ELEC2400 Signals & Systems

Chapter 7. $\mathcal{Z}$-Transforms

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7. $\mathcal{Z}$-Transforms

- For continuous time systems, we analysed the response $y(t)$ using its Laplace Transform $Y(s)$:
  - The differential equation was transformed in an algebraic equation.
  - The qualitative information on $y(t)$ was determined using the poles of $Y(s)$.
- For discrete time systems, we use the **difference equation**:

$$y_{k+n} + a_{n-1}y_{k+n-1} + \ldots + a_1 y_{k+1} + a_0 y_k = b_m u_{k+m} + \ldots + b_1 u_{k+1} + b_0 u_k.$$

*We do not analyse the output sequence $\{y_k\}$ directly in the time domain, using instead the $\mathcal{Z}$-**Transform**.*

Relationship between $\mathcal{Z}$-Transforms and Laplace Transforms

- Consider the continuous time system:

$$\frac{d^2y(t)}{dt} + 2\frac{dy(t)}{dt} + y(t) = 3\frac{du(t)}{dt}$$

Now in frequency domain, considering sinusoidal components in the input $u(t)$ and the output $y(t)$:

$$(j\omega)^2 Y(\omega)e^{j\omega t} + 2(j\omega)Y(\omega)e^{j\omega t} + Y(\omega)e^{j\omega t} = 3(j\omega)U(\omega)e^{j\omega t}$$

Now cancelling $e^{j\omega t}$ on both sides, we get the **Frequency Response** of the system:

$$\frac{Y(\omega)}{U(\omega)} = \frac{3}{(j\omega)^2 + 2j\omega + 1}$$

which, however, does not consider *transient responses* of the system.

- On the other hand, Laplace Transform considers the exponential components e^{st} of the input $u(t)$ and the output $y(t)$. Then:

$$s^2 Y(s)e^{jst} + 2sY(s)e^{jst} + Y(s)e^{st} = 3s \cdot U(s)e^{st}$$

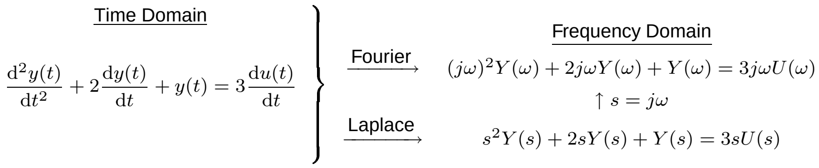
and we obtain the **Transfer Function** of the system:

$$\frac{Y(s)}{U(s)} = \frac{3s}{s^2 + 2s + 1} \triangleq H(s)$$

This is a *generalisation* of Fourier analysis since sine wave signals $e^{j\omega t}$ are special cases of exponential signals e^{st}, when $s = j\omega$.

Relationship between $\mathcal{Z}$-Transforms and Laplace Transforms

- Illustrative diagram:


Time Domain

$$\frac{\mathrm{d}^2 y(t)}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t) = 3\frac{\mathrm{d}u(t)}{\mathrm{d}t}$$

Frequency Domain

$\xrightarrow{\text{Fourier}}$ $(j\omega)^2 Y(\omega) + 2j\omega Y(\omega) + Y(\omega) = 3j\omega U(\omega)$

$\uparrow s = j\omega$

$\xrightarrow{\text{Laplace}}$ $s^2 Y(s) + 2sY(s) + Y(s) = 3sU(s)$

  - This gives us a complete bag of tools to work with in continuous time.
  - What about discrete time?

Relationship between $\mathcal{Z}$-Transforms and Laplace Transforms

- Consider, for example, the discrete time system:

$$y_{k+2} - 1.5y_{k+1} + 0.56y_k = 3u_{k+1}$$

We assume an input $u_k = Ue^{j\omega k\Delta}$ (Δ: sampling period) which generates an output at the same frequency $y_k = Ye^{j\omega k\Delta}$. Then:

$$e^{j\omega 2\Delta} Y e^{j\omega k\Delta} - 1.5e^{j\omega\Delta} Y e^{j\omega k\Delta} + 0.56Y e^{j\omega k\Delta} = 3e^{j\omega\Delta} e^{j\omega k\Delta}$$

Cancelling $e^{j\omega k\Delta}$ on both sides, we find the **discrete time Frequency Response**:

$$\frac{Y}{U} = \frac{3e^{j\omega\Delta}}{e^{j2\omega\Delta} - 1.5e^{j\omega\Delta} + 0.56}$$

As for the continuous time case, this doesn't tell us anything about the *transient response* of the system.

- Considering an exponential in the input $u_k = Ue^{sk\Delta}$, which produces an exponential output $y_k = Ye^{sk\Delta}$, we have that:

$$e^{2s\Delta}Ye^{sk\Delta} + 1.6e^{s\Delta}Ye^{sk\Delta} + 0.64Ye^{5k\Delta} = 3e^{s\Delta}Ue^{sk\Delta}$$

If we denote $z = e^{s\Delta}$ then we obtain the **Transfer Function**:

$$\frac{Y}{U} = \frac{3z}{z^2 - 1.5z + 0.56} \triangleq H(z)$$

If $s = j\omega$ then we have $z^k = e^{j\omega k\Delta}$, and then $H(z)$ for the special choice $z = e^{j\omega\Delta}$ gives the sinusoidal steady state frequency response.

