Chapter 6

Sampling, Reconstruction and the Discrete Time Fourier Transform

As the previous chapter has highlighted via the example of filter synthesis, an essential difficulty in areas of signal processing, feedback control and telecommunications (to name but a few) is that physical component tolerances can hamper the accuracy and robustness of any design. Furthermore, any adaptation (for example, to implement a changed low-pass cutoff frequency) requires altered component values.

Largely because of these limitations, most modern signal processing schemes are implemented 'digitally'. That is, computer processing is employed in a manner in which (for example) filter specification corresponds to co-efficient values stored in random-access memory, and hence may be represented accurately (independent of component tolerances) and in a manner which is easily adapted.

However, unlike the op-amp based filter circuits of the previous chapter, computer processing of data cannot proceed in a continuously evolving manner (ie. in 'continuous time') since all processing operations must be synchronized with respect to an internal clock signal. Therefore, computer-based signal processing, which is more commonly termed 'Digital Signal Processing' (DSP), can only work with discrete *samples* of signals where the samples themselves are taken (much) less frequently than the underlying clock frequency of any processing unit.

This necessary process of working with samples of signals, rather than the signals themselves, raises several important questions. How might these samples be obtained? How much 'information' do the samples carry about the underlying continuous time signals they represent? How might a continous time signal be constructed from samples produced by a processing unit? This chapter is devoted to addressing these and further related issues.

6.1 Sampling

The process of taking samples of an underlying continous time (ie. continously evolving) signal is presented diagramatically in figure 6.1. There it is illustrated that the sampling itself is achieved via a component known as an Analog to Digital (A/D) converter. Its function is to sample a continous valued (analog) signal at times determined by the rising edge of a clock signal, and to then convert these samples to an n-bit binary representation. Typically n will be between 6 and 24 bits depending on the quality of the A/D converter. This binary-format 'sample' of a continuous time 'real world' signal can then be transfered to a central processing unit (CPU) register, stored in random access memory (RAM) or written to disk (amongst other things).

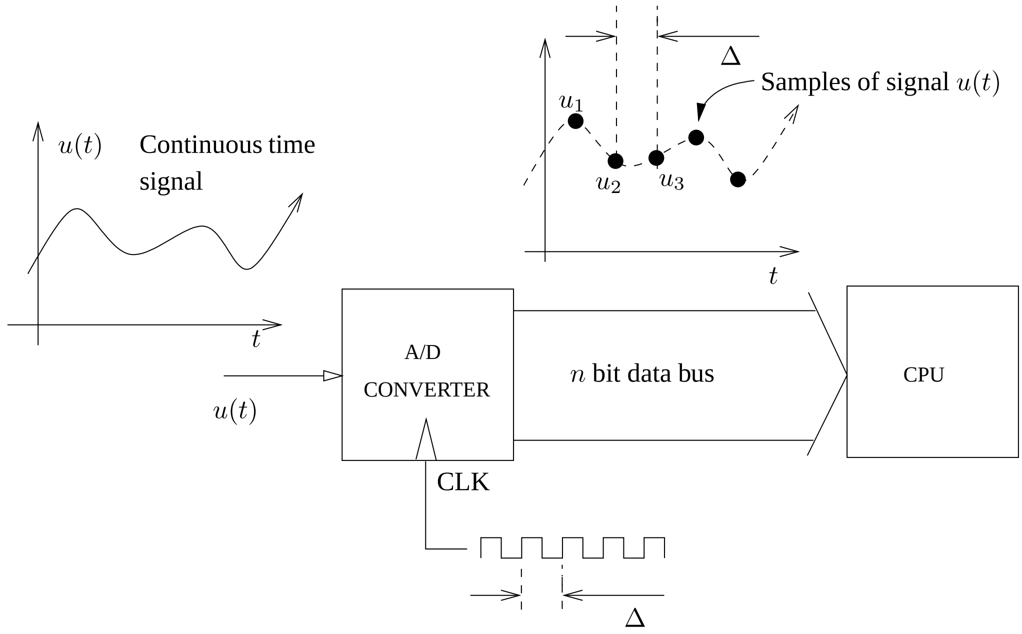

Figure 6.1: *Graphical representation of sampling process.*

The process of sampling a signal and converting it to digital form typically happens many times per second, and is determined by the time Δ between rising edges of the clock signal fed to the A/D converter.

The quantity Δ in seconds between samples is termed the 'sampling period', and the rate $f_s = 1/\Delta$ in Hertz (Hz) or (equivalently) $\omega_s = 2\pi f_s = 2\pi/\Delta$ in rad/s are known as the 'sampling frequency'.

Therefore, a continous time signal $u(t)$ gives rise to a sequence of samples $\cdots, u_{-2}, u_{-1}, u_0, u_1, u_2, \cdots$ as follows

$$u_k \triangleq u(k\Delta) = u(t)|_{t=k\Delta} \qquad ; k = \cdots, -2, -1, 0, 1, 2, \cdots.$$

This sequence of samples now provides information about $u(t)$ only at **discrete** (as opposed to continuously evolving) time instants. It is therefore commonly referred to as a 'discrete time signal'.

A sequence of values $\{u_k\} = \cdots, u_{-2}, u_{-1}, u_0, u_1, u_2, \cdots$ is known as a **discrete time signal**. It may arise via the sampling of a continuous time signal, or it may exist independently of any underlying continuous signal.

As an example of a discrete time signal that does not arise from an underlying continuous one, consider the value of a share in a publically listed company. Values are only defined at discrete instants when the share is traded, and there is no clear underlying continuously evolving value that is being sampled.

However, the vast majority of discrete signals (sample sequences) that are of engineering interest *do* arise from the sampling of an underlying continuous signal, and the remainder of this chapter is devoted to studying the relationship between these continuous signals and discrete samples.

6.2 Spectral relationship between a signal and its sampled version

Clearly, there is a key issue in this process of examining the relationship between continuous signals and discrete samples. Namel, how much 'information' exists in the samples $u_k = u(k\Delta)$ relative to the information in the original continuous signal $u(t)$. This chapter addresses this issue by measuring signal 'information' in terms of the signal spectrum.

However, a potential difficulty with this approach is that the samples $\{u_k\}$, being a discrete set of values, are of a fundamentally different nature to the continously evolving values represented by the underlying signal $u(t)$. This problem can be solved by associating the samples $\{u_k\}$ with an *imaginary* signal $u_p(t)$, which is a sequence of impulses which are weighted by the sample values.

The *perfectly sampled* version $u_p(t)$ of a signal $u(t)$ is the continous-time signal

$$u_p(t) = \cdots + u_{-1}\delta(t+\Delta) + u_0\delta(t) + u_1\delta(t-\Delta) + u_2\delta(t-2\Delta) + \cdots = \sum_{k=-\infty}^{\infty} u_k\delta(t-k\Delta). \quad (6.1)$$

This signal $u_p(t)$ is also known as the 'idealy sampled' version of $u(t)$, and a pictorial representation of its formation is shown if figure 6.2.

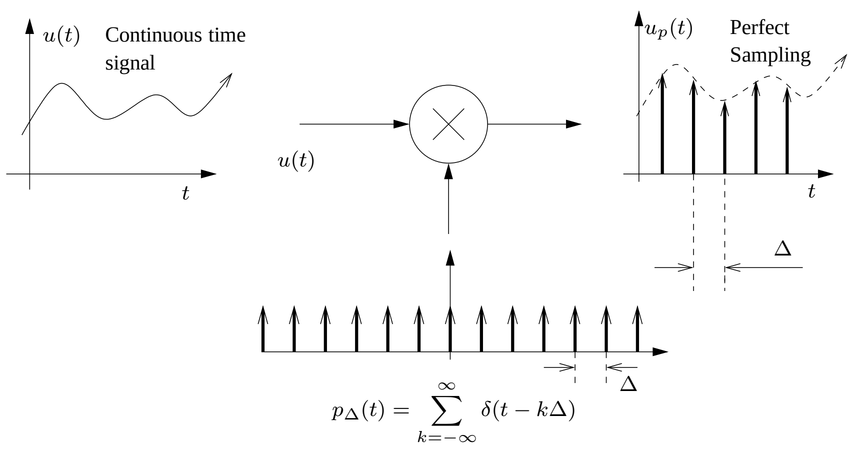

Figure 6.2: *Graphical representation of perfect sampling.*

This diagram introduces a new signal $p_\Delta(t)$, which is a sequence of period Δ regularly spaced Dirac delta and signals, and which is defined as

$$p_\Delta(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t-k\Delta). \quad (6.2)$$

This allows $u_p(t)$ to be derived from $u(t)$ via the multiplication

$$u_p(t) = u(t) \cdot p_\Delta(t). \quad (6.3)$$

The point of the imaginary signal $u_p(t)$ and its generation via multiplication with the 'sampling signal' $p_\Delta(t)$ is that it provides a means for relating the information in $u(t)$ relative to that in the samples u_k.

Specifically, by the property that multiplication in the time domain translates to convolution in the frequency domain (see table 5.2), the Fourier transforms $U_p(\omega) = \mathcal{F}\{u_p(t)\}$ and $U(\omega) = \mathcal{F}\{u(t)\}$ are related as

$$U_p(\omega) = \frac{1}{2\pi}\left[U \circledast P_\Delta\right](\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_\Delta(\sigma)U(\omega-\sigma)\,\mathrm{d}\sigma \tag{6.4}$$

where $P_\Delta(\omega) = \mathcal{F}\{p_\Delta(t)\}$. Now, $p_\Delta(t)$ is a periodic repetition of the signal $\delta(t)$ and with period $T = \Delta$. Therefore, by the discussion of section 5.3.1, because $p_\Delta(t)$ is periodic, $P_\Delta(\omega) = \mathcal{F}\{p_\Delta(t)\}$ must consist of an infinite sum of Dirac δ impulses separated by the fundamental rate $\omega_s = 2\pi/\Delta$ at which $p_\Delta(t)$ repeats.

Furthermore, again by the results of section 5.3.1, the weight associated with each Dirac delta impulse in $P_\Delta(\omega)$ centred at $\omega = k\omega_s$ is equal to 2π times that the k'th Fourier co-efficient c_k of the repeated signal component $\delta(t)$. Therefore, since according to (5.44) this c_k'th co-efficient is

$$c_k = \frac{1}{T}\int_0^T \delta(t)e^{-jk\omega_s t}\,\mathrm{d}t = \frac{1}{T} = \frac{1}{\Delta} \qquad ;\text{for all } k$$

then the weight on each Dirac delta in $P_\Delta(\omega)$ is $2\pi c_k = 2\pi/\Delta$ and hence

$$P_\Delta(\omega) = \frac{2\pi}{\Delta}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s) \qquad ;\omega_s = \frac{2\pi}{\Delta}. \tag{6.5}$$

Consequently, substituting (6.5) into (6.4) and using the fundamental definition (2.4) of the Dirac δ leads to a simple relationship between the spectrum $U_p(\omega)$ of the imaginary ideally sampled signal $u_p(t)$ given by (6.1) and the spectrum $U(\omega)$ of the underlying continuous time signal $u(t)$.

$$\begin{aligned}
U_p(\omega) &= \frac{1}{\Delta}\int_{-\infty}^{\infty} U(\omega-\sigma)\left[\sum_{k=-\infty}^{\infty}\delta(\sigma - k\omega_s)\right]\mathrm{d}\sigma \\
&= \frac{1}{\Delta}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} U(\omega-\sigma)\delta(\sigma-k\omega_s)\,\mathrm{d}\sigma \\
&= \frac{1}{\Delta}\sum_{k=-\infty}^{\infty} U(\omega - k\omega_s).
\end{aligned}$$

This result is known as the Poisson summation formula.

The spectrum $U(\omega) = \mathcal{F}\{u(t)\}$ of a signal $u(t)$ is related to the spectrum $U_p(\omega) = \mathcal{F}\{u_p(t)\}$ of its perfectly sampled version $u_p(t) = u(t)\cdot p_\Delta(t)$ according to the fact that $U_p(\omega)$ is a *repeated* and *scaled* version of $U(\omega)$ given as

$$U_p(\omega) = \frac{1}{\Delta}\sum_{k=-\infty}^{\infty} U(\omega - k\omega_s) \qquad ;\omega_s \triangleq \frac{2\pi}{\Delta}. \tag{6.6}$$

Clearly, the spacing between the repetitions is $\omega = \omega_s = 2\pi/\Delta$ rad/s, and the magnitude scaling factor is $1/\Delta$.

This relationship is illustrated in figure 6.3 where a hypothetical spectrum $|U(\omega)| = |\mathcal{F}\{u(t)\}|$ is shown in the upper diagram, followed by the spectrum $|U_p(\omega)| = |\mathcal{F}\{u_p(t)\}|$ of its perfectly

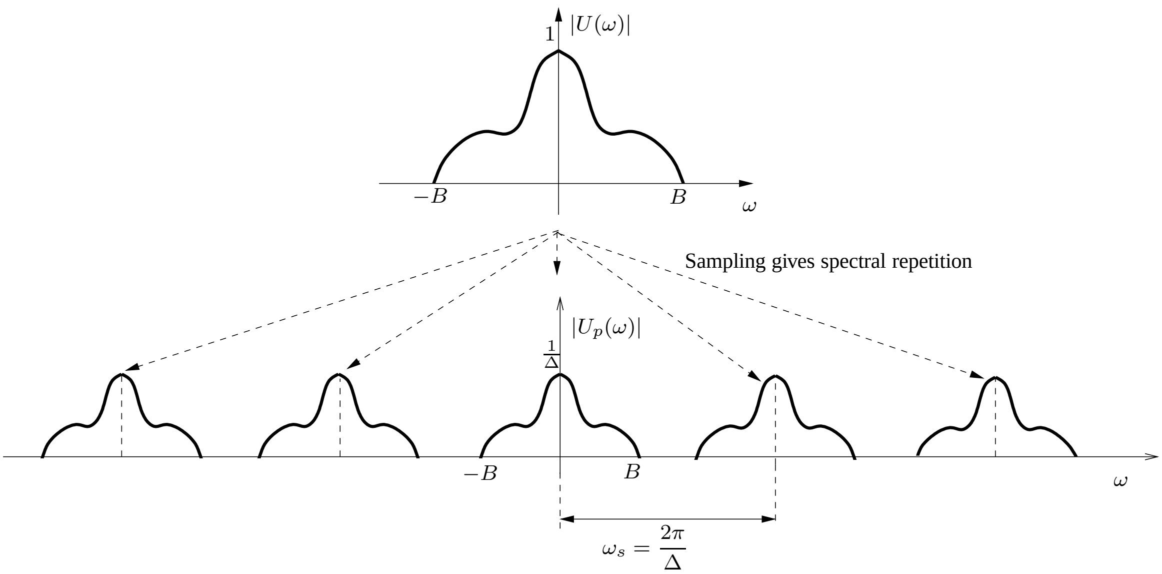

Figure 6.3: *Relationship between the spectrum of a signal $u(t)$, and the spectrum of an ideally sampled version of it $u_p(t)$.*

sampled version in the lower diagram. Note that $|U_p(\omega)|$ is simply $|U(\omega)|$ scaled in magnitude by a factor of $1/\Delta$ and then repeatedly copied at shifted frequencies which are integer multiples of the sampling frequency $\omega_s = 2\pi/\Delta$.

There are several ways in which the relationship (6.6) and hence figure 6.3 can be interpreted. Firstly, and quite formally, the relationship (6.6) simply states that the perfectly sampled signal $u_p(t)$ given by (6.3) contains spectral components that are not decaying in size even as the frequency of those components extends to infinity. This is reasonable. Since $u_p(t)$ consists of infinitely sharp 'spikes', it should be expected to contain spectral components at infinitely high frequencies.

