

# Tutorial # 7 Solutions

①

$$\sum_{k=0}^{\infty} \lambda^k = \frac{1}{1-\lambda} ; \quad \sum_{k=0}^{\infty} k \lambda^k = \lambda \frac{d}{d\lambda} \left\{ \sum_{k=0}^{\infty} \lambda^k \right\} = \frac{\lambda}{(1-\lambda)^2}$$

$$(a) \quad Y(z) = \sum_{k=0}^{\infty} (0.2z^{-1})^k = \frac{1}{1-0.2z^{-1}} = \frac{z}{z-0.2}$$

$$(b) \quad Y(z) = 0.4 \sum_{k=0}^{\infty} (e^{-0.1} z^{-1})^k = \frac{0.4}{1-e^{-0.1} z^{-1}} = \frac{0.4z}{z-0.90484}$$

$$(c) \quad Y(z) = 0.4 \sum_{k=0}^{\infty} (k) (e^{-0.1} z^{-1})^k = \frac{0.4 e^{-0.1} z^{-1}}{(1-e^{-0.1} z^{-1})^2} = \frac{0.36193z}{(z-0.90484)^2}$$

$$(d) \quad Y(z) = 3\Delta \sum_{k=0}^{\infty} \cos(0.1\Delta k) z^{-k}$$

$$= 3\Delta \sum_{k=0}^{\infty} \left\{ \frac{e^{j0.1\Delta k} + e^{-j0.1\Delta k}}{2} \right\} z^{-k}$$

$$= 1.5\Delta \sum_{k=0}^{\infty} (e^{j0.1\Delta} z^{-1})^k + 1.5\Delta \sum_{k=0}^{\infty} (e^{-j0.1\Delta} z^{-1})^k$$

$$= \frac{1.5\Delta}{1-e^{j0.1\Delta} z^{-1}} + \frac{1.5\Delta}{(1-e^{-j0.1\Delta} z^{-1})}$$

$$= \frac{z 1.5\Delta [2z - (e^{j0.1\Delta} + e^{-j0.1\Delta})]}{z^2 - (e^{j0.1\Delta} + e^{-j0.1\Delta})z + 1}$$

$$= \frac{z 3\Delta [z - \cos 0.1\Delta]}{z^2 - (2\cos 0.1\Delta)z + 1}$$

$$(e) \frac{d}{d\theta} \left\{ \sum_{k=0}^{\infty} (\sin \theta k) z^{-k} \right\} = \sum_{k=0}^{\infty} k (\cos \theta k) z^{-k}$$

And by a similar procedure to part (d)

$$\sum_{k=0}^{\infty} (\sin \theta k) z^{-k} = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}$$

$$\text{So } \sum_{k=0}^{\infty} k (\cos \theta k) z^{-k} = \frac{d}{d\theta} \left\{ \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1} \right\} = \frac{z(z^2+1) \cos \theta - 2z^2}{(z^2 - 2z \cos \theta + 1)^2}$$

$$\text{So } Y(z) = 3\Delta \sum_{k=0}^{\infty} k (\cos 0.1\Delta k) z^{-k} = \frac{3\Delta z(z^2+1) \cos 0.1\Delta - 2z^2}{(z^2 - 2z \cos 0.1\Delta + 1)^2}$$

$$y_k = \frac{1}{2\pi j} \oint_C Y(z) z^k \frac{dz}{z} = \sum_{k=1}^P \operatorname{Res}_{z=z_n} \frac{Y(z) z^k}{z} = \sum_{n=1}^P \lim_{z \rightarrow z_n} (z - z_n) \frac{Y(z) z^k}{z}$$

$$(a) \frac{Y(z)}{z} = \frac{1}{z^2(z - 0.3z + 0.02)} = \frac{1}{z^2(z - 0.2)(z - 0.1)}$$

$$\operatorname{Res}_{z=0.2} \frac{Y(z) z^k}{z} = \lim_{z \rightarrow 0.2} \frac{(z - 0.2)}{z^2(z - 0.2)(z - 0.1)} z^k = 250(0.2)^k$$

$$\operatorname{Res}_{z=0.1} \frac{Y(z) z^k}{z} = \lim_{z \rightarrow 0.1} \frac{z^k}{z^2(z - 0.2)} = -1000(0.1)^k$$

