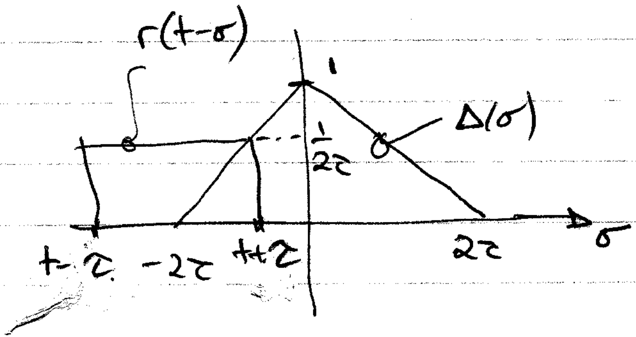
$$= \frac{e^{-j2\omega\tau} - e^{j2\omega\tau}}{-j\omega} + \frac{2j\omega\tau e^{j2\omega\tau} - e^{j2\omega\tau} + 1}{2\tau\omega^2} + \frac{1 - 2j\omega\tau e^{-j2\omega\tau} - e^{-j2\omega\tau}}{2\tau\omega^2}$$

$$= \frac{2 - \left(e^{j2\omega\tau} + e^{-j2\omega\tau}\right)}{2\tau\omega^2}$$

$$= \frac{2 - 2\cos 2\omega\tau}{2\tau\omega^2} = \frac{4\sin^2\omega\tau}{2\tau\omega^2} = 2\tau\left(\frac{\sin\omega\tau}{\omega\tau}\right)^2$$

3/ Already answered in "fast way" version of previous question.

4/ $$x(t) = \int_{-\infty}^{\infty} \Delta(\sigma)\, r(t-\sigma)\, d\sigma$$



When $|t| \geq 3\tau$, $x(t) = 0$

Hence for $-3\tau < t \leq -\tau$

$$x(t) = \frac{1}{2\tau}\int_{-2\tau}^{t+\tau}\left(1 + \frac{\sigma}{2\tau}\right) d\sigma = \frac{1}{2\tau}\left[\sigma + \frac{\sigma^2}{4\tau}\right]_{-2\tau}^{t+\tau}$$

$$= \frac{1}{2\tau}\left[+2\tau - \frac{4\tau^2}{4\tau} + (t+\tau) + \frac{t^2 + 2t\tau + \tau^2}{4\tau}\right]$$

$$= \frac{1}{8\tau^2}(t + 3\tau)^2$$

When $-\tau \leq t \leq \tau$

$$x(t) = \frac{1}{2\tau}\int_{t-\tau}^{0}\left(1+\frac{\sigma}{2\tau}\right)d\sigma + \frac{1}{2\tau}\int_{0}^{t+\tau}\left(1-\frac{\sigma}{2\tau}\right)d\sigma$$

$$= \frac{1}{2\tau}\left[\sigma+\frac{\sigma^2}{4\tau}\right]_{t-\tau}^{0} + \frac{1}{2\tau}\left[\sigma-\frac{\sigma^2}{4\tau}\right]_{0}^{t+\tau}$$

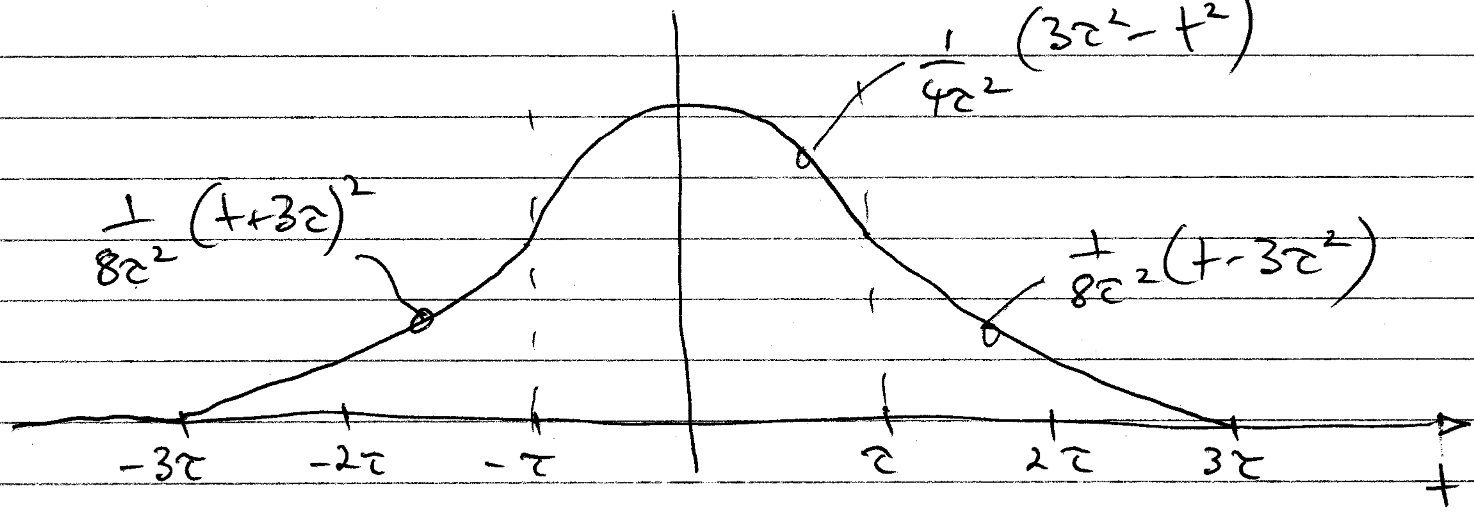
$$= \frac{1}{2\tau}\left[-t+\tau-\frac{(t-\tau)^2}{4\tau}+(t+\tau)-\frac{(t+\tau)^2}{4\tau}\right]$$