However, there is another interpretation that arises by remembering that the signal $u_p(t)$ does not actually exist, it is simply an imagined signal used to represent the sequence of samples $u_k = u(k\Delta)$. Namely, the repeated nature of $U_p(\omega)$ represents that fact that, purely on the basis of the samples $\{u_k\}$, it is impossible to be unambiguous about the frequency of sine-wave components that may have generated them.

For example, figure 6.4 shows a situation in which it is impossible to say which of two sinusoidal signals was responsible for the observed samples shown as black dots. In this situation, when representing the information carried by the samples $\{u_k\}$ via a spectrum, then that spectrum should allow for both possibile sine wave components shown in figure 6.4. In fact, since $e^{j2\pi mk} = 1$ for all integer m, k then

$$\begin{aligned}
u_k &= \cos \omega k\Delta \\
&= \text{Real}\left\{e^{j\omega k\Delta}\right\} \\
&= \text{Real}\left\{e^{j\omega k\Delta} \cdot e^{j2\pi mk}\right\} \\
&= \text{Real}\left\{e^{j(\omega + m2\pi/\Delta)k\Delta}\right\} \\
&= \cos\left(\omega + m\frac{2\pi}{\Delta}\right) k\Delta.
\end{aligned}$$

Therefore if it is believed that $\{u_k\}$ may be samples of a signal with a component at ω rad/s, then it is also a possibility that the samples could also have components at $\omega + m\omega_s$ rad/s for any integer m. There are therefore an infinite number of sine waves that could pass through the sampling points shown in figure 6.4, all of them harmonically related by the sampling frequency $\omega_s = 2\pi/\Delta$.

This further explains the repeated nature of the spectrum $U_p(\omega)$ representing the sampled signal. The repetitions at spacings of $\omega_s = 2\pi/\Delta$ simply reflects a process of accounting for all possibilities for the underlying sine-wave components.

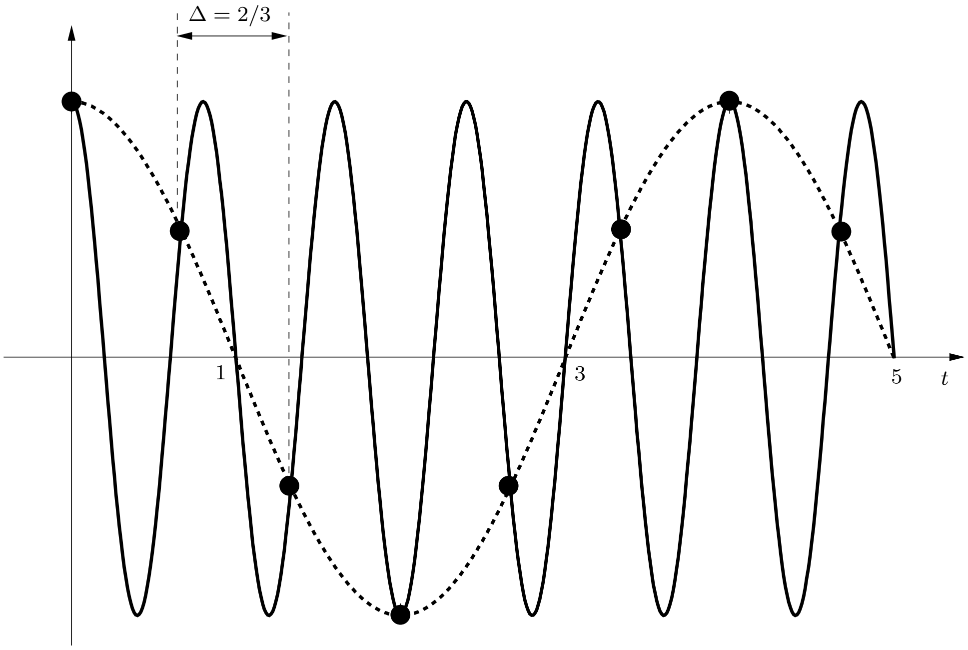

Figure 6.4: *Example illustrating the ambiguity of representing a signal via samples of it. Here, it is impossible to tell from the samples which of the two cosine waves was actually responsible for the samples represented by black dots.*

6.3 The Nyquist Sampling Criterion

Associated with the idea of sampling is that of reconstruction. That is, given samples $u_k = u(k\Delta)$ of a signal, how might a continous time version $u(t)$ be reconstructed from these samples? Of course, this reconstruction will involve computing the values of $u(t)$ between the sample points $u(k\Delta)$, which is known as ‘interpolation’. While this appears to be a time domain question, it arises that the issue of how this interpolation may be achieved is most simply addressed in the frequency domain.

Specifically, from a frequency domain perspective, the problem of reconstructing $u(t)$ from $u_p(t)$ becomes one of reconstructing $U(\omega)$ from $U_p(\omega)$. However, recalling the relationship (6.6) that $U_p(\omega)$ is nothing more than $U(\omega)$ repeated as illustrated in figure 6.3, it is clear that $U(\omega)$ may be recovered from $U_p(\omega)$ by merely selecting it from $U_p(\omega)$ via the low-pass filtering operation shown in figure 6.5. That is, $U(\omega)$ is selected from the repeated versions of it in $U_p(\omega)$ via the filtering operation

$$U(\omega) = H(\omega) \cdot U_p(\omega) \tag{6.7}$$

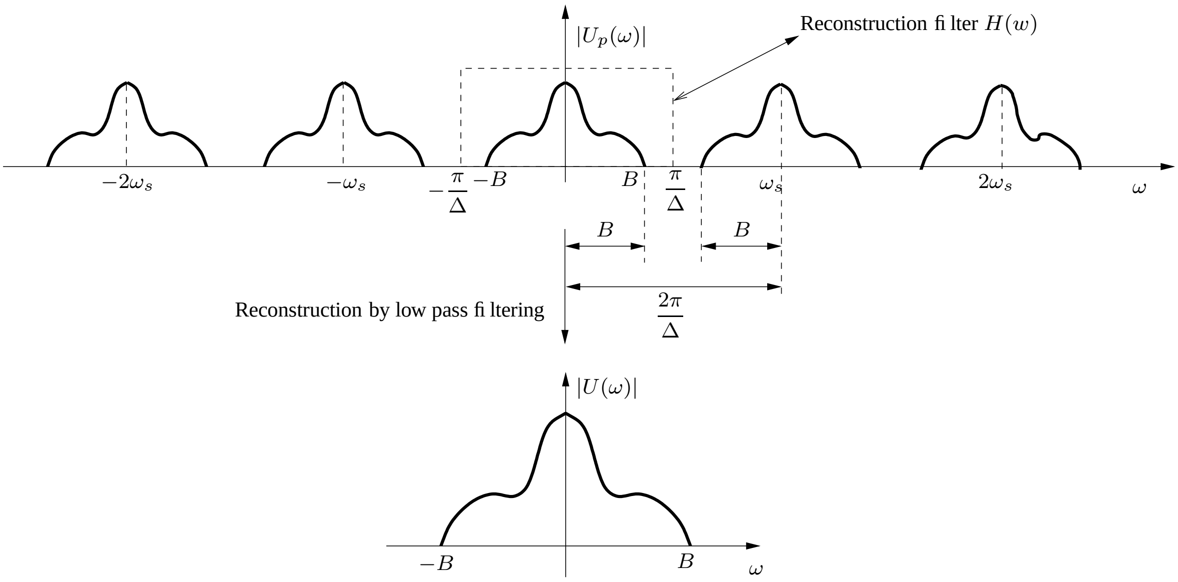

Figure 6.5: *Reconstruction of Sampled signal by low pass filtering.*

where the filter $H(\omega)$ is the perfect low pass filter

$$H(\omega) = \begin{cases} \Delta & ; |\omega| \leq \pi/\Delta, \\ \\ 0 & ; |\omega| > \pi/\Delta. \end{cases} \tag{6.8}$$

Notice that the pass-band gain of this filter is equal to the sampling period Δ since, via (6.6), the sampling process leads to $U_p(\omega)$ containing $U(\omega)$ only after scaling by a factor $1/\Delta$, and hence the filter gain of Δ in $H(\omega)$ is chosen to cancel this scaling.

However, also observe that this scheme for reconstruction of $U_p(\omega)$ from $U(\omega)$ is only possible if the sampling period Δ has been chosen small enough that when $U(\omega)$ is repeated every $\omega_s = 2\pi/\Delta$ rad/s (see figure 6.5) there is no 'overlap' of $U(\omega)$ with $U(\omega - \omega_s)$. More specifically, consider the following definition of 'bandwidth'.

For a given signal $u(t)$ with Fourier transform $U(\omega)$, suppose that a frequency B rad/s exists such that

$$U(\omega) = 0 \quad ; |\omega| \geq B.$$

That is, suppose that $U(\omega)$ is zero for frequencies $|\omega|$ beyond B rad/s. Then the signal $u(t)$ and the spectrum $U(\omega)$ are both defined to have bandwidth of B rad/s.

With this definition in mind, then the strategy (6.7) results in reconstruction of $U(\omega)$ if, and only if, (again, see figure 6.5)

$$\frac{2\pi}{\Delta} \geq 2 \times B.$$

This is known as the Nyquist Sampling Criterion, and is one of the most important principles of digital signal processing.

In order that a signal $u(t)$ can be reconstructed from samples $u_k = u(k\Delta)$, the Nyquist Sampling Criterion must be satisfied, which states that it is necessary that the sampling period be chosen such that the sampling frequency $f_s = 1/\Delta$ is *at least twice* the bandwidth of the signal (in Hz)

$$\frac{1}{\Delta} = f_s > 2\, B|_{\text{Hz}}\,. \tag{6.9}$$

Alternatively, multiplying both sides of the above by 2π, it may be expressed in rad/s as

$$\frac{2\pi}{\Delta} = \omega_s > 2\, B|_{\text{rad/s}}\,. \tag{6.10}$$

6.4 Reconstruction

With the Nyquist sampling criterion and its derivation in mind, to further understand the reconstruction principle (6.7), note that since multiplication of the Fourier transforms of signals corresponds to the convolution of those original signals in the time domain (see table 5.2), then the reconstructed signal $u(t) = \mathcal{F}^{-1}\left\{H(\omega)\cdot U_p(\omega)\right\}$ is given as

$$u(t) = \mathcal{F}^{-1}\left\{H(\omega)\cdot U_p(\omega)\right\} = \int_{-\infty}^{\infty} h(\sigma)u_p(t-\sigma)\,\mathrm{d}\sigma \tag{6.11}$$

where $h(t) = \mathcal{F}^{-1}\left\{H(\omega)\right\}$ which is

$$h(t) = \frac{\Delta}{2\pi}\int_{-\pi/\Delta}^{\pi/\Delta} e^{j\omega t}\,\mathrm{d}\omega = \Delta\frac{\sin\pi f_s t}{\pi t} \qquad ; f_s = \frac{1}{\Delta}. \tag{6.12}$$

Therefore, since

$$u_p(t) = \sum_{k=-\infty}^{\infty} u(t)\delta(t-k\Delta) \tag{6.13}$$

then substituting (6.12) and (6.13) into (6.11) and swapping the order of summation and integration provides

$$\begin{aligned}
u(t) &= \Delta\int_{-\infty}^{\infty}\frac{\sin\pi f_s\sigma}{\pi\sigma}\sum_{k=-\infty}^{\infty} u(t-\sigma)\delta(t-\sigma-k\Delta)\,\mathrm{d}\sigma\\
&= \Delta\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\sin\pi f_s\sigma}{\pi\sigma}u(t-\sigma)\delta(t-\sigma-k\Delta)\,\mathrm{d}\sigma\\
&= \Delta\sum_{k=-\infty}^{\infty} u(k\Delta)\frac{\sin\pi f_s(t-k\Delta)}{\pi(t-k\Delta)}
\end{aligned}$$

where the fundamental Dirac δ property (2.4) has been used in progressing to the last line. This formula for the reconstructed $u(t)$ is known as “Shannon’s Reconstruction Theorem”

Given samples $u_k = u(k\Delta)$ of a signal $u(t)$ which is band-limited to B rad/s, then if $2\pi/\Delta = \omega_s > 2B$, the signal $u(t)$ can be re-constructed from the samples $u(k\Delta)$ via the following interpolation formula:

$$u(t) = \sum_{k=-\infty}^{\infty} u(k\Delta)\frac{\sin \pi f_s(t-k\Delta)}{\pi f_s(t-k\Delta)} \qquad ; f_s = \frac{1}{\Delta}. \tag{6.14}$$

An intuitive interpretation of this result is that if the rate at which a signal $u(t)$ can vary between the samples is limited (in that the signal bandwidth is limited via $U(\omega) = 0$ for $|\omega| > B$, then provided the samples are sufficiency closely spaced at distance $\Delta < 2\pi/B$, it is theoretically possible to reconstruct the signal between the samples by using an interpolating function $\sin \pi f_s t/\pi f_s t$.

The detail of this interpolation process is illustrated in figure 6.6. In particular, notice that the interpolating function satisfies (m is an integer, and the sinc$(\cdot)$ function was defined in (5.12) on page 181)

$$\frac{\sin \pi f_s(t-k\Delta)}{\pi f_s(t-k\Delta)} = \text{sinc}\left(\frac{t}{\Delta} - k\right) = \begin{cases} 1 & ; t = k\Delta \\ 0 & ; t = m\Delta, m \neq k \end{cases}$$

so that a single term

$$u(k\Delta)\frac{\sin \pi f_s(t-k\Delta)}{\pi f_s(t-k\Delta)}$$

in the interpolant (6.14), is exactly equal to $u(k\Delta)$ at $t = k\Delta$, but is then zero at all the other sampling points $t = m\Delta, m \neq k$, and hence does not upset the total reconstruction at those sampling instants.

However, as is also shown by figure 6.6, in between the sampling instants the signal $u(t)$ is formed by interpolation which involves an infinite summation over the 'tails' of an infinite number of $\sin \pi f_s(t-k\Delta)/(\pi f_s(t-k\Delta) = \text{sinc}(t/\Delta - k)$ functions.

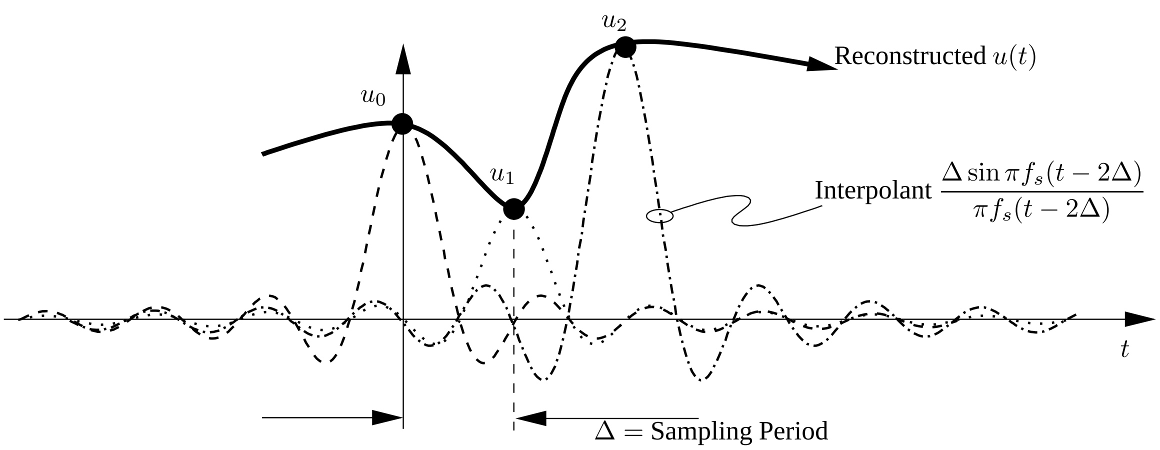

Figure 6.6: *Graphical representation of the interpolation performed by the Shannon Reconstruction Theorem. Only the terms for $k = 0, 1, 2$ in the reconstruction formula (6.14) are shown.*

In practice, this perfect reconstruction can never be attained for two reasons. Firstly, the Shannon reconstruction filter (6.8) is purely real valued, and hence its impulse response given by (6.12) is an even (symmetric) function, and therefore corresponds to a non-causal system. That is $h(t) \neq 0$ for $t < 0$ as illustrated by the explicit formula (6.12). As a result, it can never be physically realised, although approximations formed by Butterworth or Chebychev techniques (for example) can be used.