Relationship between $\mathcal{Z}$-Transforms and Laplace Transforms

- Illustrative diagram:

Time Domain

Frequency Domain

$$y_{k+2}-1.5y_{k+1}+0.56y_k=3u_{k+1}\left\{\begin{array}{l} \xrightarrow{\text{DFT}} e^{j2\omega\Delta}Y_p(\omega)-1.5e^{j\omega\Delta}Y_p(\omega)+0.56Y_p(\omega)=3e^{j\omega\Delta}U_p(\omega)\\ \qquad\qquad\qquad\uparrow z=e^{j\omega\Delta}\\ \xrightarrow{\mathcal{Z}} z^2Y(z)-1.5zY(z)+0.56Y(z)=3zU(z)\end{array}\right.$$

Similar to the continuous time case, however:

- *Sampling* has been assumed in order for the discrete time sequences to exist
- In continuous time, e^{st} is a general exponential and making $s = j\omega$ we get sinusoidal analysis $\left(e^{j\omega t} = \cos\omega t + j\sin\omega t\right)$.
- In discrete time, $z = e^{s\Delta} = e^{st}|_{t=\Delta}$ is introduced, and to get sinusoidal analysis we use $z^k = e^{j\omega\Delta k} = \cos\omega k\Delta + j\sin\omega k\Delta$.

Relationship between $\mathcal{Z}$-Transforms and Laplace Transforms

- Illustrative diagram:

Time Domain

Frequency Domain

$$y_{k+2}-1.5y_{k+1}+0.56y_k=3u_{k+1}\left\{\begin{array}{l} \xrightarrow{\text{DFT}} e^{j2\omega\Delta}Y_p(\omega)-1.5e^{j\omega\Delta}Y_p(\omega)+0.56Y_p(\omega)=3e^{j\omega\Delta}U_p(\omega)\\ \qquad\qquad\qquad\uparrow z=e^{j\omega\Delta}\\ \xrightarrow{\mathcal{Z}} z^2Y(z)-1.5zY(z)+0.56Y(z)=3zU(z)\end{array}\right.$$

- In continuous time $\quad \dfrac{\mathrm{d}^p}{\mathrm{d}t^p}e^{st} = (s)^p e^{st} \quad$ into the differential equation relates the strength $Y(s)$ of the component in $y(t)$ at e^{st} to the strength $U(s)$ of the component in $u(t)$ at e^{st}.
- In discrete time $\quad q^p z^k = z^{k+p} \quad$ into the difference equation relates the strength $Y(z)$ of the component of $\{y_k\}$ at $z^k = e^{sk\Delta}$ to the strength $U(z)$ of the component in $\{u_k\}$ at z^k.

- The strength of e^{st} in $y(t)$ is $Y(s)$, given by:

$$Y(s) = \mathcal{L}\{y(t)\} = \int_{-\infty}^{\infty} y(t)e^{-st}\mathrm{d}t$$

To know the strength of e^{st} the sequence $\{y_k\}$ we consider again the signal $y_p(t)$:

$$\begin{aligned} Y_p(s) = \mathcal{L}\{y_p(t)\} &= \int_{-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} y_k \delta(t - k\Delta) \right) e^{-st}\mathrm{d}t \\ &= \sum_{k=-\infty}^{\infty} y_k \int_{-\infty}^{\infty} \delta(t-k\Delta)e^{-st}\mathrm{d}t = \sum_{k=-\infty}^{\infty} y_k e^{-sk\Delta} \end{aligned}$$

Replacing $z = e^{-s\Delta}$ we define the **$\mathcal{Z}$-Transform** of $\{y_k\}$:

$$Y(z) \triangleq \mathcal{Z}\{y_k\} = \sum_{k=-\infty}^{\infty} y_k z^{-k}$$

Calculation of $\mathcal{Z}$-Transforms

- If we substitute $z = e^{j\omega\Delta}$ we get

$$Y\left(e^{j\omega\Delta}\right) = \sum_{k=-\infty}^{\infty} y_k e^{-j\omega k\Delta} = Y_p(\omega)$$

which gives Discrete Fourier Transform (DFT).

- Most text books define $Y(z)$, with k starting from 0:

$$Y(z) = \sum_{k=0}^{\infty} y_k z^{-k}$$

implicitly assuming all *causal* signals ($y_k = 0$ for $k < 0$).

