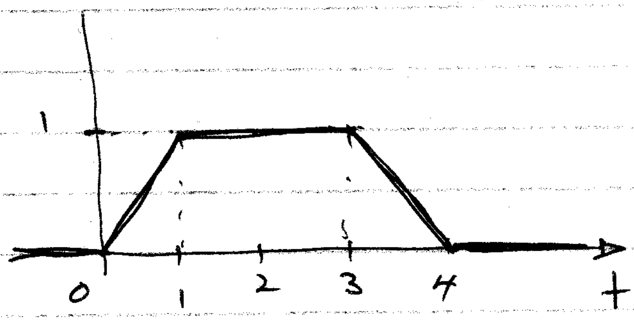
Tutorial #4 Solutions

1/ (a)



$$f(t) = t \cdot 1(t) - (t-1)\,1(t-1) - (t-3)\,1(t-3) + (t-4)\,1(t-4)$$

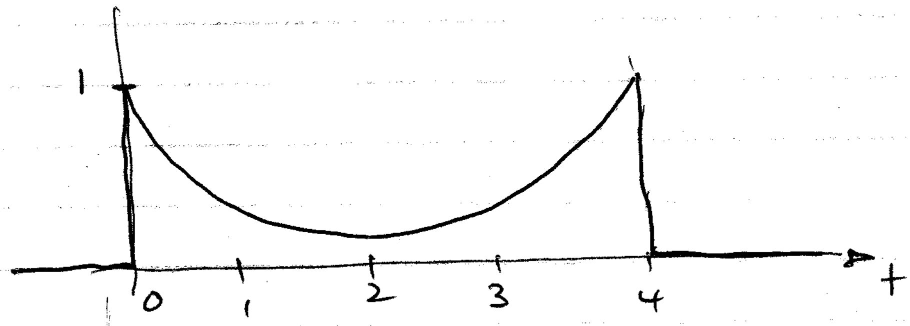
$$\mathcal{L}\{1(t)\, t^n e^{\alpha t}\} = \frac{n!}{(s-\alpha)^{n+1}}$$

$$\mathcal{L}\{g(t-T)\} = e^{-st}\,\mathcal{L}\{g(t)\}$$

Using these principles

$$F(s) = \mathcal{L}\{f\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-3s}}{s^2} + \frac{e^{-4s}}{s^2}$$

(b)



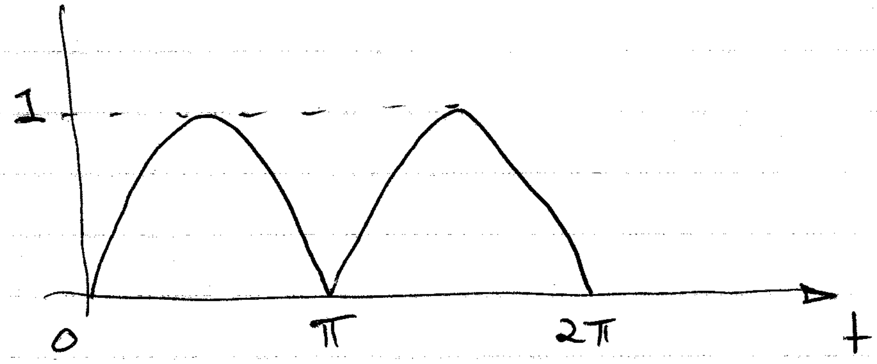
$$f(t) = e^{-t} \cdot 1(t) + \underbrace{\left(e^{t-4} - e^{-t}\right) 1(t-2)} - e^{t-4}\, 1(t-4)$$

$$\mathcal{L}\{e^{\alpha t}\} = \frac{1}{s-\alpha}; \qquad \underbrace{e^{-2} e^{t-2}\, 1(t-2)} + \underbrace{e^{-2} e^{-(t-2)}\, 1(t-2)}$$

So

$$F(s) = \frac{1}{s+1} + \frac{e^{-2} e^{-2s}}{s-1} - \frac{e^{-2}}{s+1} e^{-2s} - \frac{e^{-4s}}{s-1}$$

(c)



$$f(t) = \sin t \,.\, 1(t) - 2\sin t \; 1(t-\pi) + \sin t \,.\, 1(t-2\pi)$$

$$= \sin t \,.\, 1(t) + 2\sin(t-\pi) + \sin(t-2\pi) . 1(t-2\pi)$$

$$\mathcal{L}\{1(t)\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

So

$$\mathcal{L}\{f\} = \frac{1}{s^2+1}\left[1 + 2e^{-\pi s} + e^{-2\pi s}\right]$$

2/ (a)

$$\mathcal{L}\{f(3t)\} = \frac{1}{3} F\left(\frac{s}{3}\right)$$

$$= \frac{1}{3}\left\{\frac{s/3 + 1}{(s/3)^2 + 5s/3 + 7}\right\}$$

$$= \frac{1}{3}\left\{\frac{3s+9}{s^2+15s+63}\right\}$$

$$= \frac{s+3}{s^2+15s+63}$$

So

$$\mathcal{L}\{f[3(t-\tfrac{4}{3})]\} = e^{-\frac{4s}{3}}\left[\frac{s+3}{s^2+15s+63}\right]$$