Secondly, the underling signal $u(t)$ can never have the perfectly band-limited spectrum shown in figure 6.5. This was examined in example 5.10 of the previous chapter, where it was exposed that any signal that is of finite duration in one domain, must be of infinite duration in the other domain. Consequently, if if $u(t)$ is to be perfectly bandlimited in the frequency domain, it must be of infinite duration in the time domain.

Clearly this is never the case in reality. That is, any signal $u(t)$ of interest will only ever exist over a finite duration, and hence it will contain non-zero spectral components $U(\omega) = \mathcal{F}\{u(t)\}$ for arbitrarily large ω. Nevertheless, although these components will be non-zero, they will also be extremely small for large enough ω, and hence for any practical purposes a finite bandwidth B can always be chosen such that the resultant distorting effects due to components beyond $\omega = B$ are neglibible.

With these practical limitations in mind, the actual (approximate) reconstruction of a signal $u(t)$ from samples of it is performed via a component known as a digital to analog (D/A) converter, which performs an opposite operation to that of an A/D converter. Namely, as illustrated in figure 6.7, a D/A

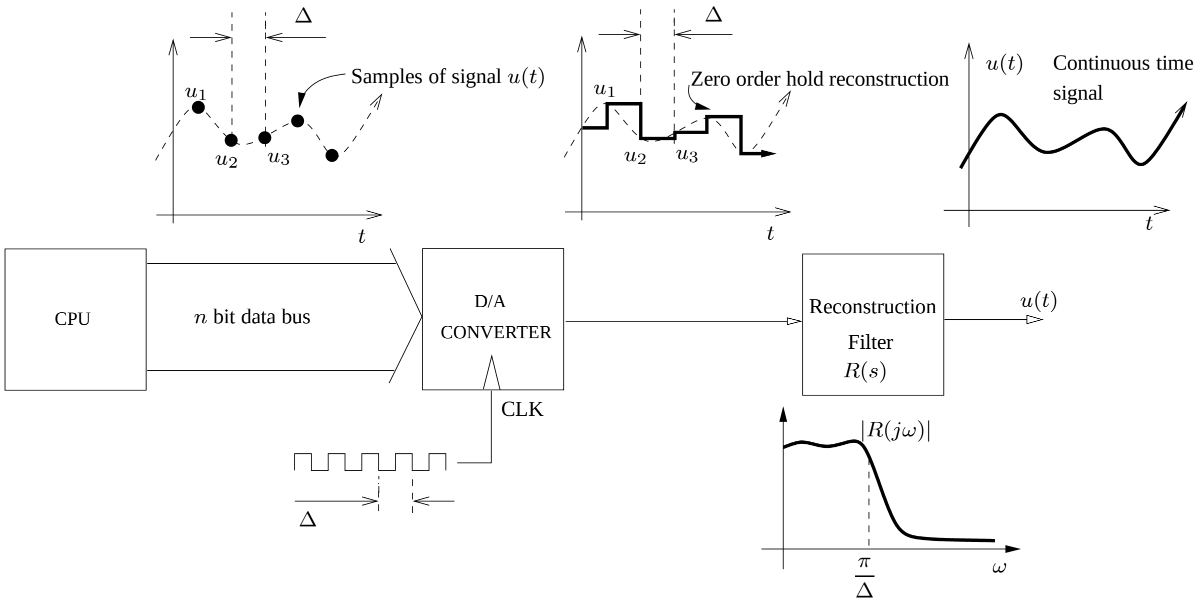

Figure 6.7: *Graphical representation of the reconstruction process.*

converter accepts n-bit wide binary representations of samples, and then at time points synchronised to the rising edge of a clock signal, it converts these binary values to a corresponding output voltage amplitude. This voltage is then held constant until the next rising clock edge when a new binary value is converted to a voltage. The process of keeping the voltage constant between samples is known as 'zero order hold' reconstruction.

The reconstruction filter then acts on this zero-order-hold signal. As already mentioned, this will be some approximation to the perfect low-pass Shannon reconstruction filter that is derived by a Butterworth, Chebychev, or other approximation procedure.

Furthermore, note that this filter does not act on the perfect sampling sequence $u_p(t)$ which is only an imaginary construction. Instead it acts on the real sample-and-held output of the D/A converter. Intuitively then, the reconstruction filter does nothing more than smooth the signal produced by the D/A converter, in order that the interpolation between samples is reasonable in the sense that the final reconstruction signal bandwidth is limited to B rad/s.

However, to understand the effect of the hold function on the reconstruction process, consider the diagrammatic representation of it shown in figure 6.8. There it is illustrated that the relationship

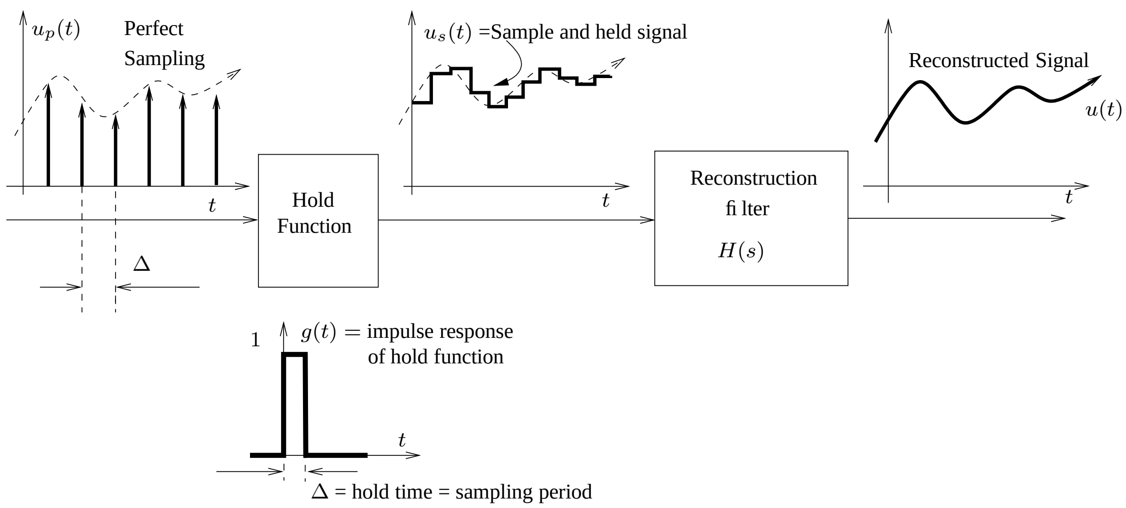

Figure 6.8: *Practical and approximate reconstruction scheme using zero-order hold function.*

between the (imaginary) perfectly sampled signal $u_p(t)$, and the real, practically achievable, sample-and-held one $u_s(t)$ is

$$u_s(t) = [u_p \circledast g](t) \tag{6.15}$$

where

$$g(t) = \begin{cases} 1 & ; t \in [0, \Delta], \\ \\ 0 & ; \text{Otherwise} \end{cases}$$

is the sample and hold function shape. Since the input to the hold function in figure 6.8 is a sequence of impulses, the convolution operation (6.15) outputs one impulse response $g(t)$ for each of these incoming impulses, which then produces the zero-order hold output $u_s(t)$ shown in figure 6.8 from the perfectly sampled version $u_p(t)$.

The relationship between $u_p(t)$ and $u_s(t)$ can then be studied in the frequency domain by taking Fourier transforms of both sides of (6.15) to provide

$$U_s(\omega) = \mathcal{F}\left\{[g \circledast u_p](t)\right\} = \mathcal{F}\left\{g(t)\right\} \cdot \mathcal{F}\left\{u_p(t)\right\} = G(\omega) \cdot U_p(\omega) \tag{6.16}$$

where

$$
\begin{aligned}
G(\omega) &= \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,\mathrm{d}t\\
&= \int_0^{\Delta} e^{-j\omega t}\,\mathrm{d}t\\
&= \left.\frac{e^{-j\omega t}}{-j\omega}\right|_{t=0}^{t=\Delta}\\
&= \frac{1-e^{-j\omega\Delta}}{-j\omega}\\
&= \frac{e^{-j\omega\Delta/2}[e^{j\omega\Delta/2}-e^{-j\omega\Delta}/2]}{j\omega}\\
&= 2e^{-j\omega\Delta/2}\cdot\frac{\sin\omega\Delta/2}{\omega} \qquad (6.17)
\end{aligned}
$$

so that

$$
|G(\omega)| = \Delta\left|\text{sinc}\left(\frac{\omega}{\omega_s}\right)\right| \qquad ;\omega_s = \frac{2\pi}{\Delta} \qquad (6.18)
$$

which is illustrated in figure 6.9. The spectrum $U_s(\omega)$, being the product of $U_p(\omega)$ and $H(\omega)$, will

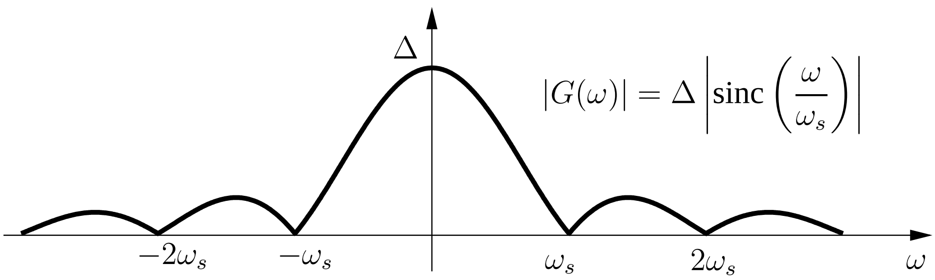

Figure 6.9: *Frequency response of Zero Order hold function*

then be a distorted version of $U_p(\omega)$ as shown in figure 6.10.

Note that because of the curved nature of the main (central) lobe of $|G(\omega)|$, the central part of $|U_s(\omega)|$ in the region $\omega \in [-\omega_s/2, \omega_s/2]$ is a distorted version of $|U(\omega)|$ formed as $U(\omega)\Delta\text{sinc}(\omega/\omega_s)$.

The degree of the distortion will depend on the ratio of the sampling frequency ω_s to the signal bandwidth B. If ω_s is very much larger than B, then as illustrated in figure 6.10, most of the signal spectrum $U(\omega)$ will be contained in a range of frequencies for which the distorting 'envelope' $\Delta\text{sinc}(\omega/\omega_s)$ is relatively constant, and hence there is little alteration between the spectra of the ideal samples, and that reconstructed using a zero-order hold circuit.

For this reason, in practical digital signal processing systems it is desirable (but not always possible) to use a sampling frequency ω_s that is significantly higher than the Nyquist rate lower limit of $\omega_s = 2B$.

6.5 Aliasing

Aliasing refers to the situation in which the Nyquist sampling criterion is not satisfied. In this case, the phenomenon of $U_p(\omega)$ being a repeated and shifted versions of $U(\omega)$ leads to these shifted $U(\omega)$

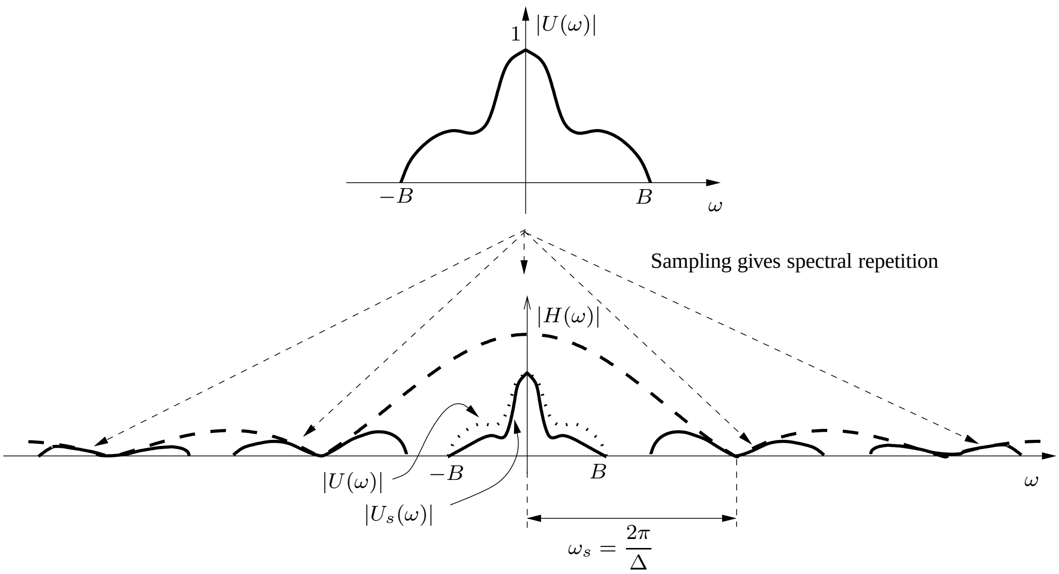

Figure 6.10: *Distortion of ideally sampled signal caused by passing through zero-order hold circuit.*

overlapping one another. In turn, and as illustrated in in figure 6.11, this implies distortion of these components in the formation of $U_p(\omega)$.

To be more specific and provide another example illustrating the point, referring back to figure 6.4, the higher frequency cosine wave shown there has fundamental frequency, and hence bandwidth $B = 5\pi/2$ rad/s. Therefore, the Nyquist sampling criterion requires a sampling frequency at twice this bandwith. That is, at $\omega_s = 10\pi/2$ rad/s. This further implies a required sampling period of $\Delta = 2\pi/\omega_s = 0.4s$. However, as shown in figure 6.4, the sampling period is only $\Delta = 2/3 > 0.4$, which therefore violates the Nyquist sampling criterion.

As a result, the lower frequency cosine wave of fundamental frequency (and hence bandwidth) $B = \pi/2$ rad/s passes through exactly the same sample points as the high frequency one, and thus acts as an *alias* (or imposter) for that higher frequency component.

This situation represented in the time domain in figure 6.4 has a frequency domain interpretation as shown in figure 6.12. Here, the top diagram shows the spectrum $U(\omega) = \pi\delta(\omega \pm \omega_\circ)$ of the underlying high frequency cosine signal $\cos\omega_\circ t$ with $\omega_\circ = 5\pi/2$. The lower diagram then shows the spectrum

$$U_p(\omega) = \sum_{k=-\infty}^{\infty} U(\omega - k\omega_s) \qquad ;\omega_s = \frac{2\pi}{\Delta} = \frac{2\pi}{2/3} = 3\pi$$

of the perfectly sampled signal. The spectral repetition involved in this process leads to the component $U(\omega - \omega_s)$ having overlap with that that of $U(\omega)$ in the formation of $U_p(\omega)$. In turn, this leads to a spectral component in $U_p(\omega)$ at

$$\omega_s - \omega_\circ = 3\pi - \frac{5\pi}{2} = \frac{\pi}{2}\,\text{rad/s.}$$

This indicates that, on the basis of the samples $\{u_k\}$, there appears to be a spectral component at $\pi/2$ rad/s. Certainly, according to figure 6.4, such as sinewave is consistent with those samples.