Calculation of $\mathcal{Z}$-Transforms

- Recall that the sum of a geometric series:

$$S_N = \sum_{k=0}^{N-1} ar^k$$

is given by:

$$S_N = \frac{a(1-r^N)}{1-r}$$

If $|r| < 1$ then $r^N \to 0$ as $N \to \infty$. This yields:

$$S_\infty = \sum_{k=-\infty}^{\infty} = \frac{a}{1-r}$$

This result will be often used to compute $\mathcal{Z}$-Transforms.

- Consider the sequence:

$$y_k = \begin{cases} \lambda^k & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$$

Then, its $\mathcal{Z}$-transform is:

$$Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} = \sum_{k=0}^{\infty} \lambda^k z^{-k} = \sum_{k=0}^{\infty} (\lambda/z)^k$$

Which provided $|\lambda/z| < 1$, gives:

$$Y(z) = \frac{1}{1-\lambda/z} = \frac{z}{z-\lambda} \quad ; \; |z| > |\lambda|$$

Calculation of $\mathcal{Z}$-Transforms

- Consider the sequence $y_k = \begin{cases} \cos\omega k\Delta & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$

Then, its $\mathcal{Z}$-transform is:

$$\begin{aligned} Y(z) &= \sum_{k=0}^{\infty} (\cos\omega k\Delta) z^{-k} = \frac{1}{2}\sum_{k=0}^{\infty} \left(e^{j\omega k\Delta} + e^{-j\omega k\Delta}\right) z^{-k} \\ &= \frac{1}{2}\sum_{k=0}^{\infty} \left(e^{j\omega\Delta} z^{-1}\right)^k + \frac{1}{2}\sum_{k=0}^{\infty} \left(e^{-j\omega\Delta} z^{-1}\right)^k \\ &= \frac{1}{2}\left\{\frac{1}{1-z^{-1}e^{j\omega\Delta}} + \frac{1}{1-z^{-1}e^{-j\omega\Delta}}\right\} \quad ; |z| > 1 \\ &= \frac{1}{2}\left\{\frac{2 - z^{-1}(e^{j\omega\Delta} + e^{-j\omega\Delta})}{1 - z^{-1}(e^{j\omega\Delta} + e^{-j\omega\Delta}) + z^{-2}}\right\} \end{aligned}$$

Which gives $\quad Y(z) = \dfrac{z(z-\cos\omega\Delta)}{z^2 - 2z\cos\omega\Delta + 1} \quad ; |z| > 1$

Calculation of $\mathcal{Z}$-Transforms

- We have found that $\displaystyle\sum_{k=0}^{\infty} (\lambda z^{-1})^k = \frac{z}{z-\lambda} \; ; |z| > \lambda$

Differentiating both sides with respect to $\lambda > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\sum_{k=0}^{\infty} (\lambda z^{-1})^k = \sum_{k=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}\lambda}\lambda^k z^{-k} = \sum_{k=0}^{\infty} k\lambda^{k-1} z^{-k} = \frac{1}{\lambda}\sum_{k=0}^{\infty} k\lambda^k z^{-k}$$

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(\frac{z}{z-\lambda}\right) = \frac{z}{(z-\lambda)^2}$$

Therefore:

$$y_k = \begin{cases} k\lambda^k & ; k \geq 0 \\ 0 & ; k < 0 \end{cases} \quad \Longleftrightarrow \quad Y(z) = \frac{\lambda z}{(z-\lambda)^2} \; ; |z| > \lambda$$

And, in general:

$$y_k = \begin{cases} k^p\lambda^k & ; k \geq 0 \\ 0 & ; k < 0 \end{cases} \quad \Longleftrightarrow \quad Y(z) = \frac{p!\lambda^p z}{(z-\lambda)^{p+1}} \; ; |z| < \lambda$$

Time Domain Sequence $\{y_k\}$	$\mathcal{Z}$ **Transform of** $\{y_k\}$
$y_k = \begin{cases} \lambda^k & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$	$Y(z) = \frac{z}{z-\lambda} \quad ; \lvert z \rvert > \lvert \lambda \rvert$
$y_k = \begin{cases} \cos \omega k \Delta & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$	$Y(z) = \frac{z(z - \cos \omega \Delta)}{z^2 - 2z \cos \omega \Delta + 1} \quad ; \lvert z \rvert > 1$
$y_k = \begin{cases} \sin \omega k \Delta & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$	$Y(z) = \frac{z \sin \omega \Delta}{z^2 - 2z \cos \omega \Delta + 1} \quad ; \lvert z \rvert > 1$
$y_k = \begin{cases} k^p \lambda^k & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$	$Y(z) = \frac{p! \lambda^p z}{(z-\lambda)^{p+1}} \quad ; \; \lvert z \rvert < \lambda$