$$\operatorname{Res}_{z=0} \frac{Y(z) z^k}{z} = \lim_{z \rightarrow 0} \frac{d}{dz} \frac{z^2 Y(z) z^k}{z} = \lim_{z \rightarrow 0} \frac{(z - 0.2)(z - 0.1) k z^{k-1} - (2z - 0.3) z^k}{(z - 0.2)^2 (z - 0.1)^2}$$

$$= \begin{cases} 750 & ; k=0 \\ 50 & ; k=1 \\ 0 & ; k>1 \end{cases}$$

So

$$y_k = \begin{cases} 0 & ; k = 0, 1 \\ 250(0.2)^k - 1000(0.1)^k & ; k > 1 \end{cases}$$

$$(b) \frac{Y(z)}{z} = \frac{2(z+1)}{z(z-2)}$$

$$\text{Res}_{z=2} \frac{Y(z)z^k}{z} = \lim_{z \rightarrow 2} \frac{2(z+1)z^k}{z} = 3 \cdot 2^k$$

$$\text{Res}_{z=0} \frac{Y(z)z^k}{z} = \lim_{z \rightarrow 0} \frac{2(z+1)z^k}{(z-2)} = \begin{cases} -1 & ; k=0 \\ 0 & ; k > 0 \end{cases}$$

So

$$y_k = \begin{cases} 2 & ; k=0 \\ 3(2)^k & ; k > 0 \end{cases}$$

$$(c) \frac{Y(z)}{z} = \frac{1}{z^2(z-1)(z-0.5)}$$

$$\text{Res}_{z=1} \frac{Y(z)z^k}{z} = \lim_{z \rightarrow 1} \frac{z^k}{z^2(z-0.5)} = 2$$

$$\text{Res}_{z=0.5} \frac{Y(z)z^k}{z} = \lim_{z \rightarrow 0.5} \frac{z^k}{z^2(z-1)} = -8(0.5)^k$$

$$\text{Res}_{z=0} \frac{Y(z)z^k}{z} = \lim_{z \rightarrow 0} \frac{d}{dz} \frac{z^2 Y(z) z^k}{z} = \lim_{z \rightarrow 0} \frac{kz^{k-1}}{(z-1)(z-0.5)} - \frac{(2z-1.5)z^k}{(z-1)^2(z-0.5)^2}$$

$$= \begin{cases} 6 & ; k=0 \\ 2 & ; k=1 \\ 0 & ; k > 1 \end{cases}$$

$$So \quad y_k = \begin{cases} 0 & ; k = 0, 1 \\ 2 - 8(0.5)^k & ; k > 1 \end{cases}$$

$$(d) \quad \frac{Y(z)}{z} = \frac{1}{(z - 0.1e^{j2\pi/3})(z - 0.1e^{-j2\pi/3})}$$

$$Res_{z=0.1e^{j2\pi/3}} \frac{Y(z)z^k}{z} = \frac{(0.1)^k e^{j\frac{2\pi}{3}k}}{(0.1) 2j \sin 2\pi/3}$$

$$Res_{z=0.1e^{-j2\pi/3}} \frac{Y(z)z^k}{z} = \frac{-(0.1)^k e^{-j\frac{2\pi}{3}k}}{(0.1) 2j \sin 2\pi/3}$$

$$So \quad y_k = \frac{(0.1)^{k-1}}{2j \sin 2\pi/3} \left\{ e^{j\frac{2\pi}{3}k} - e^{-j\frac{2\pi}{3}k} \right\}$$

$$= \frac{2(0.1)^{k-1}}{\sqrt{3}} \sin \frac{2\pi k}{3}$$

$$(e) \quad \frac{Y(z)}{z} = \frac{4}{z(z+1)(z-0.3)(z+0.2)}$$

$$Res_{z=0} \frac{Y(z)z^k}{z} = \begin{cases} -66.67 & ; k=0 \\ 0 & ; k > 0 \end{cases}$$

$$Res_{z=-0.5} \frac{Y(z)z^k}{z} = \frac{-33.33}{2} (-0.5)^k$$

$$Res_{z=0.3} \frac{Y(z)z^k}{z} = 16.67 (0.3)^k$$

$$Res_{z=-0.2} \frac{Y(z)z^k}{z} = 66.67 (-0.2)^k$$

$$y_k = \begin{cases} 0 & ; k=0 \\ -16.67(-0.5)^k + 16.67(0.3)^k + 66.67(-0.2)^k & ; k > 0 \end{cases} \quad (5)$$

