered trivial. For example, the circuit diagram specifying the implementation of such a response would be nothing more than a straight section of wire.

Another, more sophisticated candidate as a counter example, is the filter $H(s) = \frac{1}{s^2}$ for which $H(j\omega) = 1/(j\omega)^2 = 1/\omega^2$ (5.121) and this is clearly purely real valued. Furthermore, according to table 4.1 the causal impulse response $h(t) = -1(t) \cdot t$ (5.122) has Laplace transform $H(s) = \mathcal{L}\{h(t)\} = -1/s^2$. However, a crucial point to note is that the region of convergence of that transform is for Real $\{s\} > 0$. Therefore, in fact, the Laplace transform of (5.122) does not exist (the associated integral evaluates to infinity) for $s = j\omega$, and hence $\mathcal{F}\{1(t) \cdot t\}$ also does not exist. That is, there does not exist a $h(t)$ for which $\mathcal{F}\{h\} = 1/\omega^2$, and hence while (5.121) may be posed as an abstract mathematical counter example, it does not correspond to any physical system which might actually be constructed.

Finally, in the context of analysing constraints on allowable phase response, note that it is also essential that the achieved impulse response $h(t)$ is real valued, since it is not physically reasonable that a filter could be constructed that would emit a complex-valued signal, given a real-valued input. However, via the results of section 5.2.2 this will hold if and only if the phase and magnitude response of $H(j\omega)$ is Hermitian.

The achieved $H(j\omega)$ must satisfy the Hermitian condition

$$H(j\omega) = H(-j\omega)$$

in order that its impulse response $h(t) = \mathcal{F}^{-1}\{H(j\omega)\}$ is real valued. Combining this constraint with the previous one that $H(s)$ have the rational form (5.118) further implies that the co-efficients $\{a_k\}, \{b_k\}$ in this rational form must be real valued.

### 5.5.2 Filter Types and Brick Wall Specification

In general, a filter can be designed to a rather arbitrary specification $S(\omega)$ for it magnitude frequency response $|H(j\omega)|$. However, in practice there are four key types of $S(\omega)$ response that are so commonly required, that they are worth profiling separately.
In doing so, it is important to also recognise that their specification is such that, according to the
discussion of the previous section, they cannot be achieved via a transfer function \( H(s) \) that is rational,
and has an associated causal and real valued impulse response \( h(t) \). Therefore, is is important to not
only present these common filter types, but also the means whereby the quality of the necessary
approximation \( |H(j\omega)| \) to a specification \( S(\omega) \) can be dictated. This leads to what is called a ‘brick
wall’ specification.

\[
\begin{align*}
\text{Ripple} & \quad \text{Stop Band Attenuation} \\
\omega_c & \quad \omega_s
\end{align*}
\]

**Figure 5.30: Brick wall specification for a low-pass filter.**

**Low-Pass Filter**

The simplest and most canonical of these specifications is known as a ‘low-pass’ filter. The name is
chosen to imply that the specification \( S(\omega) \) is one in which spectral components at low frequencies are
‘passed’ with their magnitude unchanged, while those components at higher frequencies are ‘stopped’,
in that their magnitude is significantly attenuated.

This general requirement can be presented in more quantitative detail via the brick wall specification show in figure 5.30. Here the idea is that a region is defined in which a designed filter frequency response \( |H(j\omega)| \) must pass, and regions beyond this area are thought of as an impenetrable brick
wall.

In defining the allowable region between the brick wall, there is some important terminology and
assumptions. Firstly, the frequency region in which low-frequency spectral components are passed is
called the ‘pass band’, and extends from \( \omega = 0 \) rad/s up to a cut-off frequency \( \omega_c \) rad/s.

In this pass-band region, it is desirable that \( |H(j\omega)| \approx 1 \), and the deviation from this is known as
the allowable ‘ripple’. Finally, the amount of attenuation required of higher frequency components is
specified via a stop-band attenuation figure, together with a frequency $\omega_s$ that dictates the start of the stop-band region $\omega \in [\omega_s, \infty)$. The region $[\omega_c, \omega_s]$ in which $|H(j\omega)|$ falls from its pass-band values to its stop-band values is known as the ‘transition band’.

![Diagram](image)

Figure 5.31: Brick wall specification for a high-pass filter.

**High-Pass Filter**

Here the filter name is chosen to imply the opposite of the preceding low-pass case. Spectral components at high frequencies are passed, while those at low frequencies are attenuated, as illustrated in figure 5.31. All the terminology just introduced of pass-band, stop-band, transition band, pass-band ripple, cut-off frequency $\omega_c$ and stop-band frequency $\omega_s$ applies to this high-pass case with the same meanings as before.

**Band-Pass Filter**

The band-pass filter is a more complicated scenario than the previous ones, in the pass-band in which spectral components are to remain unaltered in magnitude is one defined by both an upper $\omega_u$ and lower $\omega_l$ cut-off frequency as shown in figure 5.33. This also implies two stop-band frequencies, a lower one $\omega_s'$ and an upper one $\omega_u'$, also shown in figure 5.33. Beyond these extra necessary specifications, the ideas of stop-band, pass-band, pass-band ripple and stop-band attenuation are as before.
Figure 5.32: *Brick wall specification for a band-pass filter.*

**Band-Stop Filter**

The final important case to consider is the band-stop filter, which is dual to the band-pass filter in that instead of a pass-band, a range \( \omega \in [\omega_c^l, \omega_c^u] \) specified by lower \( \omega_c^l \) and upper \( \omega_c^u \) cutoff frequencies defines a stop-band in which spectral components are attenuated (instead of passed), as shown in figure 5.33.
Figure 5.33: *Brick wall specification for a band-stop filter.*
5.5.3 Butterworth Low-Pass Approximation

With these ideas of filter specification, and filter design constraints in mind, we now turn to the question of how a rational transfer function \( H(s) \) corresponding to causal and real valued impulse response \( h(t) = L^{-1}\{H(s)\} \) might be designed such that \( |H(j\omega)| \) satisfies a brick wall specification.

In addressing this question, attention will first be focussed only on how low-pass specifications might be met, although later sections will illustrate how these low-pass designs may then be used to obtain high-pass, band-pass and band-stop solutions.

One of the most important and widely used filter design methods is that of ‘Butterworth approximation’, with the name deriving from the engineer who developed the method. This approach involves considering the perfect (but unattainable) low-pass response

\[
|H(j\omega)| = \begin{cases} 
1 & |\omega| \leq \omega_c, \\
0 & |\omega| > \omega_c 
\end{cases}
\]

and approximating it with

\[
|H(j\omega)|^2 = \frac{1}{1 + \left(\frac{\omega}{\omega_c}\right)^{2n}}.
\]

(5.124)

Here the idea is that, depending on the choice of the integer \( n \) which is a design variable, when \( \omega < \omega_c \) then \( \omega/\omega_c^{2n} \) is much less than one, and hence (5.124) is approximately equal to one. On the other hand, when \( \omega > \omega_c \) then \( \omega/\omega_c^{2n} \) is much greater than one, and hence (5.124) is approximately equal to zero. In this way, a low-pass response is achieved, which is illustrated for a range of choices of \( n \) in figure 5.13a. Clearly, as the integer design variable \( n \), which is known as the filter ‘order’ is increased, the quality of the approximation (5.124) to the perfect low-pass response (5.123) is also increased. However, as will shortly be made clear, this comes at a penalty of increased filter complexity as \( n \) is increased.