Calculation of $\mathcal{Z}$-Transforms

Time Domain Sequence $\{y_k\}$	$\mathcal{Z}$ **Transform of** $\{y_k\}$
$y_k = \begin{cases} \lambda^k \cos \omega k \Delta & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$	$Y(z) = \frac{z(z - \lambda \cos \omega \Delta)}{z^2 - 2\lambda \cos \omega \Delta z + \lambda^2}$
$y_k = \begin{cases} \lambda^k \sin \omega k \Delta & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$	$Y(z) = \frac{\lambda z \sin \omega \Delta}{z^2 - 2\lambda z \cos \omega \Delta + \lambda^2}$
$y_k = \begin{cases} k\lambda^k \sin \omega k \Delta & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$	$Y(z) = \frac{\lambda z (z^2 - \lambda^2) \sin \omega \Delta}{(z^2 - 2\lambda z \cos \omega \Delta + \lambda^2)^2}$
$y_k = \begin{cases} k\lambda^k \cos \omega k \Delta & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$	$Y(z) = \frac{\lambda z [\cos \omega \Delta (z^2 + \lambda^2) - 2\lambda z]}{(z^2 - 2\lambda z \cos \omega \Delta + \lambda^2)^2}$

Properties of $\mathcal{Z}$-Transforms

- Using the $\mathcal{Z}$-Transforms $U(z)$ and $Y(z)$ we can now obtain the strength of the sequence z^k in $\{u_k\}$ and $\{y_k\}$.

  Example: Consider the basic filter

  $$y_{k+2} - 1.5y_{k+1} = 0.56y_k = 3u_{k+1}$$

  Then the transfer function is:

  $$Y(z) = \frac{3z}{z^2 - 1.5z + 0.56} U(z)$$

  If $\{u_k\}$ is a step function, then $U(z) = \dfrac{z}{z-1}$, and so

  $$\begin{aligned} Y(z) &= \left(\frac{3z}{z^2 - 1.5z + 0.56} \right) \frac{z}{(z-1)} \\ &= \frac{70z}{(z-0.7)} - \frac{120z}{(z-0.8)} + \frac{50z}{(z-1)} \end{aligned}$$

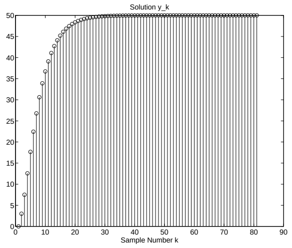
Example: (continued)

$$Y(z) = \frac{70z}{(z-0.7)} - \frac{120z}{(z-0.8)} + \frac{50z}{(z-1)}$$

$$\Rightarrow \; y_k = \mathcal{Z}^{-1}\{Y(z)\}$$

$$= 70(0.7)^k - 120(0.8)^k + 50(1)^k$$

$$; \; k \geq 0$$



- However, in the calculations we glossed over a few things:
  - We assumed the $\mathcal{Z}$-Transform was linear.
  - We assumed that the $\mathcal{Z}$-Transform was unique.
  - We assumed that $\{y_k\}$ was causal ($y_k = 0$ for $k < 0$).

Properties of $\mathcal{Z}$-Transforms

- **Linearity of $\mathcal{Z}$-Transform**
  Suppose $\{u_k\}$ and $\{x_k\}$ are two sequences and α and β two real numbers. Then for the sequence:

  $$y_k = \alpha u_k + \beta x_k$$

  The $\mathcal{Z}$-Transform is:

  $$Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} = \sum_{k=-\infty}^{\infty} (\alpha u_k + \beta x_k) z^{-k}$$

  $$= \alpha \sum_{k=-\infty}^{\infty} u_k z^{-k} + \beta \sum_{k=-\infty}^{\infty} x_k z^{-k}$$

  $$= \alpha U(z) + \beta X(z)$$

  so that the $\mathcal{Z}$-Transform is **linear**.

Properties of $\mathcal{Z}$-Transforms

- **Uniqueness of $\mathcal{Z}$ Transform**
  Suppose two sequences $\{u_k\}$ and $\{x_k\}$ with $\mathcal{Z}$-transform $U(z)$ and $X(z)$. If $U(z) = X(z)$, then by definition:

  $$U(z) = \sum_{k=-\infty}^{\infty} u_k z^{-k} = \sum_{k=-\infty}^{\infty} x_k z^{-k} = X(z)$$

  - $U(z)$ and $X(z)$t are polynomials in z^{-1}.
  - The two polynomials are equal *if and only if* all its co-efficients are equal.
  - Therefore $Y(z) = U(z)$ *if and only if* $y_k = u_k$ for every k.

  This means each $\mathcal{Z}$-transform is **uniquely associated** with only one discrete time sequence.