$$= \frac{1}{4\tau^2}\left(3\tau^2-t^2\right)$$

Similarly, for $t \in [\tau, 3\tau]$

$$x(t) = \frac{1}{8\tau^2}\left(t-3\tau^2\right)$$

Hence



Since $x(t) = [r \circledast \Delta](t)$

Then $X(\omega) = R(\omega)\,\Delta(\omega) = 2\tau\left(\dfrac{\sin\omega\tau}{\omega\tau}\right)^3$

5/ Consider $r(t) = \begin{cases} 1/\tau & ; \ |t| \leq \tau/2 \\ 0 & ; \ |t| > \tau/2 \end{cases}$

Then $R(\omega) = \displaystyle\int_{-\tau/2}^{\tau/2} e^{-j\omega t}\,dt = \frac{e^{-j\omega\tau} - e^{j\omega\tau}}{-j\omega} = \frac{2\sin\omega\tau/2}{\omega\tau}$

Furthermore

$$g(t) = A\tau\left[r(t+T) + r(t-T)\right]$$

Hence

$$G(\omega) = 2A\tau\,\frac{\sin\omega\tau/2}{\omega\tau}\left[e^{j\omega T} + e^{-j\omega T}\right]$$

$$= \frac{4A\tau}{\omega}\cos\omega T \cdot \sin\left(\frac{\omega\tau}{2}\right)$$

Similarly $f(t) = A\tau\left[r(t+2T) + r(t-2T) + 2r(t)\right]$

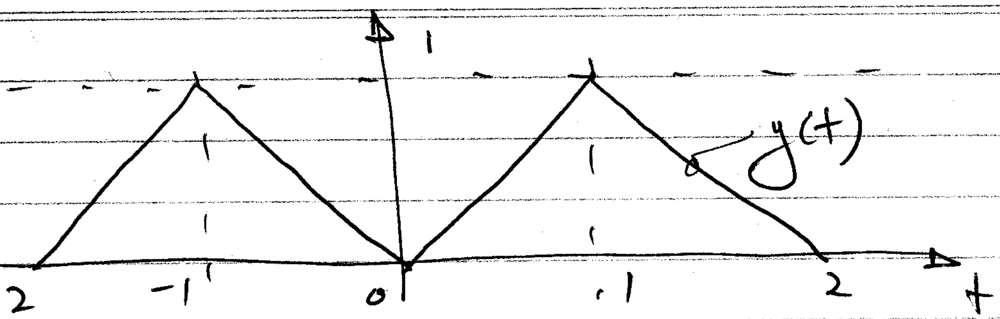
Hence $F(\omega) = A\tau\,R(\omega)\left[e^{j2T\omega} + e^{-j2T\omega} + 2\right]$

$$= A\tau\,R(\omega)\left[2\cos(2T\omega) + 2\right]$$

$$= 2A\tau\,R(\omega)\cos^2(T\omega)$$

$$= 4A\tau\left(\frac{\sin\omega\tau/2}{\omega\tau}\right)\cos^2(T\omega)$$

6/



$$y(t) = \Delta(t+1)\Big|_{\tau=\frac{1}{2}} + \Delta(t-1)\Big|_{\tau=1/2}$$

Where $$\Delta(\omega)\Big|_{\tau=\frac{1}{2}} = \frac{4}{\omega^2}\sin^2\left(\frac{\omega}{\tau}\right)$$