To appreciate this latter point, it is necessary to explain how the transfer function \( H(s) \) is found such that its frequency response \( H(j\omega) \) satisfies (5.124). To this end, note that if we assume that all the filter co-efficients are real valued (this is necessary to ensure that \( H(j\omega) = H(-j\omega) \) and hence \( h(t) \in R \)) then

\[
|H(j\omega)|^2 = \left| H(s) \right|_{s=j\omega}^2
\]

\[
= \left( H(s) \right)_{s=j\omega} \left( \frac{H(s)}{H(s)} \right)_{s=j\omega}
\]

\[
= \left( H(s) \right)_{s=j\omega} \left( H(-s) \right)_{s=j\omega}
\]

\[
= H(s)H(-s)_{s=j\omega}.
\]

(5.125)

Re-expressing the approximation (5.124) as

\[
|H(j\omega)|^2 = \frac{\omega_c^{2n}}{\omega_c^{2n} + (-s^2)^n}_{s=j\omega}
\]

(5.126)

and substituting (5.125) into (5.126) then provides

\[
H(s)H(-s) = \frac{\omega_c^{2n}}{\omega_c^{2n} + (-s^2)^n}.
\]
The poles of $H(s)H(-s)$ are then those values of $s$ for which

$$\omega_c^{2n} + (-s^2)^n = 0$$

which implies

$$(-s^2)^n = -\omega_c^{2n} = \omega_c^{2n} e^{j\pi+2\pi k}; \quad k = 0, 1, 2, 3, \ldots$$

Taking the $n$th root of both sides of this equation then indicates that

$$-s^2 = \omega_c^{2} e^{j(2k+1)\pi/n}$$

and therefore, on taking square roots of both sides of this last equation, the product $H(s)H(-s)$ has poles at

$$s = j\omega_c e^{j(2k+1)\pi/2n} = e^{j\pi/2} \omega_c e^{j(2k+1)\pi/2n} = \omega_c e^{j(2k+n+1)\pi/2n}; \quad k = 0, 1, 2, \ldots, 2n - 1.$$

These poles are located at equally spaced positions around a circle of radius $\omega_c$ as shown for $n = 1, 2, 3, 4$ in figure 5.35.

This solution for the poles of $H(s)H(-s)$ provides quantities from which the poles of $H(s)$ need to be extracted. To achieve this, notice that if $H(s)$ is to be stable, then its poles must lie in the left half plane according to section 4.8. Therefore, only the left half plane poles of $H(s)H(-s)$ are used in forming the Butterworth approximation as follows.
Figure 5.35: Diagram showing pole positions of $H(s)H(-s)$ when Butterworth low pass approximation is used. Diagrams are for approximations of order $n = 1, 2, 3, 4$.

The Butterworth approximation $H(s)$ to a low pass filter with cut-off frequency $\omega = \omega_c$ is

$$H(s) = \frac{\omega_n^2}{(s - \omega_c e^{j(\frac{1}{2} + \frac{1}{n})\pi})(s - \omega_c e^{j(\frac{1}{2} + \frac{3}{n})\pi})(s - \omega_c e^{j(\frac{1}{2} + \frac{5}{n})\pi}) \cdots (s - \omega_c e^{j(\frac{1}{2} + \frac{2n-1}{2n})\pi})}$$ (5.128)

While this has a somewhat complicated form, for the cases of $n = 1, \cdots, 5$ it amounts to

\begin{align*}
n = 1; & \quad H(s) = \frac{\omega_c}{s + \omega_c} \\
n = 2; & \quad H(s) = \frac{\omega_n^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} \\
n = 3; & \quad H(s) = \frac{\omega_n^3}{(s + \omega_c)(s^2 + \omega_c s + \omega_c^2)} \\
n = 4; & \quad H(s) = \frac{\omega_n^4}{(s^2 + (\sqrt{2} + \sqrt{2})\omega_c s + \omega_c^2)(s^2 + (\sqrt{2} - \sqrt{2})\omega_c s + \omega_c^2)} \\
n = 5; & \quad H(s) = \frac{\omega_n^5}{(s + \omega_c)(s^2 + 2\cos \frac{2\pi}{5} \cdot \omega_c s + \omega_c^2)(s^2 + 2\cos \frac{2\pi}{3} \cdot \omega_c s + \omega_c^2)}
\end{align*} (5.129)
which can be obtained from the general expressions

\[
H(s) = \begin{cases} 
\prod_{k=0}^{n/2-1} \frac{\omega_c^2}{s^2 + s2\omega_c \sin(k + 1/2)\pi/n + \omega_c^2} & ; n \text{ even} \\
\left(\frac{\omega_c}{s + \omega_c}\right)^{(n-1)/2} \prod_{k=0}^{(n-1)/2} \frac{\omega_c^2}{s^2 + s2\omega_c \sin(k + 1/2)\pi/n + \omega_c^2} & ; n \text{ odd.}
\end{cases}
\]

This process of extracting a stable transfer function \(H(s)\) such that \(|H(j\omega)|^2\) has a prescribed form is known as ‘spectral factorisation’.

There are several important features of the Butterworth approximation. Firstly, note that regardless of the order \(n\) of the filter, the response at the cut-off frequency \(\omega_c\) is

\[
10 \log_{10} |H(j\omega)|^2|_{\omega=\omega_c} = -10 \log_{10} \left(1 + \left(\frac{\omega}{\omega_c}\right)^{2n}\right) = -10 \log_{10} 2 = -3 \text{ dB.}
\]

Since the response at \(\omega = 0\) is also, regardless of \(n\), always equal to \(10 \log_{10} |H(j0)| = 10 \log_{10} 1 = 0\text{ dB}, then this implies that a Butterworth filter has a pass-band ripple of \(|H(j\omega)| \in [0.707, 1] = [-3 \text{ dB, 0 dB}]\) regardless of the filter order \(n\).

Secondly, from the definition (5.124), note that

\[
\frac{d}{d\omega} |H(j\omega)|^2 = \frac{d}{d\omega} \left[1 + \left(\frac{\omega}{\omega_c}\right)^{2n}\right]^{-1} = -\left[1 + \left(\frac{\omega}{\omega_c}\right)^{2n}\right]^{-2} \frac{2n}{\omega_c} \left(\frac{\omega}{\omega_c}\right)^{2n-1}
\]

so that

\[
\frac{d}{d\omega} |H(j\omega)|^2 \bigg|_{\omega=0} = 0.
\]

That is, the Butterworth approximation begins at \(\omega = 0\) with zero slope, and hence is smooth. Furthermore, it is clear from (5.130) that it will be possible to factor a term \(\omega^{2n-k}\) from the \(k\)’th derivative of \(|H(j\omega)|^2\), and hence

\[
\frac{d^k}{d\omega^k} |H(j\omega)| = 0, \quad k = 1, 2, 3, \ldots, 2n - 1.
\]

Therefore, as the filter order \(n\) is increased, the response near \(\omega = 0\) becomes smoother, which since the passband ripple must be 3dB, implies a sharper roll-off near \(\omega = \omega_c\). This feature (5.131) leads to a Butterworth approximation also being called a ‘maximally flat’ approximation.

Finally, again using the definition (5.124), for \(\omega > \omega_c\) the denominator term can be approximated as \(1 + (\omega/\omega_c)^{2n} \approx (\omega/\omega_c)^{2n}\) so that

\[
\frac{20 \log_{10} \left|\frac{H(j\omega_c)}{H(j\omega_s)}\right|}{\log_{10} \frac{\omega_s}{\omega_c}} \approx 20 \left(\frac{\log_{10} \frac{1}{\sqrt{2}} - n \log_{10} \frac{\omega_c}{\omega_s}}{\log_{10} \frac{\omega_s}{\omega_c}}\right).
\]

Solving for \(n\) then gives a method for choosing the filter order \(n\) according to the required roll-off in the transition band

\[
dn \approx \frac{3 + 20 \log_{10} \left|\frac{H(j\omega_c)}{H(j\omega_s)}\right|}{20 \log_{10} \frac{\omega_s}{\omega_c}}.
\]
For sharp roll-offs, the 3 term in the numerator has little effect, and a rule of thumb is that $n$ must be the nearest integer not less than the required roll-off in dB/decade divided by 20.

**Example 5.27** Butterworth Filter Design Consider the case of designing a low-pass filter to satisfy a requirement of a stop-band attenuation of 30dB for a cut-off frequency $\omega_s = 7\omega_c$. Then according to (5.133), if a pass-band ripple of 3dB is acceptable then a Butterworth approximation solution will need to be of order $n$ given as

$$n = \frac{3 + 30}{20 \log_{10}\frac{7}{\omega_c}} = 1.9524.$$  

That is, a 2nd order Butterworth response

$$H(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2} \quad (5.134)$$

will be necessary, which is shown in figure 5.36 together with the above-mentioned brick-wall specification. This transfer function can then be physically implemented via the circuit shown in figure 5.37.