(b) $g(t) = t f(t) \Rightarrow G(s) = -\dfrac{d}{ds} F(s)$

So $G(s) = -\dfrac{d}{ds}\left\{\dfrac{s+1}{s^2+5s+7}\right\} = \dfrac{s^2+2s-2}{(s^2+5s+7)^2}$

(c) $g(t) = \dfrac{d^2}{dt^2} f(t) \Rightarrow G(s) = s^2 F(s) - s f(0) - \dot{f}(0)$

Via initial value Theorem $f(0) = \lim\limits_{s\to\infty} sF(s) = 1$

Also $\mathcal{L}\{\dot{f}\} = sF(s) - f(0) = sF(s) - 1 = \dfrac{-4s-7}{s^2+5s+7}$

Hence by initial value theorem

$$\dot{f}(0) = \lim_{s\to\infty} \frac{-4s^2-7s}{s^2+5s+7} = -4$$

So Finally

$$G(s) = \frac{s^2(s+1)}{s^2+5s+7} - s + 4 = \frac{13s+28}{s^2+5s+7}$$

(d) $G(s) = \dfrac{1}{s} F(s) = \dfrac{s+1}{s^3+5s^2+7s}$

(e) $g(t) = \sin 2t\, f(t)$

$$= \left(\frac{e^{2jt} - e^{-2jt}}{2j}\right) f(t)$$

$$= \frac{1}{2j} e^{2jt} f(t) - \frac{1}{2j} e^{-2jt} f(t)$$

Now $\mathcal{L}\{e^{\alpha t} f(t)\} = \int e^{\alpha t} f(t) e^{-st} dt = \int f(t) e^{(\alpha - s)t} dt = F(s-\alpha)$

So

$$\mathcal{L}\{g\} = \frac{1}{2j}\left[F(s-2j) - F(s+2j) \right]$$

$$= \frac{j}{2}\left[F(s+2j) - F(s-2j) \right]$$

$$= \frac{2(s^2+2s+2)}{s^4+10s^3+47s^2+110s+109}$$

(f) By fact that $\mathcal{L}\{e^{\alpha t} f(t)\} = F(s-\alpha)$

$$G(s) = F(s+3) = \frac{s+4}{s^2+11s+31}$$

(g) $$G(s) = F(s).F(s) = \left(\frac{s+1}{s^2+5s+7} \right)^2$$

3 (a) $$F(s) = \frac{s+2}{(s+3)(s+4)}$$, poles at $s=-3$, $s=-4$

So $$f(t) = \left(\operatorname*{Res}_{s=-3} F(s)\, e^{-st} + \operatorname*{Res}_{s=-4} F(s)\, e^{-st} \right).1(t)$$

$$= \left[\left(\frac{-3+2}{-3+4} \right) e^{-3t} + \frac{(-4+2)}{(-4+3)} e^{-4t} \right].1(t)$$

$$= \left(2e^{-4t} - e^{-3t} \right).1(t)$$

(b)

$$F(s) = \frac{s+1}{s(s+\beta)(s+\bar{\beta})} \qquad \beta = 2.5 + j\frac{\sqrt{3}}{2}$$

So

$$f(t) = \frac{0+1}{\beta \cdot \bar{\beta}} + \frac{(1-\beta)}{\beta(\bar{\beta}-\beta)} e^{-\beta t} + \frac{(1-\bar{\beta})}{\bar{\beta}(\beta-\bar{\beta})} e^{-\bar{\beta} t}$$

$$= \frac{1}{|\beta|^2} + \frac{1}{(\bar{\beta}-\beta)} \left[\frac{(1-\beta)}{\beta} e^{-\beta t} + \frac{(1-\bar{\beta})}{\bar{\beta}} e^{-\bar{\beta} t} \right]$$

~~Now~~

$$= \frac{1}{7} + \frac{1}{\sqrt{7}} e^{j1.76} e^{-(2.5+j\sqrt{3}/2)t} + \frac{1}{\sqrt{7}} e^{-j1.76} e^{-(2.5-j\sqrt{3}/2)t}$$

$$= \frac{1}{7} + \frac{2}{\sqrt{7}} e^{-2.5t} \cos\left(\frac{\sqrt{3}}{2}t - 1.76\right)$$