Notice that from figure 6.12, if a Shannon reconstruction filter $H(\omega)$ is applied, then the resulting

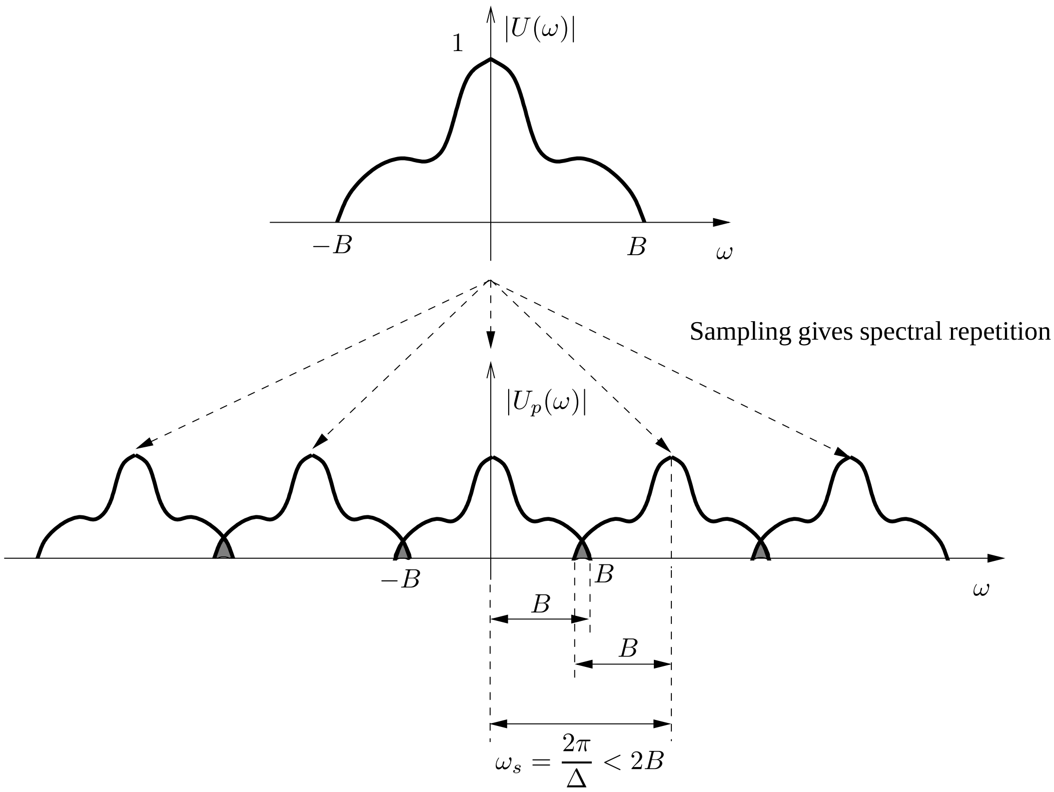

Figure 6.11: *Violation of the Nyquist Criterion causing spectral overlap, which is known as 'aliasing'.*

signal will be a cosine signal at frequency $\omega = \omega_s - \omega_\circ = \pi/2$ rad/s, and not at the original $\omega_\circ = 5\pi/2$ rad/s frequency.

In order to avoid this sort of aliasing problem, practical digital signal processing schemes include a low-pass filter that precedes the analog to digital converter as shown in figure 6.13. This filter is known as an 'anti-aliasing' filter and its cut-off frequency ω_c is set at half the sampling frequency $\omega_s/2 = \pi/\Delta$ (or less) in order to ensure that no spectral components that would violate the Nyquist bandwidth criterion of $B < \omega_s/2$ are allowed to progress to the analog to digital converter input, and hence be sampled.

As a final comment, the reader may have noticed that when watching older movies, particularly westerns, it is common to observe fast moving wagon wheels which have spokes that appear to be moving quite slowly compared to the expected wheel rotation. This situation is, in fact, exactly the aliasing one just presented, once it is recognised that the sampling process is imposed by the discrete frames that, projected at speed, make up the moving picture.

In this case, when the frequency $\omega_\circ$ of the wagon wheel rotation is high, then the sampling frequency ω_s imposed by the rate at which the movie frames are exposed may be such that the Nyquist sampling frequency is violated, and hence a lower frequency component $\omega_s - \omega_\circ$ is also consistent with the frames.

The actual reconstruction filter is provided by the human visual system, which interpolates between the frames to form the impression of a contiously moving image. To a first approximation, this system is a low pass operation which will simply select the $\omega_s - \omega_\circ$ component, hence the observation of a slowly moving wheel, rather than the 'real life' fast moving one.

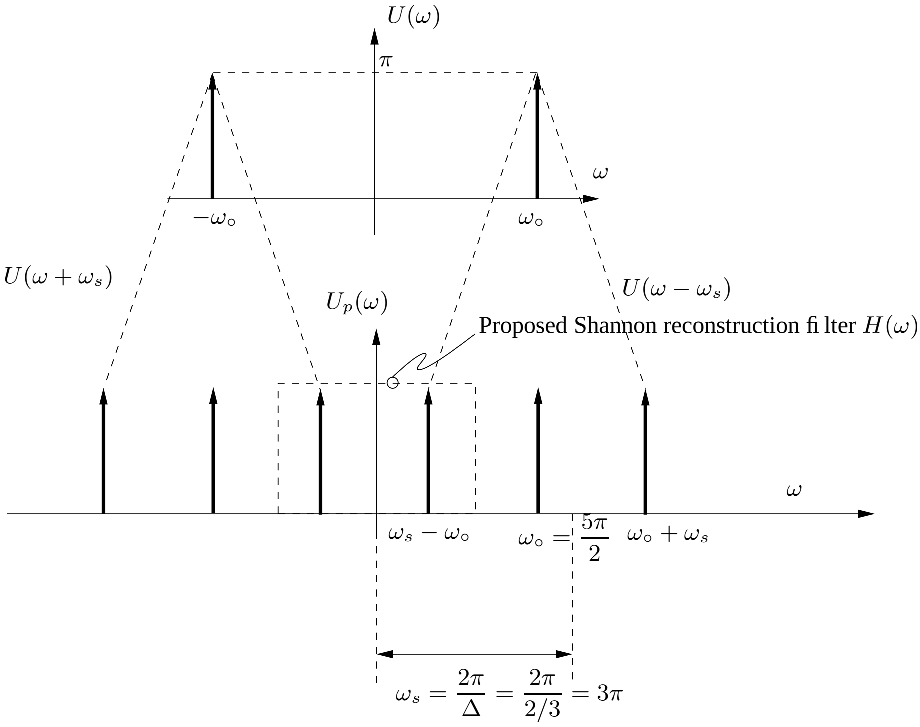

Figure 6.12: *Cosinusoidal case of figure 6.4 represented in the frequency domain.*

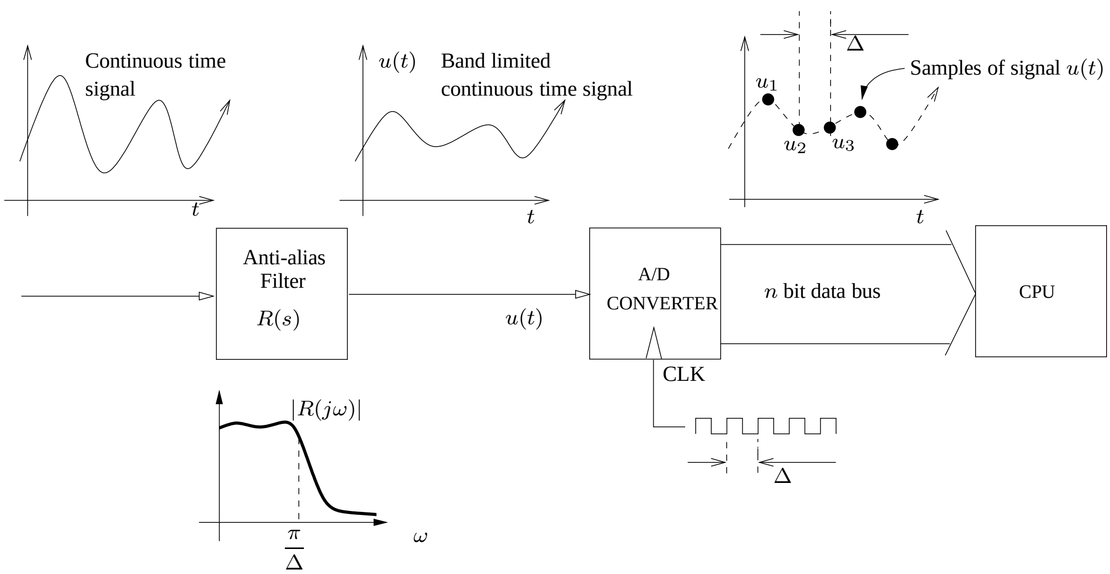

Figure 6.13: *Practical sampling scheme that includes an anti-aliasing filter*

6.6 The Discrete Time Fourier Transform

An essential message of this chapter is that the study of the relationship between a signal $u(t)$ and samples of it taken as $u_k = u(k\Delta)$ is greatly facilitated by the idea of the perfectly sampled signal

$$u_p(t) = u(t)p_\Delta(t) = \sum_{k=-\infty}^{\infty} u_k\delta(t - k\Delta). \tag{6.19}$$

The utility of this perfectly sampled signal $u_p(t)$ arises from the relationship between the spectrum $U_p(\omega) = \mathcal{F}\{u_p(t)\}$ and that of the underlying signal; viz. $U(\omega) = \mathcal{F}\{u(t)\}$. In turn, this relationship is given by the Poisson summation formula of equation (6.6) as

$$U_p(\omega) = \frac{1}{\Delta}\sum_{k=-\infty}^{\infty} U(\omega - k\omega_s) \qquad ; \omega_s = \frac{2\pi}{\Delta}. \tag{6.20}$$

Of course, the quantity $U_p(\omega)$ can also be formulated by direct substition of (6.19) into the definition (5.8) of the Fourier transform to provide

$$\begin{aligned} U_p(\omega) &= \int_{-\infty}^{\infty} u_p(t)e^{-j\omega t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} u(t)\left(\sum_{k=-\infty}^{\infty}\delta(t-k\Delta)\right)e^{-j\omega t}\,\mathrm{d}t \\ &= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} u(t)\delta(t-k\Delta)e^{-j\omega t}\,\mathrm{d}t \\ &= \sum_{k=-\infty}^{\infty} u(k\Delta)e^{-j\omega k\Delta} = \sum_{k=-\infty}^{\infty} u_k e^{-j\omega k\Delta}. \end{aligned} \tag{6.21}$$

This direct formulation of $U_p(\omega)$ in terms of the sample values $\{u_k\}$ is known as the 'Discrete Time Fourier Transform'.

The Discrete Time Fourier Transform (DTFT) $U(e^{j\omega\Delta})$ of the sequence of samples $\{u_k\}$ is defined as

$$U(e^{j\omega\Delta}) \triangleq \sum_{k=-\infty}^{\infty} u_k e^{-j\omega k\Delta}. \tag{6.22}$$

The term 'discrete' arises since the samples u_k are a discrete (as opposed to continuous) set of values. Note that in this definition, the notation $U_p(\omega)$ for the DTFT is replaced with $U(e^{j\omega\Delta})$ for two reasons.

Firstly, although the DTFT is motivated as the Fourier transform of the perfectly sampled signal $u_p(t)$, once the DTFT is derived, it applies for more general cases where the samples $\{u_k\}$ need not necessarily be formed as $u_k = u(k\Delta)$.

Secondly, the argument $e^{j\omega\Delta}$ is chosen so that the discrete time Fourier transform $U(e^{j\omega\Delta})$ can be discriminated from the Fourier transform $U(\omega)$.

Of course, in the case where $u_k = u(k\Delta)$ are samples of $u(t)$ so that $U(e^{j\omega\Delta}) = U_p(\omega) = \mathcal{F}\{u_p(t)\}$, then the relationship between $U(e^{j\omega\Delta})$ and the spectrum $U(\omega) = \mathcal{F}\{u(t)\}$ of the underlying continuous time signal is the same as that already derived in section 6.1. Namely, according to the Poisson summation formula (6.20)

$$U(e^{j\omega\Delta}) = U_p(\omega) = \frac{1}{\Delta}\sum_{k=-\infty}^{\infty} U(\omega - k\omega_s) \qquad ;\omega_s = \frac{2\pi}{\Delta}.$$

That is, as illustrated in figure 6.14, the DTFT $U(e^{j\omega\Delta})$ is simply $U(\omega)$, repeated at intervals of the sampling frequency $\omega_s = 2\pi/\Delta$ rad/s, and scaled by a factor of $1/\Delta$.

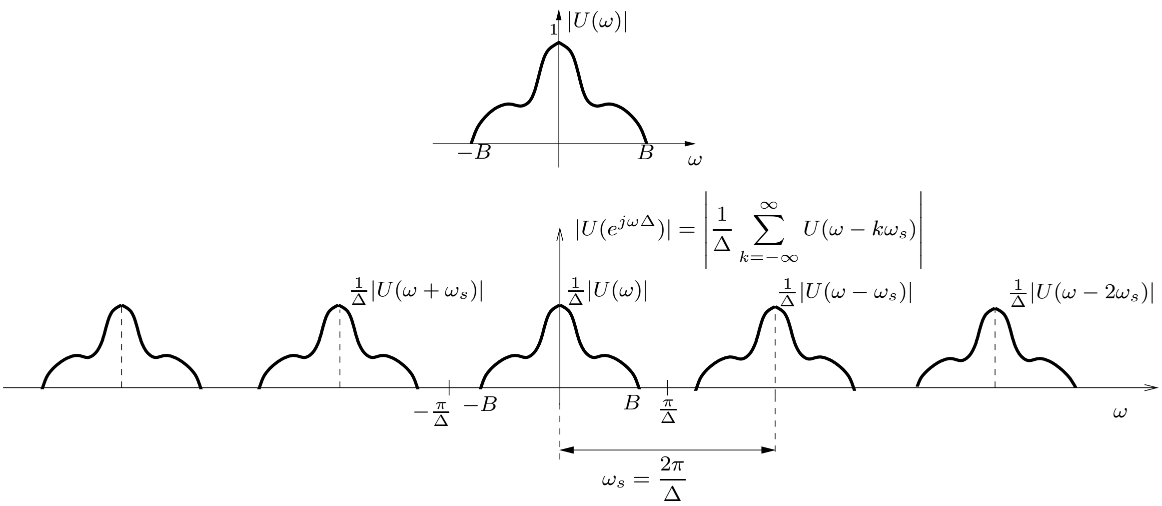

Figure 6.14: *Relationship between the spectrum of a signal $U(\omega) = \mathcal{F}\{u(t)\}$, and the Tiscrete Time Fourier Transform (DTFT) $U(e^{j\omega\Delta})$ of the sampled verions $u_k = u(k\Delta)$ of the signal $u(t)$.*

On the other hand, when the values u_k are not formed as samples of an underlying continuous time signal - for example they could be stock market prices which are only quoted at discrete instants - then DTFT $U(e^{j\omega\Delta})$ of those samples will still repeat at intervals of the sampling frequency $\omega_s = 2\pi/\Delta$.

To see this, note that for m being an arbitrary integer

$$U(e^{j(\omega+m\omega_s)\Delta}) = \sum_{k=-\infty}^{\infty} u_k e^{-jk(\omega+m\omega_s)\Delta} = \sum_{k=-\infty}^{\infty} u_k e^{-jk\omega\Delta}e^{-jkm\omega_s\Delta}. \tag{6.23}$$

However,

$$e^{-jkm\omega_s\Delta} = e^{-jkm(2\pi/\Delta)\Delta} = e^{-jkm2\pi} = 1 \tag{6.24}$$

where the equality with 1 follows since *both* of k and m are integers, and hence $km2\pi$ is an integer multiple of 2π. Substituting (6.24) into (6.23) then implies that regardless of how the samples u_k are obtained, the DTFT is a periodic function.