![Figure 5.36: Butterworth low pass filter approximation together with brick-wall specification - see Example 5.27](image)

Using the techniques introduced in section 4.10, the transfer function $H(s)$ between $V_i(s)$ and $V_o(s)$ may be found as

$$H(s) = -\left(\frac{R_f}{R_i}\right)\frac{\omega_c^2}{s^2 + 2\zeta\omega_c s + \omega_c^2}$$
where

\[ \omega_c = \frac{1}{C_f C_i R_f R} \]

\[ \zeta = \frac{C_f R_f R_i + C_f R R_i + C_f C_i R_f}{2R_i \sqrt{C_f C_i R_f R}} \]

Therefore, in order to implement the Butterworth low pass filter approximation \((5.134)\), the component values \(R_f, R_i, C_f, C_i\) in the circuit of figure \((5.37)\) need to be chosen to satisfy the above equations with \(\zeta = \sqrt{2} \), \(R_f / R_i = 1\) and with \(\omega_c\) the desired cutoff frequency. This implies three constraints, with five degrees of freedom in the choice of component values. This allows many possible solutions for the component values, and typically this needs to be exploited so that a quantities are found near those which may be commercially obtained.

Note that other circuit implementations are also possible, such as the so called ‘Sallen and Key’ one shown in figure \((5.38)\) which has transfer function \(H(s) = V_o(s)/V_i(s)\) given by

\[ H(s) = \left(1 + \frac{R_f}{R_i}\right) \frac{1}{R_1 R_2 C_1 C_2 s^2 + (R_2 C_2 + R_1 C_2 - R_1 C_1 / R_i) s + 1} \]

Due to a positive feedback loop in the circuit, this transfer function can be less sensitive than other circuit designs to changes (or inaccuracies) in component values.
Figure 5.38: Alternative ‘Sallen and Key’ Low Pass Filter Circuit.
5.5.4 Chebychev Low-pass Approximation

A further widely used filter design method is that of Chebychev approximation. The key difference relative to a Butterworth approach is that the passband ripple is not fixed at 3dB. Instead it becomes a design parameter that can be traded-off with the choice of filter order \(n\) in order to achieve a brick-wall specification with a lower filter order than can be provided by a Butterworth approach.

Despite these differences, the Chebychev approximation method retains the original idea introduced in (5.124) of approximating a perfect low-pass response (5.123) by using a polynomial \(p(x)\) evaluated at \(x/c\). In the Butterworth case, this polynomial was of the form \(p(x) = x^{2n}\), which was chosen since it provided small values for \(x < 1\), and then rapidly escalated in value for \(x > 1\). However, this is perhaps not the optimal polynomial to choose.

For example, it would seem sensible to instead use a polynomial \(p(x)\) such that \(|p(x)| < 1\) for \(x \in [0, 1]\), and such that \(p(x)\) is then as large as possible for \(x > 1\). The class of polynomials which satisfy these requirements were discovered by the Russian mathematician Chebychev, and hence are called ‘Chebychev polynomials’. For order \(n\) they are denoted as \(C_n(x)\) and given as

\[
C_n(x) = \begin{cases} 
\cos(n \cos^{-1} x) & ; |x| \leq 1 \\
\cosh(n \cosh^{-1} x) & ; |x| > 1.
\end{cases}
\]

At first glance, these expression would not appear to be polynomial, but in fact they are as the cases \(n = 1, 2, 3, 4\) illustrate.

\[
C_1(x) = x \\
C_2(x) = 2x^2 - 1 \\
C_3(x) = 4x^3 - 3x \\
C_4(x) = 8x^4 - 8x^2 + 1
\]

These functions are plotted in figure 5.5.4 and clearly satisfy the requirements mentioned above. Higher order cases continuing from (5.136) may be computed using the recursive formula

\[
C_{n+1}(x) = 2xC_n(x) - C_{n-1}(x).
\]

The idea now is to use these polynomials in place of \(p(x) = x^{2n}\) in (5.124) so as to provide a so-called ‘Chebychev approximation’ to a perfect low-pass filter response as follows.

\[
|H(j\omega)|^2 = \frac{1}{1 + \epsilon^2 C_n^2 \left(\frac{\omega}{\omega_c}\right)}.
\]

The nature of this low pass filter approximation is illustrated in figure 5.39. Of particular importance is the inclusion of a new design variable \(\epsilon\) in (5.137) that is in addition to that of the filter order \(n\). Its effect is to control the amount of ripple in the pass-band. That is, since \(|C_n(\omega/\omega_c)| < 1\) for \(\omega < \omega_c\), then \(\epsilon^2 C_n^2(\omega/\omega_c) < \epsilon^2\) in the pass-band where \(\omega < \omega_c\). This implies that the Chebychev approximation (5.137) will wander between \(|H(j\omega)|^2 = 1\) and \(|H(j\omega)|^2 = (1 + \epsilon^2)^{-1}\) in this pass-band which implies a ripple expressed in decibels of

\[
r = \text{Ripple}_{dB} = 10 \log_{10}(1 + \epsilon^2).
\]

Therefore, if this ripple \(r\) is a fixed design specification then \(\epsilon\) needs to be chosen as

\[
\epsilon = \sqrt{10^r/10} - 1.
\]
The spectral factorisation calculations necessary to extract a stable $H(s)$ such that $|H(j\omega)|^2$ satisfies (5.137) are considerably more complicated than for the Butterworth case presented in the previous section, although they are presented in Appendix 5.4.2 for the interested reader. To give some idea of the results that are obtained there, for $n = 1, 2, 3$ the Chebychev low-pass filter approximations are

\[
\begin{align*}
n = 1; & \quad H(s) = \frac{\omega_c/\varepsilon}{s + \frac{1}{\varepsilon}} \\
n = 2; & \quad H(s) = \frac{\omega_c^2/2\varepsilon}{s^2 + (\sqrt{1 + 1/\varepsilon^2} - 1)\omega_c s + \omega_c^2\sqrt{1 + \varepsilon^2}/2\varepsilon} \\
n = 3; & \quad H(s) = \frac{\omega_c^3/4\varepsilon}{(s + \gamma)(s^2 + \alpha s + \beta)}
\end{align*}
\]

While these expressions are much more complicated than the Butterworth case in that the formulae for the filter co-efficients rapidly become cumbersome, once these co-efficients are known, the transfer functions themselves are no more complex than for the Butterworth case.

However, whereas the poles of a Butterworth filter are equi-spaced around a circle of radius $\omega_c$ radians per second, the placing of Chebychev filter poles is more complicated. As derived in Appendix 5.4.2 they lie on the ellipse shown in figure 5.40. A final important property of Chebychev
approximations is that according to (5.135) and (5.137)
\[
\frac{d}{d\omega} \log_{10} |H(j\omega)|^2 \bigg|_{\omega=\omega_c} = \frac{2n\varepsilon^2 \cos(n \cos^{-1}(\omega/\omega_c)) \sin(n \cos^{-1}(\omega/\omega_c))}{\sqrt{1 - (\omega/\omega_c)^2}} \left(\varepsilon^2 \cos^2(n \cos^{-1}(\omega/\omega_c)) + 1\right)_{\omega=\omega_c} = -2n^2 \left(\frac{\varepsilon^2}{\varepsilon^2 + 1}\right)
\]
so that the initial roll-off rate near the cut-off frequency $\omega_c$ is proportional to $n^2$. This is in contrast to the Butterworth case in which
\[
\frac{d}{d\omega} \log_{10} |H(j\omega)|^2 \bigg|_{\omega=\omega_c} = -2n \left(\frac{\omega/\omega_c}{(\omega/\omega_c)^{2n} + 1}\right)_{\omega=\omega_c} = -n
\]
and hence the Butterworth roll-off rate, being proportional to $n$ is considerably slower that the Chebychev rate proportional to $n^2$. This suggests that for a given order $n$, since $n^2 > n$, then a Chebychev design will always provide sharper initial roll-off in the transition band. However, note that this roll-off is affected by the $\varepsilon^2 (\varepsilon^2 + 1)^{-1}$ term in (5.138) which is proportional to the pass-band ripple. Therefore, there is a tradeoff in Chebychev filter design between this pass-band ripple and initial roll-off in the transition band. The more the allowable ripple, the faster the roll-off, and vice versa, the less the ripple, the slower the roll-off.