- **Time Shifting Property**
  When using:

  $$\mathcal{Z}\{qy_k\} = \mathcal{Z}\{y_{k+1}\} = zY(z)$$

  we are implicitly assuming that $\{y_k\}$ is causal.
  However, $\{y_k\}$ might have non-causal components $(y_k \neq 0$ for $k < 0)$ sometimes called **initial conditions**.
  - **Backward shifts in time**

    $$\mathcal{Z}\{q^{-n}y_k\} = \sum_{k=0}^{\infty} q^{-n}y_k z^{-k} = \sum_{k=0}^{\infty} y_{k-n}z^{-k} = y_{-n}z^{-0} + y_{1-n}z^{-1} + \ldots$$

    $$= z^{-n}\left(y_{-n}z^{n} + y_{1-n}z^{-(1-n)} + \ldots + y_0 + y_1 z^{-1} + \ldots\right)$$

    $$= z^{-n}\left(\underbrace{\sum_{k=1}^{n} y_{-k}z^{k}}_{\text{init. cond.}} + \underbrace{\sum_{k=0}^{\infty} y_k z^{-k}}_{Y(z)}\right)$$

Properties of $\mathcal{Z}$-Transforms

- **Time Shifting Property**
  - **Backward shifts in time** That is:

    $$\mathcal{Z}\{q^{-n}y_k\} = z^{-n}Y(z) + z^{-n}\sum_{k=1}^{n} y_{-k}z^{k}$$

    For $n = 1, 2, 3, \ldots$ this becomes:

    $$\begin{aligned}
    \mathcal{Z}\{q^{-1}y_k\} &= z^{-1}Y(z) + y_{-1} \\
    \mathcal{Z}\{q^{-2}y_k\} &= z^{-2}Y(z) + z^{-1}y_{-1} + y_{-2} \\
    \mathcal{Z}\{q^{-3}y_k\} &= z^{-3}Y(z) + z^{-2}y_{-1} + z^{-1}y_{-2} + y_{-3} \\
    \vdots \quad & \quad \vdots \\
    \mathcal{Z}\{q^{-n}y_k\} &= z^{-n}Y(z) + z^{-n+1}y_{-1} \; + \; \ldots \; + \; y_{-n}
    \end{aligned}$$

Properties of $\mathcal{Z}$-Transforms

- **Time Shifting Property**
  - An analogous result using **Forward shifts in time** is:

    $$\mathcal{Z}\{q^{n}y_k\} = z^{n}Y(z) - z^{n}\sum_{k=0}^{n-1} y_k z^{-k}$$

    Which for $n = 1, 2$ and 3 is:

    $$\begin{aligned}
    \mathcal{Z}\{qy_k\} &= zY(z) - zy_0 \\
    \mathcal{Z}\{q^2y_k\} &= z^2Y(z) - z^2y_0 - zy_1 \\
    \mathcal{Z}\{q^3y_k\} &= z^3Y(z) - z^3y_0 - z^2y_1 - zy_2
    \end{aligned}$$

    This is not so useful in practice:
    Note that (for $n = 3$) $Y(z)$ will depend on $\{y_0, y_1, y_2\}$, which we are precisely interested to obtain !

- **Example 1**:
  Consider the case of borrowing money from the bank:
  - we borrow C dollars,
  - we repay in equal installments of P dollars,
  - the interest rate per payment period is $100\alpha\%$ of unpaid principle,
  - y_k is the amount owed after k th payment, and
  - $u_k =$ payment per period.

  Then, by the time the $(k+1)$th payment is due:
  $$y_{k+1} = y_k + \alpha y_k - u_k \quad \Longleftrightarrow \quad q y_k = (1+\alpha) y_k - u_k.$$
  Taking the $\mathcal{Z}$-Transform and using linearity and time shifting properties gives:
  $$zY(z) - zy_0 = (1+\alpha)Y(z) - U(z)$$

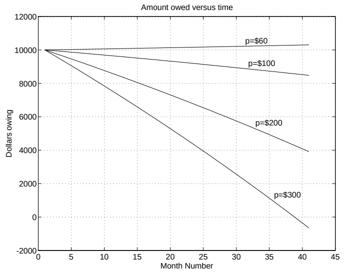
Properties of $\mathcal{Z}$-Transforms

- **Example 1**: (continued)
  So, we obtain $\quad Y(z) = \dfrac{zy_0}{z-(1+\alpha)} - \dfrac{U(z)}{z-(1+\alpha)}$
  Now $y_0 = C$ the amount initially borrowed, and $U(z)$ corresponds to the constant repayment P. Then:
  $$u_k = \begin{cases} P & ; k \geq 0 \\ 0 & ; k < 0 \end{cases} \quad \Longleftrightarrow \quad U(z) = Pz/(z-1)$$
  Substituting these values gives:
  $$\begin{aligned} Y(z) &= \frac{Cz}{z-(1+\alpha)} - \frac{Pz}{(z-1)(z-(1+\alpha))} \\ &= \left(\frac{P}{\alpha}\right)\frac{z}{z-1} + \left(C - \frac{P}{\alpha}\right)\frac{z}{z-(1+\alpha)} \\ \Rightarrow \quad y_k &= \frac{P}{\alpha} + \left(C - \frac{P}{\alpha}\right)(1+\alpha)^k \quad ; \; k \geq 0 \end{aligned}$$

Properties of $\mathcal{Z}$-Transforms

- **Example 1**: (continued)
  $$y_k = \frac{P}{\alpha} + \left(C - \frac{P}{\alpha}\right)(1+\alpha)^k \quad ; \; k \geq 0$$
  - The system is unstable unless $\alpha < 0$
  - However banks **do not pay you** for borrowing money!
  - Of course, $\alpha > 0$ and we are able to pay off the loan if $C - \frac{P}{\alpha} < 0$.