The Discrete Time Fourier transform (DTFT) is a periodic function that repeats at intervals of the sampling frequency $\omega_s = 2\pi/\Delta$. That is, for any integer m

$$U(e^{j(\omega+m\omega_s)\Delta}) = U(e^{j\omega\Delta}) \tag{6.25}$$

Example 6.1 Discrete Time Fourier Transform of an Exponential Consider the case of the samples $\{u_k\}$ being of the form

$$u_k = \begin{cases} \lambda^k & ;k \geq 0 \\ 0 & ;k < 0 \end{cases} \tag{6.26}$$

where $\lambda \in \mathbf{R}$ satisfies $|\lambda| < 1$. This signal is illustrated in the top diagram of figure 6.15. According

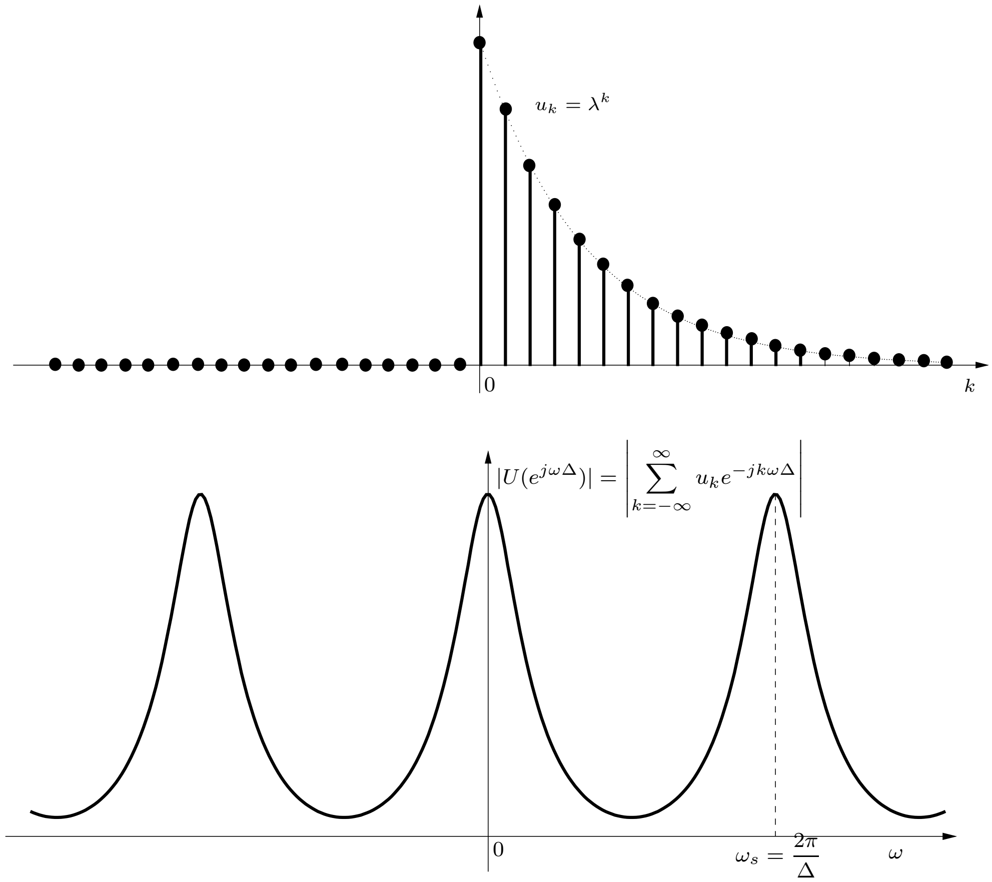

Figure 6.15: *The top diagram is an exponentially decaying sample sequence as considered in Example 6.1, the bottom diagram is the magnitude $|U(e^{j\omega\Delta})|$ of the DTFT of those samples.*

to the definition (6.22) the DTFT $U(e^{j\omega\Delta})$ of these samples is given as

$$\begin{aligned} U(e^{j\omega\Delta}) &= \sum_{k=-\infty}^{\infty} u_k e^{-jk\omega\Delta} \\ &= \sum_{k=0}^{\infty} \lambda^k e^{-jk\omega\Delta} \\ &= \sum_{k=0}^{\infty} \left(\lambda e^{-j\omega\Delta}\right)^k. \end{aligned}$$

This last quantity is a geometric series of the form

$$S_\infty = a + ar + ar^2 \cdots = \sum_{k=0}^{\infty} ar^k$$

with $a = 1$ and $r = \lambda e^{-j\omega\Delta}$. See Appendix 6.A.1 for more detail on this point, where it is established that provided $|r| < 1$ then

$$S_\infty = \frac{a}{1-r}$$

and hence

$$U(e^{j\omega\Delta}) = \frac{1}{1-\lambda e^{-j\omega\Delta}} = \frac{e^{j\omega\Delta}}{e^{j\omega\Delta}-\lambda}. \tag{6.27}$$

This DTFT is illustrated in the lower diagram of figure 6.15. Note in particular its nature of repeating every $\omega_s = 2\pi/\Delta$ rad/s.

■

Example 6.2 Poisson Summation Formula Note that in the previous example, the samples u_k could have been formed by sampling a continuous time signal as $u_k = u(k\Delta) = u(t)|_{t=k\Delta}$ where

$$u(t) = e^{(\Delta^{-1}\log\lambda)t} \tag{6.28}$$

so that

$$u_k = u(k\Delta) = e^{(\Delta^{-1}\log\lambda)k\Delta} = e^{k\log\lambda} = e^{\log\lambda^k} = \lambda^k.$$

Now, since the Fourier transform $U_p(\omega)$ of the perfectuly sampled version $u_p(t)$ of $u(t)$ is equal to the discrete time Fourier transform $U(e^{j\omega\Delta})$ of the samples, then by the previous example

$$U_p(\omega) = U(e^{j\omega\Delta}) = \frac{e^{j\omega\Delta}}{e^{j\omega\Delta}-\lambda}.$$

Furthermore, according to table 5.1 the spectrum $U(\omega) = \mathcal{F}\{u(t)\}$ defined in (6.28) is given as

$$U(\omega) = \frac{1}{j\omega - \Delta^{-1}\log\lambda}.$$

Therefore, the Possion summation formula (6.20) asserts that

$$\frac{e^{j\omega\Delta}}{e^{j\omega\Delta}-\lambda} = \frac{1}{\Delta}\sum_{k=-\infty}^{\infty}\frac{1}{j(\omega - 2\pi k/\Delta) - \Delta^{-1}\log\lambda} = \sum_{k=-\infty}^{\infty}\frac{1}{j(\omega\Delta - 2\pi k) - \log\lambda}.$$

The point is that, arithmetically (as well as conceptually), the results of the Poisson summation formula are non-trivial since it is by no means clear how the above summation equality could be established by any other means.

■

Example 6.3 Transform of a rectangular pulse Consider the case of samples

$$u_k = \begin{cases} 1 & ; |k| \le N, \\ 0 & ; |k| > N \end{cases} \tag{6.29}$$

which form a rectangular pulse shape. Then the DTFT of these samples is

$$U(e^{j\omega\Delta}) = \sum_{k=-\infty}^{\infty} u_k e^{-jk\omega\Delta} = \sum_{k=-N}^{N} e^{-jk\omega\Delta} = \sum_{k=0}^{2N} e^{-j(k-N)\omega\Delta} = e^{jN\omega\Delta}\sum_{k=0}^{2N} e^{-jk\omega\Delta}. \tag{6.30}$$

Now, (see Appendix 6.A.1), the final summation above is a geometric series with first term $a = 1$ and ratio $r = e^{-j\omega\Delta}$ and hence

$$\begin{aligned} U(e^{j\omega\Delta}) &= e^{jN\omega\Delta} \cdot \frac{a(1-r^{2N+1})}{1-r} \\ &= \frac{e^{jN\omega\Delta}(1-e^{-j(2N+1)\omega\Delta})}{1-e^{-j\omega\Delta}} \\ &= \frac{e^{jN\omega\Delta} - e^{-jN\omega\Delta}\cdot e^{-j\omega\Delta}}{1-e^{-j\omega\Delta}} \\ &= \frac{e^{-j\omega\Delta/2}\cdot\left(e^{j(N+1/2)\omega\Delta} - e^{-j(N+1/2)\omega\Delta}\right)}{e^{-j\omega\Delta/2}\left(e^{j\omega\Delta/2} - e^{-j\omega\Delta/2}\right)} \\ &= \frac{\sin(N+1/2)\omega\Delta}{\sin\omega\Delta/2}. \end{aligned} \tag{6.31}$$

■

The function (6.31) arises in mathematics via the study of the convergence of Fourier series, where it is known as the 'Dirichlet kernel $D_{N/2}$ of order $N/2$ evaluated at $\omega\Delta$'. That is (see also Appendix 5.A.1)

$$D_N(\omega) \triangleq \frac{\sin(2N+1)\omega}{\sin\omega/2} \tag{6.32}$$

is the Dirichlet kernel of order N, which is also known in engineering as the 'periodic cardinal sine (sinc)' function.

6.7 The Inverse Discrete Time Fourier Transform

Recall from section 5.11 of the previous chapter, that if a general function $f(t)$ was *periodic* so that $f(t) = f(t+T)$ for some period T, then it could be expressed via an exponential Fourier series as

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi t/T} \tag{6.33}$$

where the co-efficients $\{c_k\}$ were given by the formula

$$c_k = \frac{1}{T}\int_0^T f(t)e^{-jk2\pi t/T}\,\mathrm{d}t. \tag{6.34}$$

Now, note that according to the Poisson summation formula (6.20), which is illustrated in figure 6.3, then the Fourier transform $U_p(\omega) = \mathcal{F}\{u_p(t)\}$ of a perfectly sampled signal $u_p(t)$ is a periodic function which satisfies

$$U_p(\omega) = U_p(\omega+\omega_s) \qquad ;\omega_s = \frac{2\pi}{\Delta}.$$

Therefore, according to (6.33), with the change of variable $t \mapsto \omega$, since $U_p(\omega)$ is periodic in ω, then is has a Fourier series representation

$$U_p(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{jk2\pi\omega/T}. \tag{6.35}$$

Since the period T between repetitions in $U_p(\omega)$ is $T = \omega_s = 2\pi/\Delta$ then (6.35) may be re-written as

$$U_p(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega\Delta}. \tag{6.36}$$

However, from (6.22), for this same quantity $U_p(\omega)$, we already have the expression

$$U_p(\omega) = \sum_{k=-\infty}^{\infty} u_k e^{-j\omega k\Delta} = \sum_{k=-\infty}^{\infty} u_{-k} e^{j\omega k\Delta} \tag{6.37}$$

Therefore, comparing (6.36) and (6.37) indicates that $c_k = u_{-k}$ (which is the same as $u_k = c_{-k}$) in (6.36). Furthermore, with the substition $T = 2\pi/\Delta$, equation (6.34) provides a formula for c_k as

$$c_k = \frac{\Delta}{2\pi}\int_0^{2\pi/\Delta} U_p(\omega)e^{-jk\omega\Delta}\,\mathrm{d}\omega.$$

Therefore, remembering that as just argued $u_k = c_{-k}$, then the samples $\{u_k\}$ can be recovered from their Discrete Time Fourier Transform $U_p(\omega)$ as follows

The Inverse Discrete Time Fourier Transform (IDTFT) of

$$U(e^{j\omega\Delta}) = \sum_{k=-\infty}^{\infty} u_k e^{-jk\omega\Delta}$$

is given as

$$u_k = \frac{\Delta}{2\pi}\int_0^{2\pi/\Delta} U(e^{j\omega\Delta})e^{jk\omega\Delta}\,\mathrm{d}\omega. \tag{6.38}$$

Example 6.4 Inverse Transform of Preceding Example Consider the case of $U(e^{j\omega\Delta})$ being that derived in Example 6.1 of

$$U(e^{j\omega\Delta}) = \frac{e^{j\omega\Delta}}{e^{j\omega\Delta} - \lambda}. \tag{6.39}$$

In this case, the inverse discrete time Fourier transform of this quantity is given by (6.38) as

$$u_k = \frac{\Delta}{2\pi}\int_0^{2\pi/\Delta} \left(\frac{e^{j\omega\Delta}}{e^{j\omega\Delta} - \lambda}\right) e^{jk\omega\Delta}\,\mathrm{d}\omega. \tag{6.40}$$

In this situation, consider the variable substitution

$$z = e^{j\omega\Delta} \tag{6.41}$$

in which case

$$\mathrm{dz} = j\Delta e^{j\omega\Delta}\mathrm{d}\omega = j(\Delta z)\,\mathrm{d}\omega. \tag{6.42}$$

Then since $z = e^{j\omega\Delta}$ for $\omega \in [0, 2\pi/\Delta)$ transcribes a unit circle $\mathcal{C}$ in the complex plane, the substitutions (6.41) and (6.42) transform the inverse DTFT (6.40) into the contour integral

$$u_k = \frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{z}{z-\lambda} z^k \frac{\mathrm{d}z}{z} = \frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{z^k}{z-\lambda}\,\mathrm{d}z. \tag{6.43}$$

As established in section 4.5.1, such contour integrals depend only on the poles of the integrand via the associated residue. In particular, when $k \geq 0$, the integrand contains only one pole at $z = \lambda$ and hence (again, see section 4.5.1)

$$u_k = \operatorname*{Res}_{z=\lambda}\left(\frac{z^k}{z-\lambda}\right) = z^k\Big|_{z=\lambda} = \lambda^k.$$

On the other hand, when $k < 0$ then (6.43) becomes

$$u_k = \frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{1}{z^{|k|}(z-\lambda)}\,\mathrm{d}z. \tag{6.44}$$

Now consider the variable substitution

$$\mu = \frac{1}{z}$$

so that

$$\mathrm{d}\mu = -\frac{1}{z^2}\mathrm{d}z = -\mu^2\mathrm{d}z \tag{6.45}$$

in which case (6.44) can be expressed as

$$u_k = \frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{\mu^{|k|}\cdot\mu}{(1-\lambda\mu)}\,\frac{\mathrm{d}\mu}{\mu^2} = \frac{1}{2\pi j}\oint_{\mathcal{C}} \frac{\mu^{|k|}}{\mu(1-\lambda\mu)}\,\mathrm{d}\mu \tag{6.46}$$

where the negative sign in (6.45) has been cancelled by a further negative sign introduced by the reversed integration direction implied by the $z \mapsto 1/\mu$ substitution used in progressing to (6.46). Note that there is only one pole, and hence only one residue associated with this integrand. Therefore, for $k < 0$

$$u_k = \operatorname*{Res}_{\mu=0}\left(\frac{\mu^{|k|}}{\mu(1-\lambda\mu)}\right) = \frac{\mu^{|k|}}{(1-\lambda\mu)}\Bigg|_{\mu=0} = 0. \tag{6.47}$$

As a consequence, the complete inverse discrete time Fourier transform of (6.39) is

$$u_k = \begin{cases} \lambda^k & ; k \geq 0, \\ 0 & ; k < 0 \end{cases} \tag{6.48}$$

which is consistent with (6.26).