Finally, $n$ can be made the subject of (5.138) to provide a means for judging the required filter order to satisfy a given roll-off specification
\[
n \approx \sqrt{\left(\frac{\varepsilon^2 + 1}{\varepsilon^2}\right) \cdot \frac{1}{40} \text{ dB/dec}}
\]
where the required dB/decade roll-off can be computed in terms of the cut-off and stop-band $\omega_s$ frequencies together with the stop-band attenuation as

$$\frac{\text{dB}}{\text{dec}} = 20 \log_{10} \left| \frac{H(j \omega_c)}{H(j \omega_s)} \right|.$$

**Example 5.28** Profiling of Butterworth and Chebychev solutions Consider the case of requiring a low-pass filter with cutoff frequency 1 kHz, stop-band frequency of 2 kHz, stop-band gain of $-30$dB and pass-band ripple of 3dB. According to (5.133) a Butterworth solution will require a filter of order

$$n = \frac{3 + 30}{20 \log_{10} 2} \approx 5$$

which bysubstituting $s/\omega_c$ for $s$ (with $\omega_c = 2\pi \cdot 1000$) in (5.129) has transfer function

$$H(\omega_c s) = \frac{1.002377293}{(s^2 + 1.618803s + 1.000950)(s^2 + 0.618328s + 1.000950)(s + 1.000475)}.$$

Here the formulation $H(\omega_c s)$ is used instead of the more usual $H(s)$ in order to preserve the number of significant digits that can be conveyed. For example

$$H(s) = \frac{1}{0.00001s^2 + 0.01s + 1}.$$
is more easily written

\[ H(100s) = \frac{1}{s^2 + s + 1}. \]

With this in mind, a Chebychev approximation requires

\[ \varepsilon = \sqrt{10^{3/10}} - 1 = 0.9976 \]

in order to satisfy the 3dB ripple specification, which then further implies that the 4th order Chebychev approximation

\[ H(\omega_c s) = \frac{0.1252971616}{(s^2 + 0.411239s + 0.195980)(s^2 + 0.170341s + 0.903087)} \]

satisfies the roll-off requirements. The magnitude frequency responses of these Butterworth and Chebychev approximation solutions, together with the brick-wall design specification are shown in figure 5.41. Circuits which will implement these Butterworth and Chebychev transfer functions are shown in figures 5.43 and 5.44. Here, care has been taken to find component values near ones that are commercially available. However, this has not been totally successful. Although 10kΩ resistors can be easily sourced, 12.9nF, 19.7nF and 77.4nF capacitors and 822Ω or 976Ω resistors are not available. The nearest standard component values are 10nF, 22nF, 68nF, 820Ω and 1kΩ respectively. Even with these available quantities, manufacturers typically only guarantee resistance values to a tolerance of ±5% and capacitance values to ±10%.

The effect of this component value variation on the frequency can be gauged from Monte-Carlo analysis of the frequency responses of the circuits with ±10% tolerances on component values which are shown in figure 5.42.
A Mont-Carlo simulation is one where the simulation is done many times and with random component values (up to a tolerance) chosen on each simulation. The results are then plotted on top of each other to get an idea of sensitivity to component values.

As can be seen, there is considerable variation in frequency response, and this is a significant motivation for the employment of digital signal processing methods.

![Butterworth Low-pass approximation](image1)

**Figure 5.42:** *Spice Monte Carlo Simulation of op-amp circuit realisations of Butterworth and Chebychev approximations to ideal low pass filter.*

![Chebychev Low-pass approximation](image2)

![Opamp Circuit to realise Butterworth approximation Lowpass filter in the example.](image3)

**Figure 5.43:** *Opamp Circuit to realise Butterworth approximation Lowpass filter in the example.*

### 5.5.5 High-pass Filter Approximation

With this discussion of how low-pass filters may be designed via Butterworth and Chebychev approximation methods now complete, we now address how further filter types may be achieved, and in fact this involves now starting with a low-pass ‘prototype’ design.
That is, taking the case of high-pass filter design as an example, the idea is to take an existing low-pass design $H_{LP}(s)$ and then transform it into a required high-pass design $H_{HP}(s)$ by the use of a transformation $\kappa(s)$ such that

$$H_{HP}(s) = H_{LP}(\kappa(s)).$$

(5.140)

Clearly, the mapping $\kappa(s)$ should be one such that $\kappa(j\omega)$ exchanges high frequencies for low frequencies (and vice-versa) so that the pass-band of the low pass filter $H_{LP}(s)$ is translated to being the pass-band for the high pass filter $H_{HP}(s)$.

Furthermore, since the achieved high pass filter $H_{HP}(s)$ must be a rational function of $s$ for reasons given in section 5.5.1 then $\kappa(s)$ must also be a rational function. These requirements are met by the choice

$$\kappa(s) = \frac{\omega_c^{LP}}{\omega_c^{HP}}s$$

(5.141)

where

$$\omega_c^{LP} \triangleq \text{Low Pass Filter cutoff frequency}$$

$$\omega_c^{HP} \triangleq \text{High Pass Filter cutoff frequency}.$$

To see why this transformation is appropriate, note that it will imply a high pass filter frequency response of

$$H_{HP}(j\omega) = H_{LP}(\kappa(s))|_{s=j\omega} = H_{LP}\left(\frac{\omega_c^{LP}}{\omega_c^{HP}}j\omega\right) = H_{LP}\left(-\frac{j\omega_c^{LP}}{\omega_c^{HP}}\right) = H_{LP}\left(\frac{j\omega_c^{LP}}{\omega}\right)$$

where, in progressing to the last line we have are recognising the constraint the frequency response must be Hermitian if the associated impulse response is to be real valued.

Therefore, the magnitude frequency response $|H_{HP}|$ at $\omega$ is the same as the magnitude response $|H_{LP}|$ at $\omega\frac{\omega_c^{LP}}{\omega_c^{HP}}/\omega$. This mapping

$$\omega \mapsto \frac{\omega_c^{LP}}{\omega_c^{HP}}$$
is illustrated in figure 5.45 which makes clear that high frequency regions will be shifted to low frequency ones and vice versa. In particular

$$|H_{HP}(j\omega)|_{\omega=\omega_{c}^{HP}} = \left| H_{LP} \left( j \frac{\omega_{c}^{LP}, \omega_{c}^{HP}}{\omega} \right) \right|_{\omega=\omega_{c}^{HP}} = |H_{LP} (\omega_{c}^{LP})|$$

$$|H_{HP}(j\omega)|_{\omega=\omega_{s}^{LP}} = \left| H_{LP} \left( j \frac{\omega_{s}^{LP}, \omega_{c}^{HP}}{\omega} \right) \right|_{\omega=\omega_{s}^{LP}} = |H_{LP} (\omega_{s}^{HP})|.$$ 

Therefore, the high-pass filter derived from a low-pass prototype by means of the transformation (5.140), (5.141) will achieve the same stop-band attenuation, pass-band ripple, and transition band-width as did the low-pass prototype.

Figure 5.45: The non-linear transformation $\kappa(j\omega)$ that can be used to turn a low pass filter into a high pass filter.

**Example 5.29** Obtaining a high-pass filter from a low-pass prototype Consider the simple case of a 1st order Butterworth low pass filter

$$H_{LP}(s) = \frac{\omega_{c}^{LP}}{s + \omega_{c}^{LP}}.$$
Then a corresponding high pass filter can be designed as

\[ H_{HP}(s) = \frac{H_{LP}(s)|_{s=\omega_{c}^L \omega_{c}^H}}{\omega_{c}^L \omega_{c}^H \omega_{c}^L \omega_{c}^H / s + \omega_{c}^L \omega_{c}^L} \]

\[ = \frac{s}{s + \omega_{c}^H}. \]

Plots of the prototype low-pass filter \( H_{LP}(j\omega) \) together with the derived high-pass filter \( H_{HP}(j\omega) \) are shown in the top half of figure 5.48.

5.5.6 Band-Pass and Band-Stop filter design

The procedure for band-pass and band-stop filter design is similar to that just presented for high-pass design in that a low-pass prototype is first obtained, and then a mapping \( \kappa(s) \) is used to transform this initial design into the band-pass or band-stop one required.