In the figure:
$C = \$10,000$
$\alpha = 8/12\%$ per month
for different payments P.



- **Example 2**. Suppose a digital filter whose transfer function is:

$$H(q) = \frac{q}{q^2 - 1.7q + 0.72}$$

This is equivalent to the input-output relationship:

$$y_{k+2} - 1.7y_{k+1} + 0.72y_k = u_{k+1}$$

Taking $\mathcal{Z}$transforms of both sides of this difference equation:

$$\left(z^2Y(z) - z^2y_0 - zy_1\right) - 1.7\left(zY(z) - zy_0\right) + 0.72Y(z) = zU(z) - zu_0$$

Which on factoring out $Y(z)$ and $U(z)$ becomes:

$$Y(z) = \left(\frac{z}{z^2 - 1.7z + 0.72}\right)U(z) + \frac{z(z-1.7)y_0 + zy_1 - zu_0}{z^2 - 1.7z + 0.72}$$

Here $Y(z)$ depends on the unknowns y_0 and y_1 !
This happens because $k = 2$ was implicitly taken as the division between past and future, and not $k = 0$.

Properties of $\mathcal{Z}$-Transforms

- **Example 2**. (continued)
The problem can be *avoided* rewriting the *recursive* input-output difference equation as:

$$y_k - 1.7y_{k-1} + 0.72y_{k-2} = u_{k-1}$$

Equivalently, using the backward shift operator q^{-1}:

$$H(q) = \frac{q}{q^2 - 1.7q + 0.72} \quad \Rightarrow \quad H(q^{-1}) = \frac{q^{-1}}{1 - 1.7q^{-1} + 0.72q^{-2}}$$

Using the $\mathcal{Z}$Transform time shifting properties:

$$Y(z) - 1.7\left(z^{-1}Y(z) + y_{-1}\right) + 0.72\left(z^{-2}Y(z) + z^{-1}y_{-1} + y_{-2}\right) = z^{-1}U(z) + u_{-1}$$

Which on factoring out $Y(z)$ and $U(z)$ becomes:

$$\begin{aligned} Y(z) &= \frac{z^{-1}}{1 - 1.7z^{-1} + 0.72z^{-2}}U(z) + \frac{(1.7 - 0.72z^{-1})y_{-1} - 0.72y_{-2} + u_{-1}}{1 - 1.7z^{-1} + 0.72z^{-2}} \\ &= \frac{z}{z^2 - 1.7z + 0.72}U(z) + \frac{z(1.7z - 0.72)y_{-1} - 0.72z^2y_{-2} + z^2u_{-1}}{z^2 - 1.7z + 0.72} \end{aligned}$$

Properties of $\mathcal{Z}$-Transforms

- **Example 2**. (continued)
If we consider $\{u_k\}$ a unit step at $k = 0$, then:

$$Y(z) = \frac{z^2}{(z-0.9)(z-0.8)(z-1)} + \frac{z(1.7z - 0.72)y_{-1} - 0.72z^2y_{-2}}{(z-0.9)(z-0.8)}$$

Using partial fraction expansion, we obtain:

$$\begin{aligned} Y(z) = 50\left(\frac{z}{z-1}\right) &+ (-90 + 8.1y_{-1} - 6.48y_{-2})\left(\frac{z}{z-0.9}\right) \\ &+ (40 - 6.4y_{-1} + 5.76y_{-2})\left(\frac{z}{z-0.8}\right) \end{aligned}$$

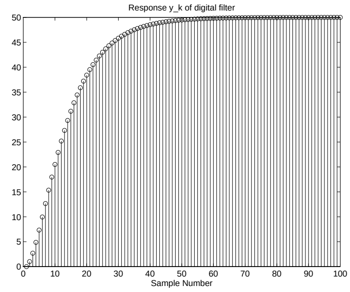
Then, the step response of the filter is given by:

$$y_k = \begin{cases} 50 + (-90 + 8.1y_{-1} - 6.48y_{-2})(0.9)^k & \\ +(40 - 6.4y_{-1} + 5.76y_{-2})(0.8)^k & ; k \geq 0 \\ y_{-1} & ; k = -1 \\ y_{-2} & ; k = -2 \end{cases}$$

- **Example 2**. (continued)

  In particular,
  if the initial conditions
  y_{-1} and y_{-2} are zero, then:

  $$y_k = 50 - 90(0.9)^k + 40(0.8)^k \quad ; \; k \geq 0$$



  - The d.c. gain of this filter is 50, as expected:

    $$\text{d.c. gain} = H(e^{j\omega\Delta})\Big|_{\omega=0} = H(1) = \frac{1}{1-1.7+0.72} = 50$$

  - The (sometimes complex) partial fraction expansion will be avoided using an Inverse $\mathcal{Z}$-Transform formula.