Hence $$Y(\omega) = \frac{4}{\omega^2}\sin^2\left(\frac{\omega}{\tau}\right)\left[e^{j\omega} + e^{-j\omega}\right]$$

$$= \frac{8}{\omega^2}\sin^2\left(\frac{\omega}{2}\right)\cos\omega$$

7/ (a) Let $$r(t) = \begin{cases} 1/\tau & ; \ |t| \leq \tau/2 \\ 0 & ; \ |t| > \tau/2 \end{cases}$$

Then $$R(\omega) = \frac{2\sin(\omega\tau/2)}{\omega\tau}$$

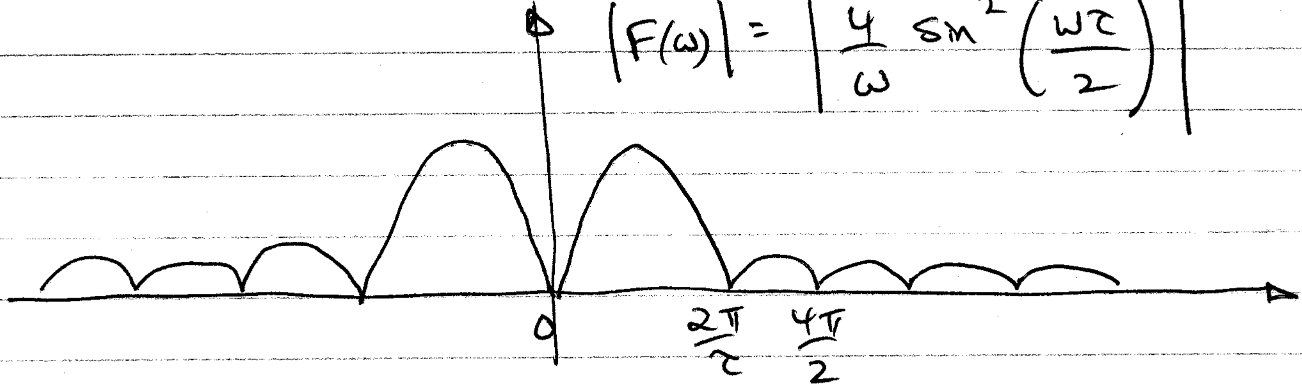
Also $$f(t) = \tau r(t-\tau/2) - \tau r(t - 3\tau/2)$$

Hence $$F(\omega) = \frac{2\tau\sin(\omega\tau/2)}{\omega\tau}\left[e^{-j\omega\tau/2} - e^{-j\frac{3}{2}\omega\tau}\right]$$

$$= \frac{4j}{\omega}e^{-j\tau\omega}\sin^2\left(\frac{\omega\tau}{2}\right)$$

(6)

$$|F(\omega)| = \left|\frac{4}{\omega}\sin^2\left(\frac{\omega\tau}{2}\right)\right|$$



(b) $$x(t) = \sum_{k=-\infty}^{\infty} f(t - 2\tau k)$$

Hence $$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_s k t} \quad ; \quad \omega_s = \frac{2\pi}{2\tau} = \frac{\pi}{\tau}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{j\frac{\pi k t}{\tau}}$$

where $$c_k = \frac{1}{2\tau}\int_0^{2\tau} f(t) e^{-j\frac{\pi k}{\tau}t}\, dt = \frac{1}{2\tau} F\left(\frac{\pi k}{\tau}\right)$$

where $F(\omega) = \frac{4j}{\omega} e^{-j\tau\omega} \sin^2\left(\frac{\omega\tau}{2}\right)$ was derived in part (a)

Hence $$x(t) = \sum_{k=-\infty}^{\infty} \frac{1}{2\tau} F\left(\frac{\pi k}{\tau}\right) e^{j\frac{\pi k t}{\tau}}$$

(c) $$X(\omega) = \sum_{k=-\infty}^{\infty} 2\pi c_k\, \delta(\omega - k\pi/\tau)$$

$$= \sum_{k=-\infty}^{\infty} \frac{\pi}{\tau} F\left(\frac{\pi k}{\tau}\right) \delta\left(\omega - \frac{\pi k}{\tau}\right)$$

8) $\omega_s = \frac{2\pi}{2} = \pi$

Hence $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\pi k t}$ where

$$c_k = \frac{1}{2}\int_{-1}^{1} e^{-2|t|} e^{-j\pi k t}\, dt$$

$$= \frac{1}{2}\int_{-1}^{0} e^{2t} e^{-j\pi k t}\, dt + \frac{1}{2}\int_{0}^{1} e^{-2t} e^{-j\pi k t}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{2t-jk\pi t}}{2-jk\pi}\right]_{t=-1}^{t=0} + \frac{1}{2}\left[\frac{e^{-2t-j\pi k t}}{-2-j\pi k}\right]_{t=0}^{t=1}$$

$$= \frac{1}{2}\left[\left(\frac{1-e^{-2+jk\pi}}{2-jk\pi}\right) - \left(\frac{e^{-2-jk\pi}-1}{2+j\pi k}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1-e^{-2}(-1)^k}{2-jk\pi} + \frac{1-e^{-2}(-1)^k}{2+j\pi k}\right]$$