**Band-Pass Transformation**

In this case the required band-stop filter \( H_{BP}(s) \) is obtained from the prototype low-pass filter design \( H_{LP}(s) \) as

\[ H_{BP}(s) = H_{LP}(\kappa(s)) \]

where

\[ \kappa(s) = \frac{\omega_{c}^L (s^2 + \omega_{c}^H \omega_{c}^L)}{s(\omega_{c}^H - \omega_{c}^L)}. \] (5.142)

This maps \( s = j\omega \) as

\[ \kappa(j\omega) = \frac{\omega_{c}^L ((j\omega)^2 + \omega_{c}^H \omega_{c}^L)}{j\omega(\omega_{c}^H - \omega_{c}^L)} = j \frac{\omega_{c}^L (\omega_{c}^H \omega_{c}^L - \omega^2)}{\omega(\omega_{c}^H - \omega_{c}^L)} \] (5.143)

which is illustrated in figure 5.46. Note from that diagram that \( \kappa(j\omega) \) maps the band-pass cut-off region as \([\omega_{c}^L, \omega_{c}^H] \mapsto [-\omega_{c}^L, \omega_{c}^L] \) so that

\[ \left| H_{BP}(j\omega) \right|_{\omega \in [\omega_{c}^L, \omega_{c}^H]} = \left| H_{LP}(j\omega) \right|_{\omega \in [-\omega_{c}^L, \omega_{c}^L]}. \]

Therefore, the pass-band ripple of the achieved band-pass filter will be the same as the pass-band ripple of the low-pass filter prototype. However, the stop-band frequencies \( \omega_{c}^L \) and \( \omega_{c}^U \) do not necessarily map to values related to the stop-band frequency \( \omega_{c}^L \) of the low-pass filter prototype. As a result, the stop-band attenuation achieved by this design method is not necessarily that of the low-pass filter prototype. Instead, it is necessary to experiment with filter order until the stop-band attenuation specification is achieved.
Band-Stop Transformation

In this final case the required band-stop filter $H_{BS}(s)$ is obtained from the prototype low-pass filter design $H_{LP}(s)$ as

$$H_{BS}(s) = H_{LP}(\kappa(s))$$

where

$$\kappa(s) = \frac{\omega_c^{LP} s(\omega_u - \omega_c^f)}{s^2 + \omega_u^u \omega_c^f}$$

(5.144)

This maps $s = j\omega$ as

$$\kappa(j\omega) = \frac{\omega_c^{LP} j\omega(\omega_u - \omega_c^f)}{(j\omega)^2 + \omega_u^u \omega_c^f} = j \frac{\omega_c^{LP} \omega(\omega_u - \omega_c^f)}{(\omega_u^u \omega_c^f - \omega^2)}$$

(5.145)

which is illustrated in figure 5.47. Note from that diagram that this selection of $\kappa(j\omega)$ maps the lower pass-band region as $[0, \omega_c^f] \mapsto [0, \omega_u^{LP}]$ and the upper pass-band region maps as $[\omega_c^u, \infty] \mapsto [-\omega_u^{LP}, 0]$. Therefore

$$|H_{BS}(j\omega)|_{\omega \in [0,\omega_c^f]} = |H_{BS}(j\omega)|_{\omega \in [\omega_u^u, \infty]} = |H_{LP}(j\omega)|_{\omega \in [0,\omega_u^{LP}]}$$

Therefore, the pass-band ripple of the achieved band-stop filter will be the same as the pass-band ripple of the low-pass filter prototype. However, the stop-band frequencies $\omega_u^f$ and $\omega_u^u$ do not necessarily map to values related to the stop-band frequency $\omega_u^{LP}$ of the low-pass filter prototype. As a result, the stop-band attenuation achieved by this design method is not necessarily that of the low-pass filter prototype. Instead, it is necessary to experiment with filter order until the stop-band attenuation specification is achieved.
Figure 5.47: The non-linear transformation $\kappa(j\omega)$ that can be used to turn a low pass filter into a band stop filter.

Example 5.30 Previous Example Revisited Suppose that we start with a Butterworth first order low pass prototype

$$H_{\text{LP}}(s) = \frac{\omega_{\text{LP}}}{s + \omega_{\text{LP}}}.$$ 

This then provides a band-pass filter design of

$$H_{\text{BP}}(s) = H_{\text{LP}}(s) \bigg|_{s = \kappa(s) = \frac{\omega_{\text{LP}}(s^2 + \omega_c^u \omega_c^\ell)}{s(\omega_c^u - \omega_c^\ell)}}$$

$$= \frac{\omega_{\text{LP}}}{s(\omega_c^u - \omega_c^\ell) + \omega_{\text{LP}}} + \frac{\omega_{\text{LP}}^2}{s^2 + s(\omega_c^u - \omega_c^\ell) + \omega_{\text{LP}}^2}. $$
Finally, a band-stop filter can be obtained as

\[ H_{BS}(s) = H_{LP}(s) \bigg|_{s = \kappa(s)} = \frac{\omega_{c}^{LP} s (\omega_{a} \omega_{c} - \omega_{c}^2)}{s^2 + \omega_{c}^2 \omega_{c}^L} \]

\[ = \frac{\omega_{c}^{LP} s (\omega_{a} \omega_{c} - \omega_{c}^2)}{s^2 + \omega_{c}^2 \omega_{c}^L} + \omega_{c}^{LP} \]

\[ = \frac{s^2 + \omega_{c}^2 \omega_{c}^L}{s^2 + \omega_{c}^2 (\omega_{a} \omega_{c} - \omega_{c}^2) + \omega_{c}^2 \omega_{c}^L}. \]

The responses \(|H_{BP}(j\omega)|\) and \(|H_{BS}(j\omega)|\) of these filter designs are shown in the bottom half of figure 5.48, where the top figures show the original low-pass prototype response together with the high pass filter derived from this prototype in Example 5.29.
5.6 Concluding Summary

The focus of this chapter has been on understanding how signals may be decomposed into spectral components, and how the response of linear systems are then affected by these components. In particular, the spectral decomposition of a signal \( f(t) \) was achieved by means of the Fourier transform

\[
F(\omega) = \mathcal{F} \{ f(t) \} \triangleq \int_{-\infty}^{\infty} f(t)e^{-j\omega t} \, dt = \left[ \int_{-\infty}^{\infty} f(t)e^{-st} \, dt \right]_{s=j\omega} = \mathcal{L} \{ f(t) \} |_{s=j\omega} = F(s)|_{s=j\omega}.
\]

Furthermore, in the case of this signal being an input to a linear time invariant (LTI) system with impulse response \( h(t) \), then the corresponding spectral decomposition \( G(\omega) = \mathcal{F} \{ g(t) \} \) of the output \( g(t) = [h \otimes f](t) \) of that LTI system is

\[
G(\omega) = H(\omega)F(\omega).
\]

Here, \( H(\omega) \), known as the frequency response of the linear system, is the Fourier transform of its impulse response, and hence also equal to the system transfer function \( H(s) = \mathcal{L} \{ h(t) \} \) evaluated at \( s = j\omega \)

\[
H(\omega) = \mathcal{F} \{ h(t) \} = \mathcal{L} \{ h(t) \} |_{s=j\omega} = H(s)|_{s=j\omega}.
\]

That is, the effect of the Fourier transform is to map the operation of convolution to the operation of multiplication. This is a property that is inherited from the corresponding Laplace transform property illustrated in the previous chapter, and by virtue of the Fourier transform being the special case of that Laplace transform.

In a similar vein, all the other important Laplace transform properties such as linearity, time shifting mapping to phase shifting, time scaling mapping to frequency scaling and so on are also all inherited by the Fourier transform after the substitution of \( s = j\omega \) in the corresponding Laplace transform property.

However, precisely because of this special (and hence restricted) choice of \( s = j\omega \) being purely imaginary, the Fourier transform has some further properties of its own. In particular, if \( f(t) \) is purely real valued, then \( F(\omega) = \mathcal{F} \{ f(t) \} \) is Hermitian, which means

\[
F(\omega) = \overline{F(-\omega)}.
\]

Importantly, this implies that \(|F(\omega)|\) is an even function of \( \omega \), and hence symmetric about the vertical \( y \) axis.

