Tutorial #2 Solutions

1/ (a)

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

Observer Canonical Form:

$$A = \begin{bmatrix} -5 & 1 \\ -3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Controller Canonical Form:

$$A = \begin{bmatrix} -5 & -3 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

(b)

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

Observer Canonical Form:

$$A = \begin{bmatrix} -5 & 1 \\ -3 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -11 \\ -8 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 3$$

Controller Canonical Form

$$A = \begin{bmatrix} -5 & -3 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} -11 & -8 \end{bmatrix} \quad D = 3$$

(c)

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

Observer Canonical Form:

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \\ -3 & 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Controller Canonical Form:

$$A = \begin{bmatrix} -1 & -2 & -5 & -3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 4 & 1 \end{bmatrix}$$

(d)

$$\frac{d}{dt}x(t) = ax(t) + bu(t)$$

$$y(t) = cx(t)$$

Observer Canonical Form:

$a = -2 \qquad b = 1 \qquad c = 1$

Controller Canonical Form:

$a = -2 \qquad b = 1 \qquad c = 1$

2/ Observer Canonical Form for differential equation:

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$

$$A = \begin{bmatrix} -8 & 1 \\ -15 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} -(\alpha+\beta) & 1 \\ -\alpha\beta & 0 \end{bmatrix} \quad \text{with} \quad \alpha = 5 \quad \beta = 3$$

Therefore

$$e^{At} = \frac{1}{(5-3)} \begin{bmatrix} 5e^{-5t} - 3e^{-3t} & e^{-3t} - e^{-5t} \\ 15(e^{-5t} - e^{-3t}) & 5e^{-3t} - 3e^{-5t} \end{bmatrix}$$

and hence:

$$h(t) = Ce^{At}B$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 5e^{-5t} - 3e^{-3t} & e^{-3t} - e^{-5t} \\ 15(e^{-5t} - e^{-3t}) & 5e^{-3t} - 3e^{-5t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2}\left(e^{-3t} - e^{-5t}\right)$$

Also, from lectures

$$\begin{bmatrix} y(0) \\ \frac{d}{dt}y(0) \end{bmatrix} = \mathcal{O}x_0 + \Gamma \begin{bmatrix} u(0) \\ \frac{d}{dt}u(0) \end{bmatrix}$$

Where

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -8 & 1 \end{bmatrix}$$

Therefore, since $u(t) = s(t-1)$ implies $u(0) = \frac{d}{dt}u(0) = 0$ then

$$x_0 = \begin{bmatrix} 1 & 0 \\ -8 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -3 \end{bmatrix}$$

so that

$$Ce^{At}x_0 = \frac{1}{2}\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 5e^{-5t} - 3e^{-3t} & e^{-3t} - e^{-5t} \\ 15(e^{-5t} - e^{-3t}) & 5e^{-3t} - 3e^{-5t} \end{bmatrix} \begin{bmatrix} -0.5 \\ -3 \end{bmatrix}$$

$$= \frac{0.5e^{-5t} - 1.5e^{-3t}}{2}.$$

The differential equation solution therefore is

$$y(t) = \frac{(0.5e^{-5t} - 1.5e^{-3t})}{2} + \frac{1}{2}\int_0^t \left(e^{-3(t-\sigma)} - e^{-5(t-\sigma)}\right) s(\sigma - 1)\, d\sigma$$

Now

$$\int_0^t \left(e^{-3(t-\sigma)} - e^{-5(t-\sigma)}\right) s(\sigma-1)\,d\sigma = 0 \quad ; t < 1$$

while for $t > 1$ it is equal to

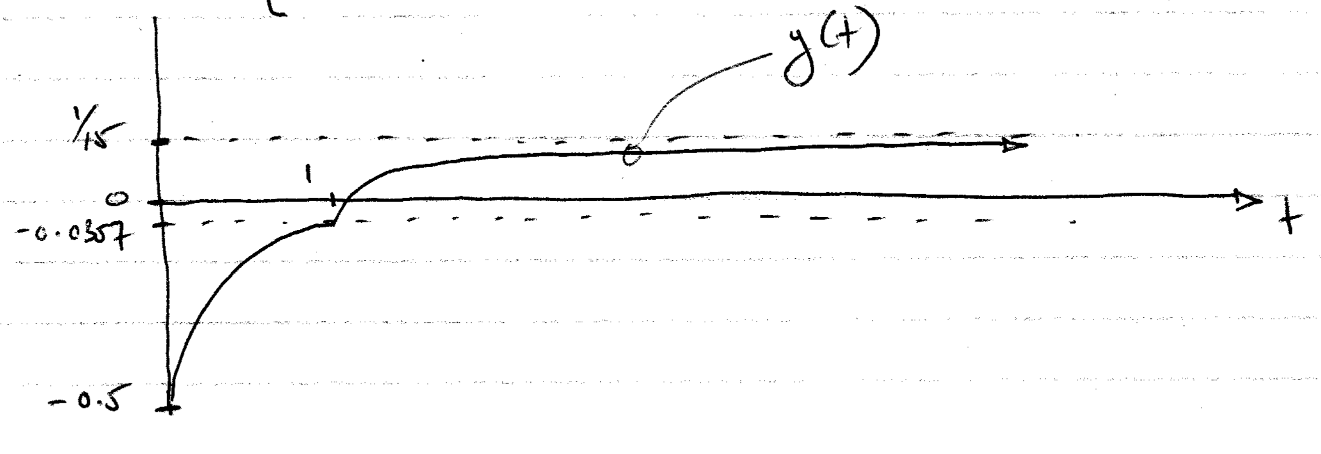
$$\int_1^t \left(e^{-3(t-\sigma)} - e^{-5(t-\sigma)}\right) d\sigma$$