Inverting $\mathcal{Z}$-Transforms Using Contour Integration

- If we make the substitution:

  $$z = e^{j\omega\Delta} \quad \Rightarrow \quad Y_p(\omega) = \sum_{k=-\infty}^{\infty} y_k z^{-k} \triangleq Y(z)$$

  And also $\quad \mathrm{d}z = j\Delta e^{j\omega\Delta}\mathrm{d}\omega = j\Delta z \mathrm{d}\omega \quad \Rightarrow \quad \mathrm{d}\omega = \dfrac{\mathrm{d}z}{zj\Delta}$

  Substituying these expressions in the IDFT:

  $$y_k = \frac{\Delta}{2\pi}\int_0^{\frac{2\pi}{\Delta}} Y_p(\omega)e^{j\omega k\Delta}\,\mathrm{d}\omega = \frac{\Delta}{2\pi}\oint_C Y(z)z^k\,\frac{\mathrm{d}z}{zj\Delta}$$

  where the contour C is the unit circle, and simplifying:

  $$y_k = \frac{1}{2\pi j}\oint_C \frac{Y(z)}{z} z^k \mathrm{d}z$$

  This formula is known as the **Inverse $\mathcal{Z}$-Transform**.

Inverting $\mathcal{Z}$-Transforms Using Contour Integration

- We recall the *Cauchy's Residue Theorem*:

  $$\frac{1}{2\pi j}\oint_C f(z)dz = \sum_{n=1}^{m} \operatorname*{Res}_{z=p_n} F(z)$$

  Where $\{p_n\}$ are the m poles of $F(z)$ inside C, and $\operatorname{Res}_{z=p_k} F(z)$ is the *residue* of $F(z)$ at p_k.
- Applying this theorem to the Inverse $\mathcal{Z}$-Transform:

  $$y_k = \frac{1}{2\pi j}\oint_C \frac{Y(z)}{z}z^k\mathrm{d}z = \sum_{n=1}^{m} \operatorname*{Res}_{z=p_n} \frac{Y(z)}{z}z^k$$

  Where $\{p_n\}$ are the m poles of $Y(z)/z$ inside the unit circle. If every pole p_n only occurs once:

  $$y_k = \sum_{n=1}^{m} \operatorname*{Res}_{z=p_n} \frac{Y(z)}{z}z^k = \sum_{n=1}^{m} \frac{(z-p_n)Y(z)}{z}\Bigg|_{z=p_n} (p_n)^k$$

- **Example 1**
  Consider the $\mathcal{Z}$-Transform:

$$Y(z) = \frac{z^2}{(z-0.8)(z-0.9)(z-1)}$$

By using the Contour integral inversion formula:

$$\begin{aligned}
y_k &= \frac{1}{2\pi j}\oint_C \frac{z^{k+1}}{(z-0.8)(z-0.9)(z-1)}\,\mathrm{d}z \\
&= \sum_{n=1}^{3} \operatorname*{Res}_{z=p_k} \frac{z^{k+1}}{(z-0.8)(z-0.9)(z-1)} \\
&= \left.\frac{z^{k+1}}{(z-0.9)(z-1)}\right|_{z=0.8} + \left.\frac{z^{k+1}}{(z-0.8)(z-1)}\right|_{z=0.9} + \left.\frac{z^{k+1}}{(z-0.8)(z-0.9)}\right|_{z=1} \\
&= \frac{0.8(0.8)^k}{(0.8-0.9)(0.8-1)} + \frac{0.9(0.9)^k}{(0.9-0.8)(0.9-1)} + \frac{1}{(1-0.8)(1-0.9)} \\
&= 40(0.8)^k - 90(0.9)^k + 50 \quad ; \ k \geq 0
\end{aligned}$$

Inverting $\mathcal{Z}$-Transforms Using Contour Integration

- From the previous example, we observe that if the poles in $Y(z)/z$ only occur once, then y_k can be obtained applying a simple set of rules:
  - Cancel each pole one by one
  - When you cancel a pole at p_n, substitute p_n into what is left.
  - Multiply this number by p_n^k
  - Add all these results up and you've found the Inverse $\mathcal{Z}$-Transform.