Furthermore, the principle of duality asserted that time and frequency domain functions could be swapped and, modulo a change of sign in the function argument and a \( 2\pi \) scaling factor, Fourier transform results are invariant to this exchange of domains. Since the above real-implies-Hermitian principle is not affected by any argument sign change or scaling factor, then its dual is that \( F(\omega) \) is purely real valued if, and only if

\[
f(t) = \overline{f(-t)}.
\]

Another important consequence of this duality principle, which will be very important in the developments of the next chapter, is that since the convolution \([f \otimes g](t)\) of two time domain signals has Fourier transform \( \mathcal{F} \{ [f \otimes g](t) \} = F(\omega)G(\omega) \) which is the multiplication of the two transforms, then by duality the swapping of time and frequency domains implies that the Fourier transform of the product of two signals will be the convolution of their transforms (plus a \( 2\pi \) factor)

\[
\mathcal{F} \{ f(t)g(t) \} = \frac{1}{2\pi} [F \otimes G] (\omega).
\]
Other important properties of the Fourier transform are the energy relationship of Parseval’s Theorem

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

and the Paley–Wiener Theorem which asserts that $f(t) = 0$ for $t < 0$ if, and only if

$$\int_{-\infty}^{\infty} \log |F(\omega)| \frac{1}{1 + \omega^2} \, d\omega > -\infty.$$

By recognising that an LTI system has frequency response $H(\omega)$ which is the Fourier transform of its impulse response $h(t)$, this Paley–Wiener condition indicates that a causal system (which must have $h(t) = 0$ for $t < 0$) cannot have frequency response $H(\omega)$ which is zero over any band of frequencies.

There is a class of signals which requires special attention in this context of spectral analysis, namely periodic signals which satisfy $f(t) = f(t + T)$ for some period $T$. Firstly, such signals have an exponential Fourier series decomposition

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_s t}; \omega_s \triangleq \frac{2\pi}{T}$$

where the Fourier co-efficients $\{c_k\}$ are given by

$$c_k = \frac{1}{T} \int_{0}^{T} f(t)e^{-j\omega_s t} \, dt.$$ 

This implies that the spectral decomposition of $f(t)$ involves only discrete spectral components at the fundamental frequencies $\pm \omega_s$ rad/s, and at an infinite number of harmonic frequencies $\pm 2\omega_s, \pm 3\omega_s, \ldots$ that are harmonic to this fundamental. Note that this fundamental frequency $\omega_s = 2\pi/T$ is simply the rate, in rad/s, at which the periodic signal repeats.

These same spectral conclusions also arise from the Fourier transform of a periodic signal, which is of the form of an infinite number of Dirac-delta’s, centred at the fundamental frequency $\omega_s$ and its harmonics as follows

$$F(\omega) = \mathcal{F}\{f(t)\} = \sum_{k=-\infty}^{\infty} b_k \delta(\omega - k\omega_s); \omega_s \triangleq \frac{2\pi}{T}$$

where the $\{b_k\}$ co-efficients are related to the $\{c_k\}$ Fourier co-efficients as

$$b_k = 2\pi c_k.$$ 

Because of this close relationship between Fourier transforms and Fourier series, there are certain relationships between the properties of periodic functions and their Fourier co-efficients $\{c_k\}$ that are direct analogues of the associated Fourier transform property. For example, linearity, convolution mapping to multiplication and so on.

However, there are some important special cases associated with the special periodicity property of the signals concerned. For example, the Parseval’s relationship between energy for Fourier transforms, in the case of periodic signals and Fourier series becomes a relationship of average power over a period being given as

$$\frac{1}{T} \int_{0}^{T} |f(t)|^2 \, dt = \sum_{k=-\infty}^{\infty} |c_k|^2.$$
With these ideas of the spectral decomposition of various types of signals in mind, and recalling that the nature in which linear and time invariant (LTI) systems affect the spectral decomposition of signals is via their frequency response, then a detailed understanding of the nature of that response is required.

In order to provide this, attention was concentrated on the most practically relevant class of LTI systems which have rational transfer functions \( H(s) \) that may be factored as

\[
H(s) = \frac{B(s)}{A(s)} = \frac{K(s - z_1)(s - z_2) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)}
\]

In this case if \( H(j\omega) \) decomposed in polar form as

\[
H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}
\]

then the Bode plot of the above polar form magnitude is the following summation

\[
20 \log_{10} |H(j\omega)| = 20 \log_{10} \left| \frac{K(j\omega - z_1)(j\omega - z_2) \cdots (j\omega - z_m)}{(j\omega - p_1)(j\omega - p_2) \cdots (j\omega - p_n)} \right|
\]

\[
= 20 \log_{10} |K| + 20 \sum_{\ell=1}^{m} \log_{10} |j\omega - z_\ell| - 20 \sum_{k=1}^{n} \log_{10} |j\omega - p_k|
\]

while the Bode phase plot of the above polar form argument is the summation

\[
\angle H(j\omega) = \angle K + \sum_{\ell=1}^{m} \angle (j\omega - z_\ell) - \sum_{k=1}^{n} \angle (j\omega - p_k).
\]

Understanding the nature of \( H(j\omega) \) thus reduces to that of understanding the individual components of the form \( \log_{10} |j\omega - z| \) and \( \angle (j\omega - z) \) that are summed to make up the magnitude \( |H(j\omega)| \) and phase \( \angle H(j\omega) \) of \( H(j\omega) \).

In turn, with respect to a \( \log_{10} \omega \) frequency axis, these components have straight line type approximations that (unfortunately) depend upon whether the zero \( z \) (or pole \( p \)) is real valued or not.

In brief, for \( z \) real, \( 20 \log_{10} |j\omega - z| \) may be approximated as starting from \( 20 \log_{10} |z| \text{dB, then rising upward with slope 20dB/decade from } \omega = |z| \text{ onwards. If } z \text{ is complex, then for the d.c. response } H(j0) \text{ to necessarily be real valued, then a conjugate zero } \overline{z} \text{ must also exist, in which case an approximation for the two terms} \)

\[
20 \log_{10} |j\omega - z| + 20 \log_{10} |j\omega - \overline{z}|
\]

is as above, but with the slope changed to 40dB/decade, and with a dip at \( \omega = |z| \text{ of } 15 \log_{10}(0.707/|\zeta|) \) where \( \zeta = |\cos^{-1} \angle z| \) is the damping ratio associated with the zero.

In the case of a pole \( p \), all the above applies with \( p \) substituted for \( z \), save that in accounting for the contribution to \( 20 \log_{10} |H(j\omega)| \) a minus sign is included which implies downward slopes of 20dB/decade or 40dB/decade for real and complex valued poles respectively.

Turning to phase, for \( z \) real, with respect to a \( \log_{10} \omega \) frequency axis, the angle contribution \( \angle (j\omega - z) \) may be approximated by a straight line beginning at 0° phase until \( \omega = |z|/10 \), then rising by \( 45°/\text{decade} \) if \( z < 0 \), or falling by \(-45°/\text{decade} \) if \( z > 0 \), until \( \omega = 10|z| \), at which point the straight line approximation settles at its terminal phase of 90° for \( z < 0 \) or \(-90° \) for \( z > 0 \).

If \( z \) is complex, then the joint phase contribution

\[
\angle (j\omega - z) + \angle (j\omega - \overline{z})
\]
has a slope of \( m = \frac{132}{\zeta} \) decade, extending over the range \( \log_{10}|\omega|/|z| \in [-\pi/(2m), \pi/(2m)] \) if \( \text{Real}\{z\} > 0 \), and with slope \( m = -\frac{132}{\zeta} \) decade over that same range if \( \text{Real}\{z\} < 0 \), so that the total phase variation is 180°.

In the case of a pole \( p \) instead of zero, again all the above principles apply with \( z \) replace by \( p \) and minus signs included so that rising phase contributions are replace by falling ones.

The final topic of this chapter was to consider that of frequency response synthesis, by way of studying how certain standard filtering profiles for \( |H(j\omega)| \) could be achieved by transfer functions \( H(s) \) that were constrained to be of rational form with real-valued co-efficients. In the case of \( |H(j\omega)| \) having a low-pass response the two methods of Butterworth approximation

\[
|H(j\omega)|^2 = \frac{1}{1 + (\omega/\omega_c)^{2n}}
\]

and Chebychev approximation

\[
|H(j\omega)|^2 = \frac{1}{1 + \varepsilon^2 C_n^2 (\omega/\omega_c)^2}
\]

were presented. Both methods involved a design parameter \( n \) which corresponds to the number of poles in the achieved filter \( H(s) \). However, the Chebychev approach also allows for pass-band ripple of \( 10 \log_{10}(1 + \varepsilon^2) \) dB to be specified my means of the design variable \( \varepsilon \), while the Butterworth strategy has this ripple fixed at 3dB regardless of \( n \).