$$= \frac{1}{3} e^{-3(t-\sigma)}\Big|_{\sigma=1}^{\sigma=t} - \frac{1}{5} e^{-5(t-\sigma)}\Big|_{\sigma=1}^{\sigma=t}$$

$$= \frac{1}{3}\left[1 - e^{-3(t-1)}\right] - \frac{1}{5}\left[1 - e^{-5(t-1)}\right]$$

Therefore

$$y(t) = \begin{cases} 0.25e^{-5t} - 0.75e^{-3t} & ; t \in [0,1) \\ 0.25e^{-5t} - 0.75e^{-3t} + \frac{1}{6}\left(1 - e^{-3(t-1)}\right) - \frac{1}{10}\left(1 - e^{-5(t-1)}\right) & ; t \geq 1 \end{cases}$$

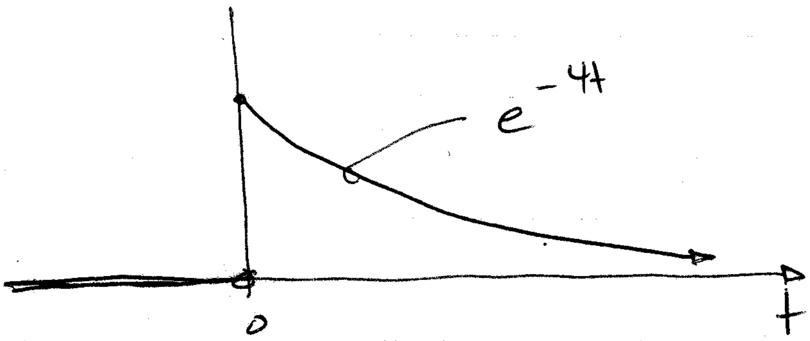


3/ (a) The differential equation has solution

$$y(t) = e^{-4t} y_0 + \int_{t_0}^{t} e^{-4(t-\sigma)} u(\sigma)\, d\sigma$$

Therefore, setting $u(\sigma) = \delta(\sigma)$ and $t_0 = -\infty$ gives impulse response

$$y(t) = h(t) = \begin{cases} e^{-4t} & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$

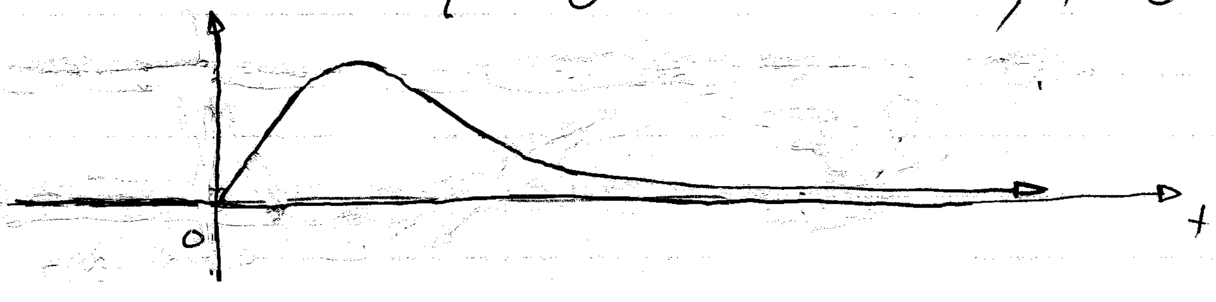


(b) As just computed in question 2/, this differential equation has solution

$$y(t) = Ce^{At}x_0 + \int_{t_0}^{t} h(t-\sigma)\, u(\sigma)\, d\sigma$$

where the impulse response $h(t)$ is

$$h(t) = Ce^{At}B = \begin{cases} \tfrac{1}{2}(e^{-3t} - e^{-5t}) & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$$



(c) The differential equation has observer canonical form realisation

$$A = \begin{bmatrix} -5 & 1 \\ -6 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Then

$$A = \begin{bmatrix} -(\alpha+\beta) & 1 \\ -\alpha\beta & 0 \end{bmatrix} \qquad \alpha = 3 \qquad \beta = 2$$

So using the methods of question 2 & the previous question, for $t \geq 0$

$$h(t) = Ce^{At}B$$

$$= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3e^{-3t} - 2e^{-2t} & e^{-2t} - e^{-3t} \\ 6\left[e^{-3t} - e^{-2t}\right] & 3e^{-2t} - 2e^{-3t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= 2e^{-3t} - e^{-2t}$$

and $h(t) = 0$ for $t < 0$