(c) $$F(s) = \frac{3s^2+2s+1}{(s+1)(s+2)^2}$$

So

$$f(t) = \left(\frac{3(-1)^2+2(-1)+1}{(-1+2)^2}\right) e^{-t} + \frac{d}{ds}\left\{\left(\frac{3s^2+2s+1}{s+1}\right)e^{st}\right\}\Bigg|_{s=-2}$$

$$= 2e^{-t} + \left(\frac{3s^2+2s+1}{s+1}\right) te^{-st}\Bigg|_{s=-2} + \left(\frac{(s+1)(6s+2)-(3s^2+2s+1)}{(s+1)^2}\right)e^{st}\Bigg|_{s=-2}$$

$$= \left[2e^{-t} - 9te^{-2t} + e^{-2t}\right] . 1(t)$$

(d) $$F(s) = \frac{s^2+1}{s^2+7s+12} = 1 - \frac{(7s+11)}{(s+4)(s+3)}$$

So $$f(t) = \delta(t) - \frac{7(-4)+11}{(-4+3)} e^{-4t} - \frac{7(-3)+11}{(-3+4)} e^{-3t}$$

$$= \delta(t) + \left(10e^{-3t} - 17e^{-4t}\right).1(t)$$

(e)

$$F(s) = \frac{s}{s^2+3s+2} + \frac{e^{-s}}{s^2+3s+2}$$

$$= \frac{s}{(s+2)(s+1)} + \frac{e^{-s}}{(s+2)(s+1)}$$

$$\Rightarrow f(t) = 1(t).\left[\frac{(-2)}{(-2+1)} e^{-2t} + \frac{(-1)}{(-1+2)} e^{-t}\right] + \left[\frac{e^{-2t}}{(-2+1)} + \frac{e^{-t}}{(-1+2)}\right]\Bigg|_{t=t-1} .1(t-1)$$

$$= \left(2e^{-2t} - e^{-t}\right)1(t) + \left[e^{-(t-1)} - e^{-2(t-1)}\right].1(t-1)$$

(4) (a)

$$H(s) = \frac{5s+1}{s^2+7s+10} = \frac{5s+1}{(s+5)(s+2)}$$

(b) $$h(t) = \mathcal{L}^{-1}\{H\}$$

$$= \left[\frac{5(-5)+1}{(-5+2)} e^{-5t} + \frac{5(-2)+1}{(-2+5)} e^{-2t}\right].1(t)$$

$$= \left[8e^{-5t} - 3e^{-2t}\right].1(t)$$

5/ (a) $\ddot{y} + 4\dot{y} + 3y = u$

$\Rightarrow \quad s^2Y - sy(0) - \dot{y}(0) + 4sY - 4y(0) + 3Y = U$

$$Y(s) = \frac{1}{s^2+4s+3}U(s) + \frac{(s+4)y(0)+\dot{y}(0)}{s^2+4s+3}$$

$u(t) = 0, \quad y(0) = -2, \quad \dot{y}(0) = 1 \quad \Rightarrow$

$$Y(s) = \frac{-2(s+4)+1}{(s^2+4s+3)} = \frac{-2s-7}{(s+3)(s+1)}$$

So $$y(t) = \left[\frac{(-2)(-3)-7}{(-3+1)}e^{-3t} + \frac{(-2)(-1)-7}{(-1+3)}e^{-t}\right].1(t)$$

$$= \left[\frac{1}{2}e^{-3t} - \frac{5}{2}e^{-t}\right].1(t)$$

(b) $u(t) = \delta(t) \Rightarrow U(s) = 1, \quad y(0) = 0, \quad \dot{y}(0) = 0 \quad \Rightarrow$

$$Y(s) = \frac{1}{(s+3)(s+1)}$$

$$\Rightarrow y(t) = \left[\frac{1}{(-1+3)}e^{-t} + \frac{1}{(-3+1)}e^{-3t}\right].1(t)$$

$$= \frac{1}{2}\left[e^{-t} - e^{-3t}\right].1(t)$$

(c) $u(t) = 1(t) \Rightarrow U(s) = 1/s$ so

$$Y(s) = \frac{1}{s(s+3)(s+1)}$$

$$\Rightarrow y(t) = \left[\frac{1}{(3)(1)} + \frac{e^{-3t}}{(-3)(-3+1)} + \frac{e^{-t}}{(-1)(-1+3)}\right] \cdot 1(t)$$

$$= \left[\frac{1}{3} + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t}\right] \cdot 1(t)$$

(d) Combine natural response of part (a) with forced response of part (d)

$$y(t) = \underbrace{\left[\frac{1}{3} + \frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t}\right] \cdot 1(t)}_{\text{Forced response}} + \underbrace{\left[\frac{1}{2}e^{-3t} - \frac{5}{2}e^{-t}\right] 1(t)}_{\text{Natural Response.}}$$