■

6.8 Normalised Frequency

With reference to figure 6.14, when the DTFT $U(e^{j\omega\Delta})$ is formed on the basis of samples $u_k = u(k\Delta)$ of an underlying continuous time signal $u(t)$, then $U(e^{j\omega\Delta})$ only contains unique information about the underlying spectrum $U(\omega) = \mathcal{F}\{u(t)\}$ within the frequency range

$$\omega \in \left[-\frac{\pi}{\Delta}, \frac{\pi}{\Delta}\right]. \tag{6.49}$$

More specifically, since in this case $U(e^{j\omega\Delta})$ is simply $U(\omega)$ repeated every $\omega_s = 2\pi/\Delta$ rad/s, then for $\omega > \pi/\Delta$ the values taken on by $U(e^{j\omega\Delta})$ are simply those it has already taken on in the range given by (6.49). For example, it is clear from figure 6.14 that (assuming $u(t)$ is real valued so that $|U(\omega)|$ is an even function)

$$\left|U(e^{j\omega\Delta})\right|_{\omega=\frac{\pi}{\Delta}+\frac{0.5\pi}{\Delta}} = \left|U(e^{j\omega\Delta})\right|_{-\frac{0.5\pi}{\Delta}}.$$

Therefore (6.49) is the useful range of frequencies that it is worth evaluating the DTFT over, which can also be expressed as

$$\omega\Delta \in [-\pi, \pi]. \tag{6.50}$$

Combined with the observation that in the preceding formulae for both the DTFT (6.23) and the IDTFT (6.40), the dependence on frequency ω occured via the presence of the product $\omega\Delta$, there exists a clear motivation to make the change of variable

$$\sigma = \omega\Delta \tag{6.51}$$

in (6.23) and (6.40). In this case, σ is known as *normalised* frequency and, perhaps confusingly, is usually given the symbol ω back again once the change of variable (6.51) has been made!

With respect to normalised frequency $\omega \mapsto \omega\Delta$ the DTFT $U(e^{j\omega})$ is given as

$$\mathrm{DTFT}\left\{u_k\right\} = U(e^{j\omega}) = \sum_{k=-\infty}^{\infty} u_k e^{-jk\omega} \tag{6.52}$$

and its inverse discrete time Fourier transform becomes

$$u_k = \mathrm{IDTFT}\left\{U(e^{j\omega})\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(e^{j\omega})e^{jk\omega}\,\mathrm{d}\omega. \tag{6.53}$$

Software packages for computing the discrete Fourier transform and its inverse typically implement this normalised frequency version. Translating results back to 'real' frequency is then very simply achieved by dividing any frequency variable (e.g. the x frequency axis on a Bode diagram) by the underyling sampling period Δ or, what is equivalent, multiplying by the sampling frequency $f_s = 1/\Delta$ in Hz.

That is, the value of $U(e^{j\omega})$ at $\omega = \omega_\star$ in normalised frequency, corresponds to the measurement of a spectral component at $\omega_\star/\Delta = \omega_\star \times f_s$ rad/s in real frequency.

6.9 Properties of the Discrete Time Fourier Transform

Recall again that the DTFTcan be viewed as a special case of the continuous Fourier transform in which the samples u_k imply a perfectly sampled signal $u_p(t)$ according to (6.19).

Because of this relationship (we are using *non*-normalised frequency here)

$$\mathrm{DTFT}\left\{u_k\right\} = U(e^{j\omega}) = U_p(\omega) = \mathcal{F}\left\{u_p(t)\right\} \tag{6.54}$$

the DTFT inherits many of the Fourier transform properties previously derived in section 5.2.2. The purpose of this section is to emphasise these features as well as to also present several further ones that stem directly from the fact that samples at discrete time points are used.

6.9.1 Properties inherited from the continous time Fourier Transform

These properties have already been established and discussed in detail in section 5.2.2 and Chapter 4 and hence they are simply listed here. In what follows, u_k, y_k, h_k, f_k and g_k are arbitrary samples, while $\alpha, \beta \in \mathbf{C}$ are arbitrary constant scalars.

Linearity

Suppose that $y_k = \alpha f_k + \beta g_k$ and that $F(e^{j\omega}) = \mathrm{DTFT}\{f_k\}$, $G(e^{j\omega}) = \mathrm{DTFT}\{g_k\}$. Then

$$Y(e^{j\omega}) = \mathrm{DTFT}\{\alpha u_k + \beta y_k\} = \alpha \mathrm{DTFT}\{u_k\} + \beta \mathrm{DTFT}\{y_k\} = \alpha F(e^{j\omega}) + \beta G(e^{j\omega}).$$

Time Shift

Suppose that $y_k = u_{k-T}$ and that $U(e^{j\omega}) = \mathrm{DTFT}\{u_k\}$. Then

$$Y(e^{j\omega}) = \mathrm{DTFT}\{u_{k-T}\} = e^{-j\omega T}\mathrm{DTFT}\{u_k\} = e^{-j\omega T}U(e^{j\omega}).$$

Modulation Property

Suppose that $y_k = e^{jk\omega_\circ} \cdot u_k$ and that $U(e^{j\omega}) = \mathrm{DTFT}\{u_k\}$. Then

$$Y(e^{j\omega}) = \mathrm{DTFT}\left\{e^{jk\omega_\circ} \cdot u_k\right\} = U(e^{j(\omega-\omega_\circ)}). \tag{6.55}$$

Symmetry and Realness

If u_k is real valued for all k, then

$$U(e^{j\omega}) = \overline{U(e^{-j\omega})}.$$

This further implies that

$$|U(e^{j\omega})|^2 = U(e^{j\omega})\overline{U(e^{j\omega})} = U(e^{j\omega})U(e^{-j\omega})$$

and by the same reasoning

$$|U(e^{-j\omega})|^2 = U(e^{-j\omega})\overline{U(e^{-\omega})} = U(e^{-j\omega})U(e^{j\omega}).$$

Therefore,

$$|U(e^{j\omega})| = |U(e^{-j\omega})|$$

so that $|U(e^{j\omega})|$ is an even function of ω if $\{u_k\}$ is real valued.

Discrete Time Convolution

When considering only samples $\{u_k\}$ of a signal, rather than an underlying signal $u(t)$ itself, it is necessary to reconsider what is meant by convolution. To address this, suppose that $h_p(t)$ and $u_p(t)$ are two perfectly sampled signals with sampling period $\Delta = 1$ second

$$u_p(t) = \sum_{k=-\infty}^{\infty} u_k\delta(t-k), \qquad h_p(t) = \sum_{n=-\infty}^{\infty} h_n\delta(t-n)$$

and that $y_p(t)$ is formed by their convolution as

$$\begin{aligned} y_p(t) &= \int_{-\infty}^{\infty} h_p(\sigma) u_p(t-\sigma)\,\mathrm{d}\sigma && (6.56) \\ &= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} h_n\delta(\sigma-n)\right)\left(\sum_{k=-\infty}^{\infty} u_k\delta(t-\sigma-k)\right)\mathrm{d}\sigma \\ &= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h_n u_k \int_{-\infty}^{\infty}\delta(\sigma-n)\delta(t-\sigma-k)\,\mathrm{d}\sigma. && (6.57) \end{aligned}$$

Applying the fundamental Dirac-delta definition (2.4) formally to the first Dirac-delta in (6.57), and without consideration of the fact that the second term in the integrand is also a Dirac-delta indicates that

$$\int_{-\infty}^{\infty}\delta(\sigma-n)\delta(t-\sigma-k)\,\mathrm{d}\sigma = \delta(t-n-k) \tag{6.58}$$

and hence $y_p(t)$ is also of the form of a perfectly sampled signal

$$y_p(t) = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h_n u_k\delta(t-n-k) = \sum_{n=-\infty}^{\infty} h_n u_{t-n}\delta(t-n). \tag{6.59}$$

Here, in progressing to the final expression, it has been recognised that the inner sum over k on the left hand side of (6.59), by virtue of the term $\delta(t-n-k)$, has only has one non-zero term at $k=t-n$. This implies that $y_p(t)$ given by the convolution (6.56) may be re-written as

$$y_p(t) = \sum_{k=-\infty}^{\infty} y_k\delta(t-k)$$

where

$$y_k = \sum_{n=-\infty}^{\infty} h_n u_{k-n}. \tag{6.60}$$

That is, the convolution of two perfectly sampled signals suggests a further definition

$$[h \circledast u]_k = \sum_{n=-\infty}^{\infty} h_n u_{k-n}$$

for the convolution of sample sequences $\{h_k\}$ and $\{u_k\}$ that specify those perfectly sampled signals.

Given two sequences $\{h_k\}$, and $\{u_k\}$ of samples, their **discrete time convolution** $h \circledast u$ to produce a third sequence $\{y_k\}$ of samples is defined as

$$y_k = [h \circledast u]_k = \sum_{n=-\infty}^{\infty} h_n u_{k-n}. \tag{6.61}$$

Furthermore, by the properties already established in section 5.2.2, since $y_p(t) = [h_p \circledast u_p](t)$ then

$$Y_p(\omega) = H_p(\omega) \cdot U_p(\omega).$$

Therefore, by recognising that the Fourier transforms of perfectly sampled signals are in fact the DTFT's of the underlying sample sequences we have established that

$$\mathrm{DTFT}\left\{[h \circledast u]_k\right\} = \mathrm{DTFT}\left\{h_k\right\} \cdot \mathrm{DTFT}\left\{u_k\right\} = H(e^{j\omega}) \cdot U(e^{j\omega}). \tag{6.62}$$

The sample sequence $\{y_k\}$ which is formed as the convolution $y_k = [h \circledast u]_k$ of two other sequences $\{h_k\}$ and $\{u_k\}$ has DTFT which is the product of the DTFT's of $\{h_k\}$ and $\{u_k\}$. That is

$$y_k = [h \circledast u]_k \quad \Longrightarrow \quad Y(e^{j\omega}) = H(e^{j\omega}) \cdot U(e^{j\omega}).$$

6.9.2 Further Properties

Because of the special Dirac-delta nature of the perfectly sampled signal, not all of the important properties of the DTFT can be derived by associating it with properties of the spectrum $U_p(\omega) = \mathcal{F}\left\{u_p(t)\right\}$. Instead, it is necessary to reason in terms of the DTFT and IDTFT definitions (6.52) and (6.53) directly.

Energy Equivalence

When considering a perfectly sampled signal such as defined by (6.19), the signal energy definition introduced in section 5.3.2 of

$$\text{Energy} = \int_{-\infty}^{\infty} |u_p(t)|^2 \, \mathrm{d}t$$

leads to an infinite, and hence ill-defined value because of the Dirac-delta components of $u_p(t)$.

In consideration of this, an alternative energy concept appropriate to samples u_k of a signal is the quantity

$$\text{Energy} = \sum_{k=-\infty}^{\infty} |u_k|^2.$$

To see how this quantity is reflected in the value of the DTFT $U(e^{j\omega})$, consider the slightly more general case of two sets of samples $\{u_k\}$ and $\{y_k\}$ for which use of the inverse DTFT definition (6.53) yields

$$\begin{aligned}
\sum_{k=-\infty}^{\infty} u_k \overline{y_k} &= \sum_{k=-\infty}^{\infty} u_k \left(\overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega}) e^{jk\omega} \, \mathrm{d}\omega}\right) \\
&= \sum_{k=-\infty}^{\infty} u_k \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{Y(e^{j\omega})} e^{-jk\omega} \, \mathrm{d}\omega\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} u_k e^{-jk\omega}\right) \overline{Y(e^{j\omega})} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} U(e^{j\omega}) \overline{Y(e^{j\omega})} \, \mathrm{d}\omega.
\end{aligned}$$

The generalised Parseval's (or Plancherel's) Theorem for samples u_k and y_k is

$$\sum_{k=-\infty}^{\infty} u_k \overline{y_k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(e^{j\omega})\overline{Y(e^{j\omega})}\,\mathrm{d}\omega \tag{6.63}$$

for which an obvious special case is $u_k = y_k$ which yields

$$\sum_{k=-\infty}^{\infty} |u_k|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |U(e^{j\omega})|^2\,\mathrm{d}\omega. \tag{6.64}$$

Transform of a product

Consider the case of two samples f_k and g_k forming a third one h_k via the product

$$h_k = f_k \cdot \overline{g_k}.$$

Then the DTFT $H(e^{j\sigma})$ is by definition

$$H(e^{j\sigma}) = \sum_{k=-\infty}^{\infty} f_k \overline{g_k} e^{-jk\sigma}.$$

However, the right hand side of this equation is nothing more than the left hand side of the generalised Parseval relationship (6.63) with the substitutions $u_k = f_k$ and $y_k = g_k e^{jk\sigma}$. Furthermore, according to the modulation property (6.55)

$$\mathrm{DTFT}\left\{g_k e^{jk\sigma}\right\} = G(e^{j(\omega-\sigma)}). \tag{6.65}$$

Substituting these quantities into (6.63) then implies that

$$H(e^{j\sigma}) = \mathrm{DTFT}\left\{f_k \cdot \overline{g_k}\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\overline{G(e^{j(\omega-\sigma)})}\,\mathrm{d}\omega. \tag{6.66}$$

Clearly, the right hand side of the above equation has the general form of a convolution, as originally defined in (3.168), but with the important obvious difference that the integration limits are finite over $\sigma \in [-\pi, \pi]$ as opposed to the orginal case (3.168) where the integration limits $\sigma \in [-\infty, \infty]$ are infinite.

A more subtle but important additional point is that the arguments in to all the functions in (6.66) are of the form $e^{j\sigma}$, which is a complex number on the unit circle, for any value of σ. As such, the form of convolution in (6.66) is termed **circular convolution**.

Consider two function $F(e^{j\omega})$, $G(e^{j\omega})$ which whose arguments lie on the unit circle. Then the circular convolution of these two functions is defined as

$$[F \circledast G](\omega) \triangleq \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\omega})\overline{G(e^{j(\omega-\sigma)})}\,\mathrm{d}\omega. \tag{6.67}$$

With these definitions in mind, then returning to (6.66), in the special case where g_k is real valued, then $\overline{g_k} = g_k$ and hence $\overline{G(e^{j(\omega-\sigma)})} = G(e^{j(\sigma-\omega)})$ so that swapping the variables σ and ω leads to the following principle.

Suppose the g_k is a real valued sample for all k. Then

$$H(e^{j\omega}) \triangleq \mathrm{DTFT}\{f_k \cdot g_k\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(e^{j\sigma})G(e^{j(\omega-\sigma)})\,\mathrm{d}\sigma = [F \circledast G](\omega). \tag{6.68}$$

The Paley–Wiener Theorem

The Paley–Wiener Theorem applicable to the DTFT is as follows

A sequence of samples u_k satisfies $u_k = 0$ for $k < 0$ if and only if its DTFT $U(e^{j\omega}) = \mathrm{DTFT}\{u_k\}$ satisfies

$$\int_{-\pi}^{\pi} \log|U(e^{j\omega})|\,\mathrm{d}\omega > -\infty. \tag{6.69}$$

As for the continuous time case, the most important application of this result is in the field of filter design, although in this situation the filters involved will digital ones, and not the analog ones studied in section 5.5.

Summary of DTFT Properties

The DTFTproperties developed in this section are summarised in table 6.1 following.