$$U(z) = \frac{z}{z-1}$$

$$z \{ q^{-1} y_k \} = z^{-1} Y(z) + y_{-1}$$

$$z \{ q^{-2} y_k \} = z^{-2} Y(z) + z^{-1} y_{-1} + y_{-2}$$

$$(a) \quad y_{k+1} + 0.4 y_k = 2u_{k+1} \quad ; y_{-1} = 2$$

$$\Rightarrow y_k + 0.4 y_{k-1} = 2u_k$$

$$\Rightarrow y_k + 0.4 q^{-1} y_k = 2u_k$$

$$\Rightarrow Y(z) + 0.4 \{ z^{-1} Y(z) + y_{-1} \} = 2U(z) \quad ; y_{-1} = 2$$

$$\Rightarrow Y(z) (1 + 0.4 z^{-1}) + 0.8 = \frac{2z}{z-1}$$

$$\text{So } Y(z) = \frac{2z^2}{(z-1)(z+0.4)} - \frac{0.8z}{z+0.4}$$

$$= \frac{10}{7} \left( \frac{z}{z-1} \right) + \left( \frac{4}{7} \right) \left( \frac{z}{z+0.4} \right) - \frac{0.8z}{z+0.4}$$

$$\text{So } y_k = \frac{10}{7} - \frac{1.6}{7} (-0.4)^k \quad ; k \geq 0$$

①

$$(b) \quad y_{k+2} - 0.2y_{k+1} - 0.08y_k = 2u_{k+2} - 2u_k \quad ; \quad y_{-1} = -2, y_{-2} =$$

$$\Rightarrow y_k - 0.2y_{k-1} - 0.08y_{k-2} = 2u_k - 2u_{k-2}$$

$$\Rightarrow Y(z) - 0.2\{z^{-1}Y(z) + y_{-1}\} - 0.08\{z^{-2}Y(z) + z^{-1}y_{-1} + y_{-2}\} \\ = 2U(z) - 2\{z^{-2}U(z) + z^{-1}u_{-1} + u_{-2}\}$$

$$\Rightarrow Y(z)(1 - 0.2z^{-1} - 0.08z^{-2}) - 0.2y_{-1} - 0.08z^{-1}y_{-1} - 0.08y_{-2} \\ = U(z)(2 - 2z^{-2}) - 2z^{-1}u_{-1} - 2u_{-2}$$

$$\text{So } Y(z) = \frac{2(1 - z^{-2})}{(1 - 0.2z^{-1} - 0.08z^{-2})} \frac{z}{z-1} + \frac{(0.2)(-2) + (0.08)(-2)z^{-1} + (0.08)(-2)z^{-2}}{1 - 0.2z^{-1} - 0.08z^{-2}}$$

$$= \frac{2z(z+1)}{(z^2 - 0.2z - 0.08)} - \frac{0.16z(z+1)}{(z^2 - 0.2z - 0.08)}$$

$$\text{So } \frac{Y(z)z^k}{z} = \frac{1.84z(z+1)}{(z-0.4)(z+0.2)} \frac{z^k}{z} = \frac{1.84(z+1)z^k}{(z-0.4)(z+0.2)}$$

$$\Rightarrow y_k = \frac{1.84(0.4+1)}{(0.4+0.2)} (0.4)^k + \frac{1.84(-0.2+1)(-0.2)^k}{(-0.2-0.4)}$$

$$= \frac{12.88}{3} (0.4)^k - \frac{7.36}{3} (-0.2)^k$$

$$(b) \quad z^2 - z + 1 = (z - (0.5 + j0.87))(z - (0.5 - j0.87))$$

(7)

$\Rightarrow$  Stable

$$(c) \quad z^2 + 2.5z + 1 = (z + 2)(z + 0.5)$$

$\Rightarrow$  Unstable

$$(d) \quad z^3 - 0.5z^2 - z + 0.5 = \underbrace{(z-1)(z+1)}_{\text{marginally stable}} (z-0.5) \Rightarrow \text{Oscillatory}$$