The chapter was then concluded by discussing how high-pass, band-pass and band-stop designs may be achieved by beginning with a low pass design satisfying the required pass-band ripple specification, and then substituting a certain rational function \( \kappa(s) \) for \( s \) in the low-pass prototype \( H(s) \).
5.7 Exercises

1. Define \( r(t) \) to be the rectangular function
\[
r(t) =\begin{cases} 1 & ; |t| \leq \tau \\ 0 & ; |t| > \tau \end{cases}
\]
Sketch \( r(t) \), derive it’s Fourier transform \( R(\omega) \), and sketch that as well.

2. Define \( \Delta(t) \) to be the triangular function
\[
\Delta(t) =\begin{cases} 1 - \frac{|t|}{2\tau} & ; |t| \leq 2\tau \\ 0 & ; |t| > 2\tau \end{cases}
\]
Sketch \( \Delta(t) \), derive it’s Fourier transform \( \triangle(\omega) \), and sketch that as well.

3. Sketch the convolution
\[
[r \hat{\otimes} r](t)
\]
How are \( r(t) \) and \( \Delta(t) \) related? From this, how should the Fourier transforms \( R(\omega) \) and \( \Delta(\omega) \) be related?

4. Consider the square signal \( r(t) \) defined in question 1 and the triangular signal \( \Delta(t) \) defined in question 2. Derive an expression for, and plot the signal \( x(t) \) given by
\[
x(t) = [r \hat{\otimes} \Delta](t)
\]
Derive an expression for the Fourier transform \( X(\omega) \).

5. Consider the signal’s \( g(t) \) and \( f(t) \) plotted in figure 5.49. Derive their Fourier transforms \( G(\omega) \) and \( F(\omega) \) and plot their amplitude response with respect to frequency \( \omega \).

6. Consider the signal
\[
y(t) =\begin{cases} |t| & ; |t| \leq 1 \\ 2 - |t| & ; |t| \in (1, 2] \\ 0 & ; Otherwise \end{cases}
\]
Sketch \( y(t) \). Derive its Fourier transform \( Y(\omega) \) and sketch that as well. Try to minimise the amount of calculation by using known results.

7. Find the spectrum (Fourier Transform) \( X(\omega) \) of the signal
\[
x(t) = 2 \text{sinc} \left( \frac{t}{2} \right) \cos(2\pi t)
\]
and plot \( X(\omega) \).
8. (a) Consider the signal \(f(t)\) shown on the left in figure 5.50. Derive its Fourier transform \(F(\omega)\) and sketch its magnitude \(|F(\omega)|\).

(b) Consider the signal \(x(t)\) shown on the right in figure 5.50. Derive its Exponential Fourier series representation.

(c) Use the answer to the preceding question to derive the Fourier transform \(X(\omega)\) of the signal \(x(t)\) shown on the right in figure 5.50. Plot its magnitude \(|X(\omega)|\).

9. Put

\[
\begin{align*}
    f(t) &= \sum_{k=-\infty}^{\infty} g(t + 2k) \\
    g(t) &= \begin{cases} 
        e^{-2|t|} & \text{if } |t| \leq 1 \\
        0 & \text{otherwise}
    \end{cases}
\end{align*}
\]

Calculate the Fourier series representation for \(f(t)\).

10. A full wave rectifier converts a 5Hz sinusoidal signal into the signal

\[y(t) = |2 \sin 10\pi t|.
\]

Sketch this signal. Calculate its Fourier transform \(Y(\omega)\) and sketch it as well. What fraction of the power in the signal \(y(t)\) is in the spectral region below 50Hz?

11. A System has frequency response

\[
H(\omega) = \frac{e^{-j0.05\omega}}{1 + j\omega/(4\pi)}
\]

(a) Prove, without deriving the impulse response \(h(t)\), that the impulse response \(h(t)\) is in fact real valued. What is the implication of this on whether or not it is possible to physically build a system with the given frequency response.

(b) (HARD) Prove using the Paley-Wiener Theorem (or otherwise) that this system has a Causal impulse response.

(c) Derive and plot the impulse response \(h(t)\) for this system.

(d) Give a design for an op-amp circuit with the magnitude frequency response \(|H(\omega)|\). Is it possible to build an op-amp circuit with the frequency response \(H(\omega)\)?

12. Consider an arbitrary continuous time filter

\[
H(s) = \frac{B(s)}{A(s)} = \frac{K \prod_{k=1}^{m} (s + z_k)}{\prod_{k=1}^{n} (s + p_k)}
\]

(a) Derive an expression for the normalised sensitivity \(S(\omega)\) of the magnitude frequency response \(|H(j\omega)|^2\) to changes in a pole position \(p_\ell\). The normalised sensitivity \(S(\omega)\) is defined by

\[
S(\omega) = \frac{1}{|H(j\omega)|^2} \frac{d|H(j\omega)|^2}{dp_\ell}
\]
(b) Using this expression for \( S(\omega) \) explain which choice of pole positions make the sensitivity large. Is large sensitivity a good or bad thing? Why?

![Figure 5.49: Signals \( g(t) \) and \( f(t) \)](image)

![Figure 5.50: Signals \( f(t) \) and \( x(t) \)](image)

13. Plot straight line approximations for the Bode Magnitude and Phase frequency responses of the following transfer functions.

(a) \[ H(s) = \frac{100}{s + 100} \]

(b) \[ H(s) = \frac{1000}{(s^2 + 110s + 1000)} \]

(c) \[ H(s) = \frac{1000(s + 1)}{(s^2 + 110s + 1000)} \]
14. Design a filter transfer function \( H(s) \) that achieves the ‘notch’ filter effect shown in figure 5.51. Is this filter unique? Plot the straight line approximation to the Bode phase response of the filter you design.

15. A system has impulse response

\[
h(t) = \begin{cases} 2e^{-3t} & ; t \geq 0 \\ 0 & ; \text{Otherwise} \end{cases}
\]

and is fed the input signal

\[
u(t) = \begin{cases} 3\cos 4\pi ft & ; t \geq 0 \\ 0 & ; \text{Otherwise} \end{cases}
\]

(a) In steady state (in the limit as \( t \to \infty \)) what is the output \( y(t) = [h \odot u](t) \) of the system?

(b) Assuming that \( y(t) \) and all its derivatives are zero for \( t < 0 \), compute the complete (not just steady state - transient component as well) response \( y(t) = [h \odot u](t) \) using any method you wish.

16. A system has impulse response

\[
h(t) = \begin{cases} \cos \omega_0 t & ; t \in [0, T] \\ 0 & ; \text{Otherwise} \end{cases}
\]

Derive and sketch the magnitude frequency response of this system. What sort of filtering action is being implemented?
17. Use MATLAB to design a bandpass filter $H(s)$ to meet the following specifications:

- $0 \leq |H(j\omega)| \leq 0.1 ; \omega \in [0, 10\text{Hz}]
- 0.7 \leq |H(j\omega)| \leq 1.5 ; \omega \in [28\text{Hz}, 52\text{Hz}]
- 0 \leq |H(j\omega)| \leq 0.1 ; \omega \geq 70\text{Hz}

Plot the Bode magnitude frequency response of your design on the same set of axes as straight lines which delineate the above ‘brick-wall’ specification. Do this for two cases:

(a) Using a Butterworth approximation method to obtain $H(s)$;
(b) Using a Chebychev approximation method to obtain $H(s)$.