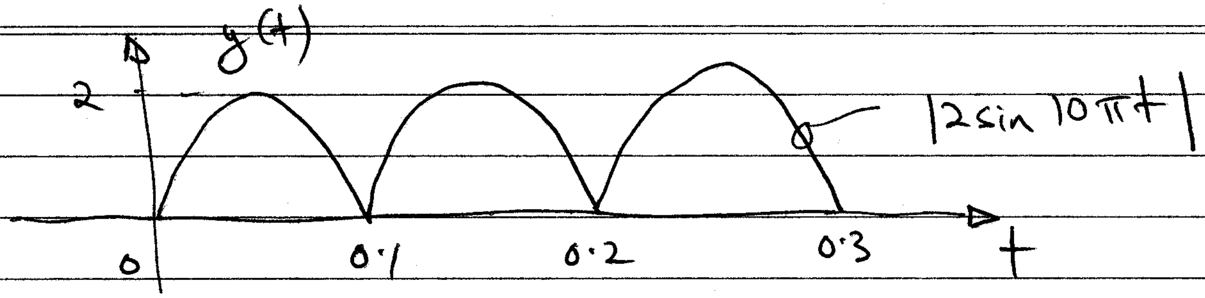
$$= \frac{1}{2}\left[\frac{(2+j\pi k)\left[1-e^{-2}(-1)^k\right] + (2-jk\pi)\left[1-e^{-2}(-1)^k\right]}{4-k^2\pi^2}\right]$$

$$= \frac{1}{2}\left[\frac{4-4e^{-2}(-1)^k}{4-k^2\pi^2}\right]$$

Hence

$$x(t) = \sum_{k=-\infty}^{\infty}\left(\frac{2-2e^{-2}(-1)^k}{4-k^2\pi^2}\right) e^{j\pi k t}$$

9/



$$\omega_s = \frac{2\pi}{0.1} = 20\pi$$

$$c_k = 10\int_0^{0.1} (2\sin 10\pi t)\, e^{-j20\pi k t}\, dt$$

$$= \frac{10}{j}\int_0^{0.1} \left(e^{j10\pi t} - e^{-j10\pi t}\right) e^{-20\pi k t}\, dt$$

$$= \frac{10}{j}\int_0^{0.1} e^{j10\pi(1-2k)t}\, dt - \frac{10}{j}\int_0^{0.1} e^{-j10\pi(1+2k)t}\, dt$$

$$= \frac{10}{j}\left[\frac{e^{j10\pi(1-2k)t}}{j10\pi(1-2k)}\right]_{t=0}^{t=0.1} + \frac{10}{j}\left[\frac{e^{-j10\pi(1+2k)t}}{10\pi(1+2k)}\right]_{t=0}^{t=0.1}$$

$$= \frac{-1}{\pi(1-2k)}\left[1 - e^{j\pi(1-2k)}\right] + \frac{1}{\pi(1+2k)}\left[1 - e^{-j\pi(1+2k)}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{1-2k} + \frac{1}{1+2k}\right]$$

$$= \frac{4}{\pi(1-4k^2)}$$

Hence $y(t) = \sum_{k=-\infty}^{\infty} \frac{4}{\pi(1-4k^2)} e^{j20\pi k t}$

Power up to & including 50 Hz = 100π rad/s = $5 \times 20\pi$ rad/s

$$= \sum_{k=-5}^{5} \left[\frac{4}{\pi(1-4k^2)} \right]^2$$

$$\approx 2 \text{ W} \qquad (1.999596 \text{ W})$$

Total power $= \frac{1}{0.1} \int_0^{0.1} y^2(t)\, dt$

$$= 10 \int_0^{0.1} 4 \sin^2 10\pi t \, dt$$

$$= 40 \int_0^{0.1} \left(\frac{1 - \cos 20\pi t}{2} \right) dt$$

$$= 40 \left[\frac{t}{2} - \frac{\sin 20\pi t}{40\pi} \right]_{t=0}^{t=0.1}$$

$$= 2 \text{ Watts}$$

Hence Fraction $\leq$ 50 Hz $= \frac{1.999596}{2} \times 100\% \approx 100\%$.