Property Name	Operation	Discrete Time Fourier Transform Property
Linearity	$y(t) = \alpha f_k + \beta\, g_k$	$Y(e^{j\omega}) = \alpha F(e^{j\omega}) + \beta\, G(e^{j\omega})$
Time Shift	$y_k = u_{k-T}$	$Y(e^{j\omega}) = e^{-j\omega T} U(e^{j\omega})$
Modulation	$y_k = e^{jk\omega_\circ} \cdot u_k$	$Y(e^{j\omega}) = U(e^{j(\omega-\omega_\circ)})$
Convolution	$y_k = \displaystyle\sum_{n=-\infty}^{\infty} h_n u_{k-n}$	$Y(e^{j\omega}) = H(e^{j\omega}) U(e^{j\omega})$
Transform of Product	$h_k = f_k \cdot g_k$	$H(e^{j\omega}) = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} F(e^{j\sigma}) G(e^{j(\omega-\sigma)})\, \mathrm{d}\sigma$
Real valued samples	$u_k \in \mathbf{R}$	$U(e^{j\omega}) = \overline{U(e^{-j\omega})}$
Hermitian samples	$u_k = \overline{u_{-k}}$	$U(e^{j\omega}) \in \mathbf{R}$
Energy Relationship	$\text{Energy} = \displaystyle\sum_{k=-\infty}^{\infty} \lvert u_k \rvert^2$	$\text{Energy} = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \lvert U(e^{j\omega}) \rvert^2\, \mathrm{d}\omega$
Paley–Wiener	$u_k = 0, \;\; k < 0$	$\displaystyle\int_{-\pi}^{\pi} \log \lvert U(e^{j\omega}) \rvert\, \mathrm{d}\omega > -\infty$

Table 6.1: *Discrete Time Fourier transform properties.*

6.10 The Discrete Fourier Transform (DFT)

An essential point of the preceding section was that there were two (compatible) views of the DTFT. On the one hand, the DTFT $U(e^{j\omega\Delta})$ could be *defined* purely in terms of a set of samples $\{u_k\}$ of an underlying continuous time signal $u(t)$ that are spaced Δ seconds apart according to (6.20) which is

$$U(e^{j\omega\Delta}) \triangleq \sum_{k=-\infty}^{\infty} u_k e^{-j\omega k\Delta}. \tag{6.70}$$

On the other hand, the DTFT could be *interpreted* as being the spectrum $U_p(\omega)$ of the perfectly sampled version $u_p(t)$ of $u(t)$, and hence could be understood as an infinite set of shifted replicas of the spectrum $U(\omega)$ of the underlying continuous time signal $u(t)$. That is, according to (6.20)-(6.22)

$$U(e^{j\omega\Delta}) \triangleq \sum_{k=-\infty}^{\infty} u_k e^{-j\omega k\Delta} = U_p(\omega) = \frac{1}{\Delta}\sum_{k=-\infty}^{\infty} U(\omega - k\omega_s) \qquad ;\omega_s = \frac{2\pi}{\Delta}. \tag{6.71}$$

which is illustrated in figure 6.14.

These dual views have an important practical consequence. Namely, that the spectrum $U(\omega)$ of a signal may be computed via software that has access to the samples u_k, uses them to compute the DTFT $U(e^{j\omega\Delta})$, and then extracts $U(\omega)$ from the DTFT $U(e^{j\omega\Delta})$ via the relationship (6.71).

The software required to evaluate (6.70) is straightforward. For a given frequency ω at which $U(j\omega\Delta)$ is to be evaluated, only a simple loop is required that, for n'th iteration, computes the product

$$u_n e^{-j\omega n\Delta} = u_n \cos\omega n\Delta - j\, u_n \sin\omega n\Delta$$

and then then adds this to a running accumulation, which after the n'th iteration will be

$$\sum_{k=-\infty}^{n} u_k e^{-j\omega k\Delta}$$

There are, however, some clear practical aspects that impinge on this strategy.

Firstly, and most obviously, the DTFT in (6.70) is an infinite sum, which in principle would require an infinite loop operating on an infinite amount of data. In practice, only a finite data record of (say) N samples will be available. This implies, that with the origin of the data record being generically labelled as the zero'th sample, the best one can hope for is to be able to make the finite loop computation

$$U_N(e^{j\omega\Delta}) = \sum_{k=0}^{N-1} u_k e^{-j\omega k\Delta}. \tag{6.72}$$

Furthermore, neither the DTFT $U(e^{j\omega\Delta})$ nor the more practical "finite data" quantity $U_N(e^{j\omega\Delta})$ can be computed at every possible frequency ω. The best that can be achieved is that a finite discrete set of frequencies are chosen for which the multiply-accummulate loop is run in order to compute (6.72).

In principle, this set of discrete frequencies could be arbitrary, but a very useful choice (for reasons pertaining to the computability of an inverse transform, as will be seen presently) is that of

$$\omega = n\omega_\circ, \qquad \omega_\circ = \frac{2\pi}{N}, \quad n = 0, 1, 2\cdots, N-1. \tag{6.73}$$

That is, if N data samples u_k are available, then N "samples" of $U_N(e^{j\omega\Delta})$ are computed that are evenly spaced from normalised frequency $\omega\Delta = 0$ up to normalised frequency $\omega\Delta = 2\pi$, which can be expressed in 'real' frequency terms as $\omega = 0$ up to $\omega = \omega_s = 2\pi/\Delta$; the latter being the sampling frequency.

This range of frequencies is appropriate since, according to the relationship (6.71), the DTFT $U(e^{j\omega\Delta})$ evaluated in this region is equal to that portion of the spectrum $U(\omega)$ which can potentially be recovered from the DTFT before the "shift and overlap" nature of (6.71) indicates that only repeated versions of $U(\omega)$ are available at any higher or lower frequencies. Again, refer to figure 6.14.

Acknowledging the dual limitations of only a finite number N of samples $\{u_k\}$ being available, only only a finite number of frequencies defined by (6.73) being feasible as computation points, then leads to the definition of what is known as the 'Discrete Fourier Transform'.

The Discrete Fourier Transform (DFT) of a set of N samples $u_0, u_1, \cdots, u_{N-1}$ is defined as

$$U_N(n\omega_\circ) = \sum_{k=0}^{N-1} u_k e^{-jkn\omega_\circ}, \qquad \omega_\circ \triangleq \frac{2\pi}{\Delta}, \quad n = 0, 1, 2, \cdots, N-1. \tag{6.74}$$

Clearly, according to the preceding discussion, the DFT $U_N(n\omega_\circ)$ computed at the normalised frequency $n\omega_\circ$ can be interpreted as an approximation of the spectrum $U(\omega)$ of the underlying continuous time signal $u(t)$ at the 'real' frequency $n\omega_\circ/\Delta$. That is

$$U_N(n\omega_\circ) \approx U(n\omega_\circ/\Delta). \tag{6.75}$$

The nature of this approximation and especially the factors that affect its accuracy, will be studied presently. Before doing so, some fundamental propeties of the DFT and an expression for its inverse will be examined.

Before moving on to this material, it is worth emphasising at this point that the fundamental quantity $\omega_\circ$ defined in (6.73) is often, in a variety of texts, referred to as ω_s. However, in this text, we have reserved $\omega_s = 2\pi/\Delta$ to denote the sampling frequency in radians per second. We therefore use the new notation $\omega_\circ$ to denote this new quantity $\omega_\circ = 2\pi/N$.

Example 6.5 DFT of Exponential Signal Consider the case of $\{u_k\}$ being samples from an exponentially evolving signal

$$u_k = \begin{cases} \lambda^k & ; k \geq 0, \\ & \\ 0 & ; k < 0. \end{cases} \tag{6.76}$$

Note that λ can be complex valued if appropriate. The DFT of N samples $u_0, \cdots, u_{N-1}$ of this signal can be computed as

$$U_N(n\omega_\circ) = \sum_{k=0}^{N-1} \lambda^k e^{-jkn\omega_\circ} = \sum_{k=0}^{N-1} [\lambda e^{-jn\omega_\circ}]^k = \frac{1 - \lambda^N e^{-jNn\omega_\circ}}{1 - \lambda e^{-jn\omega_\circ}}$$

where the last equation was obtained by recognising that the previous one represented a geometric series, for which a well known formula exists; it is given as (6.95) in Appendix 6.A.1. Furthermore, note that by the definition (6.73) of $\omega_\circ$

$$e^{-jNn\omega_\circ} = e^{-jNn2\pi/N} = e^{-jn2\pi} = 1$$

for all n. Therefore

$$U_N(n\omega_\circ) = \frac{1-\lambda^N}{1-\lambda e^{-jn\omega_\circ}}. \tag{6.77}$$

It is interesting to compare this with the normalised frequency DTFT $U(e^{j\omega})$, which uses an infinite $N = \infty$ amount of samples of u_k defined by (6.76). This DTFT was shown in equation (6.27) to be equal to

$$U(e^{j\omega}) = \frac{1}{1-\lambda e^{-j\omega}}.$$

Comparing with (6.77) indicates that the DFT, based on a finite number of N data points, is a slightly distorted version of the DTFT $U(e^{j\omega})$. Namely, it is the DTFT scaled by the constant amount $1-\lambda^N$. If, for example, $|\lambda| < 1$, then as the amount of available data N grows, $1-\lambda^N \to 1$ and hence the DFT converges to the DTFT.

A deeper discussion of how the limitation of finite data affects the DFT relative to the DTFT, and how these affects can be mitigated, is provided in the following section 6.13.2.

■

6.11 The Inverse Discrete Fourier Transform (IDFT)

As just discussed, the DFT $U_N(n\omega_\circ)$ provides a means for practical computation of an approximation to $U(n\omega_\circ/\Delta)$, where $U(\omega)$ is the spectrum of an underlying signal $u(t)$, which has been sampled with period Δ seconds in order to provided the samples $\{u_k\}$.

In applications, use of this computed spectrum often requires the ability to 'go the other way', and recover the samples $u_0, u_1, \cdots, u_{N-1}$ from the DFT $U_N(n\omega_\circ)$. This can be achieved by the Inverse Discrete Fourier Transform (IDFT).

In order to derive this inverse transform, recall that the DFT $U_N(n\omega_\circ)$ is constrained to be computed only at N frequencies $0, \omega_\circ, 2\omega_\circ, \cdots, (N-1)\omega_\circ$. If these N values are collected in a vector, and the N data samples $u_0, u_1, \cdots, u_{N-1}$ are also collected in a vector, then the N evaluations of the DFT given by (6.74) may be written as a matrix-vector multiplication as follows

$$\begin{bmatrix} U_N(0) \\ U_N(\omega_\circ) \\ U_N(2\omega_{\circ)} \\ \vdots \\ U_N((N-1)\omega_\circ) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & e^{-j\omega_\circ} & e^{-j2\omega_\circ} & \cdots & e^{-j(N-1)\omega_\circ} \\ 1 & e^{-j2\omega_\circ} & e^{-j4\omega_\circ} & \cdots & e^{-j2(N-1)\omega_\circ} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-j(N-1)\omega_\circ} & e^{-j2(N-1)\omega_\circ} & \cdots & e^{-j(N-1)^2\omega_\circ} \end{bmatrix}}_{\Omega_N} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}. \tag{6.78}$$

The point of recognising this matrix-vector interpretation is that it suggests a means for deriving the inverse discrete Fourier transform. Namely, if both sides of the relationship (6.78) are multiplied on the right by the matrix inverse Ω_N^{-1}, then the vector of data samples may be obtained from the vector of DFT evaluations.

$$\Omega_N^{-1} \begin{bmatrix} U_N(0) \\ U_N(\omega_\circ) \\ U_N(2\omega_{\circ)} \\ \vdots \\ U_N((N-1)\omega_\circ) \end{bmatrix} = \Omega_N^{-1}\Omega_N \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix}. \tag{6.79}$$

Furthermore, it so happens that the matrix inverse Ω_N^{-1} is very simply computable as (see Appendix 6.A.2 for a detailed explanation)

$$\Omega_N^{-1} = \frac{1}{N}\Omega_N^\star \tag{6.80}$$

where the notation $\cdot^\star$ notation means 'conjugate and transpose'. That is, take the complex conjugate of every element in the matrix, and then flip the matrix about its diagonal. Subsitituting this expression (6.80) into (6.78) then provides

$$\begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & e^{j\omega_\circ} & e^{j2\omega_\circ} & \cdots & e^{j(N-1)\omega_\circ} \\ 1 & e^{j2\omega_\circ} & e^{j4\omega_\circ} & \cdots & e^{j2(N-1)\omega_\circ} \\ 1 & e^{j3\omega_\circ} & e^{j6\omega_\circ} & \cdots & e^{j3(N-1)\omega_\circ} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & e^{j(N-1)\omega_\circ} & e^{j2(N-1)\omega_\circ} & \cdots & e^{j(N-1)^2\omega_\circ} \end{bmatrix}}_{\Omega_N^\star} \begin{bmatrix} U_N(0) \\ U_N(\omega_\circ) \\ U_N(2\omega_\circ) \\ \vdots \\ U_N((N-1)\omega_\circ) \end{bmatrix}. \tag{6.81}$$

Reading off one line at a time from this new matrix-vector relationship then gives the definition of the inverse DFT.

The $k'th$ sample u_k that was used in the forming of the DFT $U_N(n\omega_\circ)$ may be computed using the inverse discrete Fourier transform (IDFT) given as:

$$u_k = \frac{1}{N}\sum_{n=0}^{N-1} U_N(n\omega_\circ)e^{jkn\omega_\circ} \tag{6.82}$$

It is important to understand that there is a fundamental link between the specification of the 'bin spacing' $\omega_\circ$ and the above inverse transform. Namely, by virtue of its definition as $\omega_\circ = 2\pi/N$, it is one N'th of the normalised frequency range $[0, 2\pi]$. This is the full range of interest, since beyond it, only repeated versions will be obtained of what is already computed.

Therefore, the choice $\omega_\circ = 2\pi/N$ implies that if N data samples $u_0, \cdots, u_{N-1}$ are available, then also N 'samples' $U_N(0), \cdots, U_N((N-1)\omega_\circ)$ of the DFT are of interest. This implies that the matrix vector relationship (6.78) is one in which the 'DFT matrix' U_N is *square*, and hence that an inverse matrix U_N^{-1} exists (only square matrices have an inverse). Without this, it would not be possible to derive the inverse DFT via (6.79) which is re-expressed as (6.82).

6.12 Relationship between DFT and DTFT

By comparing the definitions (6.52) and (6.74), it is apparent that the DFT $U_N(n\omega_\circ)$ can be considered as the DTFT $U(e^{j\omega})$ of a signal which is zero for samples prior to $k = 0$, and past $k = N - 1$.

6.13 Properties of the DFT

6.13.1 Properties equivalent to those of DTFT

Linearity

Suppose that $y_k = \alpha f_k + \beta g_k$ and that $F_N(n\omega_\circ) = \text{DFT}\{f_k\}$, $G_N(n\omega_\circ) = \text{DFT}\{g_k\}$. Then

$$Y_N(n\omega_\circ) = \text{DFT}\{\alpha u_k + \beta y_k\} = \alpha\text{DFT}\{u_k\} + \beta\text{DFT}\{y_k\} = \alpha F_N(n\omega_\circ) + \beta G_N(n\omega_\circ).$$

Modulation Property

Suppose that $y_k = e^{jk\omega_\circ} \cdot u_k$ and that $U_N(n\omega_\circ) = \text{DFT}\{u_k\}$. Then

$$Y_N(n\omega_\circ) = \text{DFT}\left\{e^{jk\omega_\circ} \cdot u_k\right\} = U(e^{j(\omega-\omega_\circ)}). \tag{6.83}$$

Symmetry and Realness

If u_k is real valued for all k, then

$$U_N(n\omega_\circ) = \overline{U_N(-n\omega_\circ)}.$$

This further implies that

$$|U_N(n\omega_\circ)|^2 = U_N(n\omega_\circ)\overline{U_N(n\omega_\circ)} = U_N(n\omega_\circ)U_N(-n\omega_\circ)$$

and by the same reasoning

$$|U_N(-n\omega_\circ)|^2 = U_N(-n\omega_\circ)\overline{U(e^{-\omega})} = U_N(-n\omega_\circ)U_N(n\omega_\circ)$$

and hence

$$|U_N(n\omega_\circ)| = |U_N(-n\omega_\circ)|$$

so that $|U_N(n\omega_\circ)|$ is an even function of ω if $\{u_k\}$ is real valued.