The commands `butterord`, `butter`, `cheb2ord` and `cheby2` will be useful here.
5.A Mathematical Background

5.A.1 Dirac Delta Interpretation of Integrated Exponential

In this chapter, the reconciliation of Fourier transforms with Fourier Series (i.e. the study of Fourier transforms of periodic functions) depended crucially on the interpretation of an integrated exponential as behaving like a Dirac-delta

\[ \int_{-\infty}^{\infty} e^{j\omega t} \, dt = 2\pi \delta(\omega). \]  

(5.1)

The exploitation of this fact will also prove important in the following chapter in the context of sampling. It therefore needs to be established. For this purpose, first note that the integral in (5.1) can be written as

\[ \int_{-\infty}^{\infty} e^{j\omega t} \, dt = \lim_{n \to \infty} \int_{-(n+1/2)}^{n+1/2} e^{j\omega t} \, dt \]  

(5.2)

where \( n \) is an integer. Furthermore,

\[ \int_{-(n+1/2)}^{n+1/2} e^{j\omega t} \, dt = \frac{e^{j\omega (n+1/2)}}{j\omega} \bigg|_{t=-(n+1/2)}^{t=n+1/2} = \frac{\sin(2n+1)\omega/2}{\omega/2} = D_n(\omega). \]

This quantity \( D_n(\omega) \) is known as a ‘Dirichlet Kernel’ in the theory of Fourier series. It is plotted for the case of \( n = 50 \) in figure 5.52 where it clearly has a shape approximating a Dirac \( \delta \) function. In fact, using the geometric series summation formula

\[ \sum_{k=0}^{n} ar^k = \frac{a(1 - r^{n+1})}{1 - r} \]

then

\[ \sum_{k=-n}^{n} e^{-j\omega k} = \sum_{k=0}^{2n} e^{-j\omega(k-n)} = e^{j\omega n} \sum_{k=0}^{2n} e^{-j\omega k} = e^{j\omega n} \left( \frac{1 - e^{-j\omega(n+1)}}{1 - e^{-j\omega}} \right) = e^{j\omega n} \frac{e^{-j\omega(n+1/2)} - e^{-j\omega(n+1/2)}}{e^{-j\omega/2} - e^{-j\omega/2}} = \frac{\sin(2n+1)\omega/2}{\sin\omega/2} = D_n(\omega). \]

so that

\[ D_n(0) = \sum_{k=-n}^{n} e^{-j\omega k} \bigg|_{\omega=0} = 2n + 1 \]

and hence the height of the peak in \( D_n(\omega) \) increases without bound as \( n \to \infty \). Therefore,

\[ \int_{-\infty}^{\infty} e^{j\omega t} \, dt = \lim_{n \to \infty} D_n(\omega) = \alpha \delta(\omega) \]
where $\alpha \in \mathbb{R}$ which is the area underneath $\lim_{n \to \infty} D_n(\omega)$. However,

$$
\int_{-\infty}^{\infty} D_n(\omega) \, d\omega = \int_{-\infty}^{\infty} \sum_{k=-n}^{n} e^{-j\omega k} \, d\omega \\
= \sum_{k=-n}^{n} \int_{-\infty}^{\infty} e^{-j\omega k} \, d\omega \\
= \sum_{k=-n}^{n} \begin{cases} 
2\pi & ; \omega = 0 \\
0 & ; \text{Otherwise} 
\end{cases} \\
= 2\pi
$$

which is independent of $n$, and hence $\alpha = 2\pi$ to allow the conclusion that

$$
\int_{-\infty}^{\infty} e^{j\omega t} \, dt = 2\pi \delta(\omega),
$$

(5.3)

Figure 5.52: Dirichlet Kernel $D_n(\theta)$ for $N = 10$.

5A.2 Spectral Factorisation for Chebychev Filter

By the definition (5.137) of the Chebychev approximation to the ideal magnitude response

$$
H(s)H(-s)|_{s=j\omega} = \frac{1}{1 + \varepsilon^2 C_n^2 \left( \frac{s}{j\omega_c} \right)}
$$
whose roots are the solutions of
\[ 1 + e^{2C_n^2 \left( \frac{s}{j\omega_c} \right)} = 0 \Rightarrow C_n \left( \frac{s}{j\omega_c} \right) = \pm \frac{j}{\varepsilon}. \]
Using the definition of the \( n \)th order Chebychev polynomial then gives that the roots satisfy
\[ \cos \left( n \cos^{-1} \frac{s}{j\omega_c} \right) = \pm \frac{j}{\omega_c}. \tag{5.1} \]
Now, put \( x + jy = n \cos^{-1} s/j\omega_c \). The (5.1) becomes
\[ \pm \frac{j}{\omega_c} = \cos(x + jy) = \cos x \cos jy - \sin x \sin jy = \cos x \cosh y - j \sin x \sinh y. \]
where the last line is derived by noting that
\[ \cos j\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} = \frac{e^{-\theta} + e^{\theta}}{2} = \cosh \theta \]
and similarly for \( \sin jy \). Equating real and imaginary parts then gives
\[ \cos x \cosh y = 0 \quad \sin x \sinh y = \pm \frac{1}{\varepsilon}. \tag{5.2} \quad \tag{5.3} \]
Therefore, from (5.2)
\[ x = (2k + 1) \frac{\pi}{2} ; k = 0, \pm 1, \pm 2, \pm 3, \cdots \]
and substituting this into (5.3) gives
\[ y = \pm \sinh^{-1} \frac{1}{\varepsilon}. \]
Then using the definition of \( x \) and \( y \) also implies that
\[ n \cos^{-1} \frac{s}{j\omega_c} = (2k + 1) \frac{\pi}{2} \pm j \sinh^{-1} \frac{1}{\varepsilon} \]
to give the roots of \( H(s)H(-s) \) as
\[ s = j\omega_c \cos \left\{ \left( k + \frac{1}{2} \right) \frac{\pi}{n} \pm \frac{j}{n} \sinh^{-1} \frac{1}{\varepsilon} \right\} \]
\[ = j\omega_c \cos \left( k + \frac{1}{2} \right) \frac{\pi}{n} \cos \left( \frac{j}{n} \sinh^{-1} \frac{1}{\varepsilon} \right) \pm j\omega_c \sin \left( k + \frac{1}{2} \right) \frac{\pi}{n} \sin \left( \frac{j}{n} \sinh^{-1} \frac{1}{\varepsilon} \right) \]
\[ = \pm \omega_c \sin \left( k + \frac{1}{2} \right) \frac{\pi}{n} \sinh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right) + j\omega_c \cos \left( k + \frac{1}{2} \right) \frac{\pi}{n} \cosh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right). \]
Therefore, setting \( s = \sigma + j\omega \) implies that the roots of \( H(s)H(-s) \) are given as
\[ \sigma = \pm \omega_c \sin \left( k + \frac{1}{2} \right) \frac{\pi}{n} \sinh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right) \]
\[ \omega = \omega_c \cos \left( k + \frac{1}{2} \right) \frac{\pi}{n} \cosh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right). \]
Therefore,
\[
\frac{\sigma^2}{\sinh^2 \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right)} + \frac{\omega^2}{\cosh^2 \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right)} = \omega_c^2 \sin^2 \left( k + \frac{1}{2} \right) - \frac{\pi}{n} + \omega_c^2 \cos^2 \left( k + \frac{1}{2} \right) - \frac{\pi}{n} = \omega_c^2
\]
so that these roots lie on an ellipse as shown in figure 5.40. Incidentally, the hyperbolic sines and cosines you can be eliminated from the expressions via:
\[
\omega_c \cosh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right) = \omega_c \left\{ \frac{1 + \sqrt{1 + \varepsilon^2}}{\varepsilon} \right\}^{1/n} + \left( \frac{\varepsilon}{1 + \sqrt{1 + \varepsilon^2}} \right)^{1/n} \\
\omega_c \sinh \left( \frac{1}{n} \sinh^{-1} \frac{1}{\varepsilon} \right) = \omega_c \left\{ \frac{1 + \sqrt{1 + \varepsilon^2}}{\varepsilon} \right\}^{1/n} - \left( \frac{\varepsilon}{1 + \sqrt{1 + \varepsilon^2}} \right)^{1/n}
\]

5.A.3 The Law of Sines

The law of sines gives a relationship between the length of the sides of any triangle, and the angles in that triangle. Specifically, consider a triangle with sides of length \( a, b \) and \( c \). Denote the angle subtended by the triangle vertex opposite the side of length \( a \) as \( \theta_a \), and so on for sides \( b \) and \( c \) to yield angles \( \theta_b, \theta_c \). Then the law of sines states that
\[
\frac{a}{\sin \theta_a} = \frac{b}{\sin \theta_b} = \frac{c}{\sin \theta_c}.
\]