Inverting $\mathcal{Z}$-Transforms Using Contour Integration

- **Example 2**
  Consider the $\mathcal{Z}$-Transform:

$$Y(z) = \frac{z-1.5}{z^2-0.7z+0.1}$$

So:

$$\frac{Y(z)}{z} = \frac{(z-1.5)}{z(z-0.2)(z-0.5)}$$

This has poles at $\{0, 0.2, 0.5\}$. Therefore, applying the procedure:

$$\begin{aligned}
y_k &= \frac{(0-1.5)0^k}{(0-0.2)(0-0.5)} + \frac{(0.2-1.5)(0.2)^k}{0.2(0.2-0.5)} + \frac{(0.5-1.5)(0.5)^k}{(0.5)(0.5-0.2)} \\
&= -15(0)^k + 21.67(0.2)^k - 6.67(0.5)^k
\end{aligned}$$

Now $0^0 = 1$ and $0^k = 0$ for $k > 0$ so the final answer is:

$$y_k = \begin{cases} -15 + 21.67 - 6.67 = 0 & ; \ k = 0 \\ (21.67)(0.2)^k - 6.67(0.5)^k & ; \ k \geq 1 \end{cases}$$

- **Example 3**:
  Consider the $\mathcal{Z}$-Transform:

$$Y(z) = \frac{z^2}{(z^2 - 1.2944z + 0.64)(z-1)}$$

So:
$$\frac{Y(z)z^k}{z} = \frac{z^{k+1}}{(z-0.8e^{j\frac{\pi}{5}})(z-0.8e^{-j\frac{\pi}{5}})(z-1)}$$

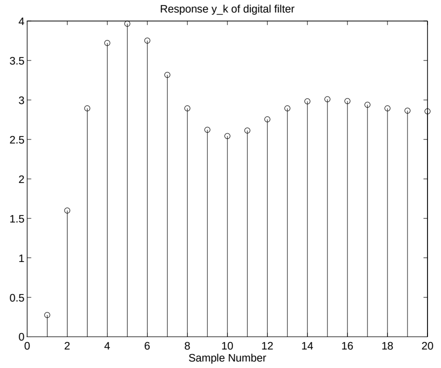
Using Contour integration the inverse $\mathcal{Z}$-transform is:

$$\begin{aligned} y_k &= \frac{1}{2\pi j}\oint_C \frac{Y(z)z^k}{z}dz \\ &= \operatorname*{Res}_{z=0.8e^{j\frac{\pi}{5}}} \frac{Y(z)z^k}{z} + \operatorname*{Res}_{z=0.8e^{-j\frac{\pi}{5}}} \frac{Y(z)z^k}{z} + \operatorname*{Res}_{z=1} \frac{Y(z)z^k}{z} \\ &= (0.8)^{k+1}\left[\frac{e^{j(k+1)\frac{\pi}{5}}}{(j1.6\sin\frac{\pi}{5})(0.8e^{j\frac{\pi}{5}}-1)} + \frac{e^{-j(k+1)\frac{\pi}{5}}}{(-j1.6\sin\frac{\pi}{5})(0.8e^{-j\frac{\pi}{5}}-1)}\right] + \frac{1}{|1-0.8e^{j\frac{\pi}{5}}|^2} \end{aligned}$$

Inverting $\mathcal{Z}$-Transforms Using Contour Integration

- **Example 3**: (continued)
  Which, after some manipulations, yields:

$$y_k = 4.923(0.8)^k\left\{0.8\sin\frac{\pi k}{5} - \sin\frac{\pi(k+1)}{5}\right\} + 2.8938$$



Inverting $\mathcal{Z}$-Transforms Using Contour Integration

- **Repeated Poles**:
  Suppose $Y(z)z^k/z$ has a m poles at $z = p$, then the residue at $z = p$ is given by:

$$\operatorname*{Res}_{z=p}\frac{Y(z)z^k}{z} = \frac{1}{(m-1)!}\left(\frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}(z-p)^m\frac{Y(z)z^k}{z}\right)\Bigg|_{z=p}$$

  - When $m = 1$ this reduces to the previous expression and set of rules.
  - When $m > 1$, after cancelling the poles at p we have to differentiate $m-1$ times **before** substituting $z = p$.

- **Repeated Poles**:

Example. Consider the $\mathcal{Z}$-Transform:

$$Y(z) = \frac{z}{(z+0.2)(z^2-0.4z+0.04)}$$

Then:

$$\frac{Y(z)z^k}{z} = \frac{z^k}{(z+0.2)(z-0.2)^2}$$

So:

$$\begin{aligned}
y_k = \frac{1}{2\pi j}\oint_C \frac{Y(z)z^k dz}{z} &= \operatorname*{Res}_{z=-0.2} \frac{z^k}{(z+0.2)(z-0.2)^2} + \operatorname*{Res}_{z=0.2} \frac{z^k}{(z+0.2)(z-0.2)^2} \\
&= \left.\frac{z^k}{(z-0.2)^2}\right|_{z=-0.2} + \left.\frac{d}{dz}\frac{z^k}{(z+0.2)}\right|_{z=0.2} \\
&= 6.25(-0.2)^k + \left.\frac{(z+0.2)kz^{k-1}-z^k}{(z+0.2)^2}\right|_{z=0.2} \\
&= 6.25(-0.2)^k - 6.25(0.2)^k + 2.5k(0.2)^{k-1} \\
&= 6.25\left[(-0.2)^k - (0.2)^k + 2k(0.2)^k\right] \\
&= 6.25\left[(-0.2)^k + (2k-1)(0.2)^k\right]
\end{aligned}$$