6.13.2 Relationship of DFT to Fourier Transform - Data Windowing

Recall from section 6.9 that a very fundamental property of the DTFT $U(e^{j\omega\Delta})$ was that it was intimately related to the Fourier Transform $U(\omega)$ of any underlying continuous signal $u(t)$ whose samples, spaced Δ seconds apart, were used to form $U(e^{j\omega\Delta})$. Namely, there was an exact equality over the frequency range between negative and positive folding frequency π/Δ:

$$U(e^{j\omega\Delta}) = U(\omega), \quad \omega \in \left[\frac{-\pi}{\Delta}, \frac{\pi}{\Delta}\right] \tag{6.84}$$

Actually, to be strictly correct, the is *not* equality in $U(\omega)$, but in $U_p(\omega)$, the spectrum of $u_p(t)$,the perfectly sampled version of $u(t)$. However, under the bandlimiting assumption that $U(\omega) = 0$ for $\omega > \pi/\Delta$ (that is, under an assumption that sampling does not produce any aliasing) then as was established in the previous section 6.6, $U(\omega) = U_p(\omega)$. In the rest of this chapter, we will assume that this is the case.

With the above equality between Fourier Transform $U(\omega)$ and DTFT $U(e^{j\omega\Delta})$ in mind, it is then natural to question how the two then relate to the DFT $U_N(n\omega_\circ)$. This is not least because it is practically feasible to compute $U_N(n\omega_\circ)$, and hence if a relationship between it and $U(\omega)$ can be established, then this provides a means for measuring the spectrum $U(\omega)$ based on samples of it.

In order to investigate this, it is first obvious that the most important difference between the DTFT and the DFT is that the latter is based on only a finite number of samples, denoted as N. This can be accounted for by a process known as "windowing", in which a truncated data series is formed from an infinite one via multiplication.

That is, a windowed sequence of samples $\{u_k^w\}$ is formed from an original sequence of samples $\{u_k\}$ according to

$$u_k^w = u_k \cdot w_k \tag{6.85}$$

where w_k is termed a "windowing sequence". If it specified as

$$w_k = \begin{cases} 1 & ; k \in [0, N-1] \\ \\ 0 & ; \text{Otherwise} \end{cases} \tag{6.86}$$

then $\{u_k^w\}$ is simply the sequence $\{u_k\}$ with all sample prior to $k = 0$ and after $k = N$ set to zero.

As such, there is therefore an immediate and obvious relationship between the DFT of $\{u_k\}$ and the DTFT of u_k^w. Namely

$$U_N(n\omega_\circ) = U^w(e^{jn\omega_\circ\Delta}). \tag{6.87}$$

That is, the DFT of $\{u_k\}$ evaluated at normalised frequency $\omega = n\omega_\circ$ is equal to the DTFT of the infinite (but largely zero) windowed sequence $\{u_k^w\}$ evaluated at real frequency $n\omega_\circ\Delta$.

Furthermore, according to (6.68), the DTFT of the product of two signals (such as $\{u_k\}$ and $\{w_k\}$) is the circular convolution of the DTFT's of those two signals. That is, denoting the DTFT's of the infinite sequence $\{u_k\}$ and the finite windowing sequence $\{w_k\}$ as

$$U(e^{j\omega}) = \text{DTFT}\{u_k\}, \qquad W(e^{j\omega}) = \text{DTFT}\{w_k\}, \tag{6.88}$$

then according to property (6.68)

$$U^w(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} U(e^{j\sigma})W(e^{j(\omega-\sigma)})\,\mathrm{d}\sigma \tag{6.89}$$

DFT of a periodic signal

As has been apparent ever since their introduction in chapter 2, the class of periodic signals is of particular importance. In particular, recall that a signal $u(t)$ is periodic of period T if

$$u(t) = u(t+T). \tag{6.90}$$

Now further suppose that samples $\{u_k\}$ of this signal have been obtained at sampling rate $\Delta = T/N$ for some integer N. Then since the underling signal $u(t)$ is periodic, so will the samples, according to

$$u_k = u_{k+N} \tag{6.91}$$

Now we recall some further spectral properties of periodic signals

Convolution

6.14 The Fast Fourier Transform (FFT)

The Fast Fourier Transform (FFT) is the same as the Discrete Fourier Transform

$$\widehat{u}_p(\omega) = \sum_{k=0}^{N-1} u_k e^{-jn\omega_s k} \tag{6.92}$$

save that the FFT only applies when N is a power of 2. That is, only when N can be written as $N = 2^M$ where M is an integer. For example $N = 32, 64, 128, 256, 512, \cdots$. The reason that N must be a power of 2 is that when it is, you can use a trick of working out the DFT not on the original sequence $\{u_k\}$ but on several shorter sequences. These new sequences are formed by successively taking every second sample in $\{u_k\}$ to get a new sequence $\{z_k\}$, then taking every second sample again etc. The DFT's of these new shorter sequences can then be cleverly combined to get the DFT for the whole sequence, but with less computation than if we just calculated the DFT directly from (6.92). Hence, it is fast to calculate the DFT using this trick and so the method is called the Fast Fourier Transform. In general, to calculate the DFT of an N point sequence you need to perform roughly N^2 calculations, but when N is a power of 2 so you can calculate the DFT using the FFT you need to perform only roughly $N \log_2 N$ calculations. You'll notice that this can make quite a difference. For example, for N = 256

$$N^2 = 65536 \qquad N \log_2 N = 256 \times 8 = 2048.$$

Most books on digital signal processing cover the FFT in a lot of detail with complicated 'butterfly' diagrams and considerations of tradeoffs between minimizing calculation time or memory requirements.

6.15 Exercises

1. Demonstrate the aliasing effect by making a careful sketch of $\cos 2\pi 10t$ and $\cos 2\pi 70t$ for $0 \leq t \leq 1/10$. Put both sketches on the same set of axes and find the sample values at $t = 0, 1/80, 2/80, \cdots, 8/80$, which corresponds to $f_s = 80$. Also, convince yourself that no other waveform bandlimited in $10 < W < 40$ can be interpolated from the sample values of $\cos 2\pi 10t$.

2. The signal $x(t)$ whose spectrum is shown if figure 6.16 is ideally sampled at $f_s = 20Hz$. Sketch the spectrum of the sampled signal for $|f| \leq 40Hz$. Can $x(t)$ be recovered ? If so, how ? Repeat with $f_s = 30Hz$.

3. A rectangular pulse of width $\tau = 2$ seconds is sampled and reconstruced using an ideal Low Pass Filter with cutoff frequency at $f_s/2$, f_s being the sampling period used. Sketch the resulting output waveforms when the sampling intervals are

   (a) $T_s = 0.8$ seconds

   (b) $T_s = 0.4$ seconds

   Assume one sample time is at the centre of the pulse.

4. Assuming an ideal system, find the minimum sampling rate that can be used to obtain samples that completely specify the following signals

   (a)
   $$x(t) = \text{sinc}(400t)$$

   (b)
   $$y(t) = 4\text{sinc}(10t)\text{sinc}(t)$$

   For part (4a), plot the ideally sampled signal and its amplitude and spectrum for sampling frequencies of 300Hz and 500Hz. Are these sampling rates acceptable for an ideal system? For part (4b) provide the same plots and answer the same question for sampling rates of 10Hz, 15Hz and 20Hz.

5. A signal consisting of sinewaves at frequencies 1kHz, 2kHz and 3kHz is sampled with period 200μs. The samples are low pass filtered with a cut-off frequency of 2.5kHz. What frequency components are in the resulting signal?

6. An FFT is to be used to estimate the spectrum of a signal based upon observations of samples of the signal. Specifications for the spectral analysis are

   (a) Resolution between spectral components 0.5Hz in separation is required.

   (b) The highest frequency spectral component to be observed is at 250Hz.

   Determine the following parameters

   (a) The minimum number N of data samples required.

   (b) The minimum sampling frequency that can be used.

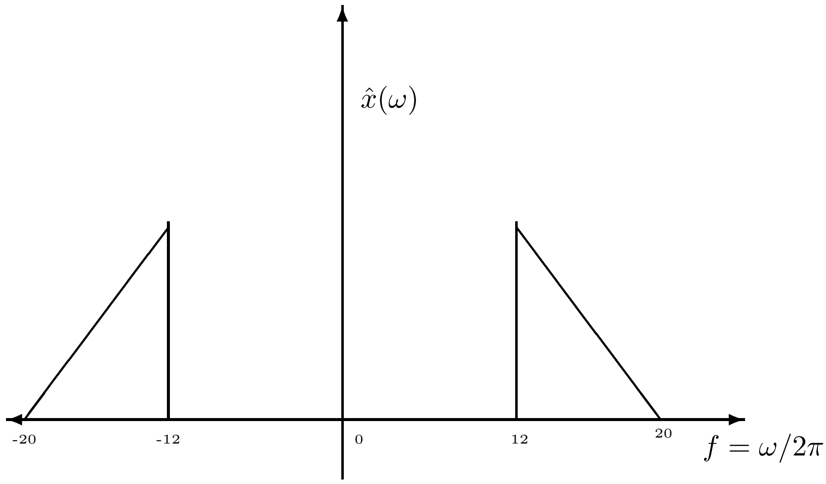

Figure 6.16: *Spectrum* $\hat{x}(\omega)$ *of* $x(t)$.

7. Show that if $\{x_k\}$ is an N point data record, and if

$$\hat{x}_N(m\omega_s) = \sum_{k=0}^{N-1} x_k e^{-jkm\omega_s} \qquad \omega_s = \frac{2\pi}{N}$$

is the DFT of of the sequence $\{x_k\}$ then the following Parsevals Formula holds

$$\sum_{k=0}^{N-1} |x_k|^2 = \frac{1}{N}\sum_{m=0}^{N-1} |\hat{x}_N(m\omega_s)|^2.$$

6.A Mathematical Background

6.A.1 Summation of a Geometric Series

A Geometric series is a summation of a list of quantities, where each quantity in the list is equal to a constant ratio r times the quantity preceding it. That is, it is a summation of the form

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} = \sum_{k=0}^{n-1} ar^k. \tag{6.93}$$

In this case, note that multiplying the above equation by r yields

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n. \tag{6.94}$$

Therefore, subtracting (6.94) from (6.93) yields

$$S_n = rS_n = a - ar^n = a(1 - r^n).$$

Making S_n the subject of this expression then yields a formula for the sum (6.93)

$$S_n = \frac{a(1 - r^n)}{1 - r}. \tag{6.95}$$

Now, if $|r| < 1$, then $r^n \to 0$ as $n \to \infty$ and hence for the infinite sum

$$S_\infty = a + ar + ar^2 + \cdots = \frac{a}{1 - r}.$$

The geometric series

$$S_n = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} = \sum_{k=0}^{n-1} ar^k \tag{6.96}$$

can be evaluated as

$$S_n = \frac{a(1 - r^n)}{1 - r} \tag{6.97}$$

and, if $|r| < 1$ then the infinite summation can be evaluated as

$$S_\infty = \frac{a}{1 - r} \tag{6.98}$$

6.A.2 Inverse of Ω_N

The matrix Ω_N defined as

$$\Omega_N \triangleq \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ e^{-j\omega_\circ} & e^{-j2\omega_\circ} & e^{-j3\omega_\circ} & \cdots & e^{-j(N-1)\omega_\circ} \\ e^{-2j\omega_\circ} & e^{-j4\omega_\circ} & e^{-j6\omega_\circ} & \cdots & e^{-j2(N-1)\omega_\circ} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ e^{-j(N-1)\omega_\circ} & e^{-j2(N-1)\omega_\circ} & e^{-j3(N-1)\omega_\circ} & \cdots & e^{-j(N-1)^2\omega_\circ} \end{bmatrix} \tag{6.99}$$

was introduces in section 6.11 as a means of describing the DFT as a matrix-vector product, and hence as a means of deriving the inverse DFT via muliplication by the matrix inverse Ω_N^{-1}. The purpose of this section is to verify that, as claimed in section 6.11, this inverse is given as the complex conjugate transpose $\Omega_N^\star$ of Ω_N divided by the scalar data length N. To establish this, consider the matrix product

$$\Omega_N^\star\Omega_N = \begin{bmatrix} 1 & e^{j\omega_\circ} & \cdots & e^{j(N-1)\omega_\circ} \\ 1 & e^{j2\omega_\circ} & \cdots & e^{j2(N-1)\omega_\circ} \\ 1 & e^{j3\omega_\circ} & \cdots & e^{j3(N-1)\omega_\circ} \\ \vdots & \vdots & & \vdots \\ 1 & e^{j(N-1)\omega_\circ} & \cdots & e^{j(N-1)^2\omega_\circ} \end{bmatrix} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{-j\omega_\circ} & e^{-j2\omega_\circ} & \cdots & e^{-j(N-1)\omega_\circ} \\ e^{-2j\omega_\circ} & e^{-j4\omega_\circ} & \cdots & e^{-j2(N-1)\omega_\circ} \\ \vdots & \vdots & \vdots & \vdots \\ e^{-j(N-1)\omega_\circ} & e^{-j2(N-1)\omega_\circ} & \cdots & e^{-j(N-1)^2\omega_\circ} \end{bmatrix}. \tag{6.100}$$

This indicates, that with the notation $[\cdot]_{\ell,m}$ denoting the ℓ,m'th element of a matrix (with indexing beginning with the $1,1$ element in the top left corner)

$$[\Omega_N^\star\Omega_N]_{\ell,m} = \sum_{k=0}^{N-1} e^{j\ell k\omega_\circ}e^{-jmk\omega_\circ}\sum_{k=0}^{N-1} e^{jk(\ell-m)\omega_\circ}. \tag{6.101}$$

Now, when $m = \ell$ so that a diagonal element of the matrix product, then $m - \ell = 0$ and hence via (6.101)

$$[\Omega_N^\star\Omega_N]_{\ell,\ell} = \sum_{k=0}^{N-1} 1 = N. \tag{6.102}$$

On the other hand, when $\ell \neq m$, then by recognising that (6.101) is a geometric sum with first term $a = 1$ and constant ratio $r = e^{j(\ell-m)\omega_\circ}$, then use of the formula (6.97) provides

$$[\Omega_N^\star\Omega_N]_{\ell,m} = \frac{1 - e^{j(\ell-m)\omega_\circ N}}{1 - e^{j(\ell-m)\omega_\circ}} \tag{6.103}$$

However, by employing the definition (6.73) of $\omega_\circ$

$$e^{j(\ell-m)\omega_\circ N} = \frac{2\pi}{N}\cdot N = 2\pi \tag{6.104}$$

and hence

$$e^{j(\ell-m)\omega_\circ N} = e^{j2\pi(\ell-m)} = 1. \tag{6.105}$$

Substituting this into (6.103) then indicates that for $\ell \neq m$

$$[\Omega_N^\star\Omega_N]_{\ell,m} = \frac{1-1}{1 - e^{j(\ell-m)\omega_\circ}} = 0. \tag{6.106}$$

Therefore,

$$[\Omega_N^\star\Omega_N]_{\ell,m} = \begin{cases} N & m = \ell \\ \\ 0 & m \neq \ell \end{cases} \tag{6.107}$$

and hence

$$\frac{1}{N}\Omega_N^\star\Omega_N = I \tag{6.108}$$

where I is the identity matrix. This confirms by direct computation that, indeed

$$\Omega_N^{-1} = \frac{1}{N}\Omega_N^\star. \tag{6.109}$$