Bounds on the Degree of Impropriety of Complex Random Vectors

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***Abstract*—A complex random vector is called improper if it is correlated with its complex conjugate. We introduce a measure for the degree of impropriety, which is a function of the canonical correlations between the vector and its complex conjugate (sometimes called the circularity spectrum). This measure is invariant under linear transformation, and it relates the entropy of an improper Gaussian random vector to its corresponding proper version. For vectors with given spectrum, we present upper and lower bounds on the attainable degree of impropriety, in terms of the eigenvalues of the augmented covariance matrix.**

***Index Terms*—Canonical correlations, circularity spectrum, improper complex random vector, noncircular random vector, strong uncorrelating transform, widely linear transformation.**

I. Introduction

A zero-mean complex random vector $\mathbf{s}$ with covariance matrix $\mathbf{R} = E\mathbf{s}\mathbf{s}^H$ is called improper if its complementary covariance matrix $\widetilde{\mathbf{R}} = E\mathbf{s}\mathbf{s}^T$ is nonzero. This letter is concerned with quantifying the degree of impropriety of $\mathbf{s}$. In [1], a vector $\mathbf{s}_1$ is called "less improper" than a vector $\mathbf{s}_2$ if the eigenvalues of the augmented covariance matrix of $\mathbf{s}_1$, which is the covariance matrix of $[\mathbf{s}_1^T, \mathbf{s}_1^H]^T$, are majorized by (less "spread out" than) those of $\mathbf{s}_2$. However, [1] does not define a numerical measure of the degree of impropriety.

Propriety is preserved by linear transformation. Hence, we require the degree of impropriety to also be invariant under nonsingular linear transformation. This means it must be a function of a complete set of invariants for $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ under linear transformation. As shown in [2], such a set is given by the canonical correlations between $\mathbf{s}$ and its conjugate $\mathbf{s}^*$. These canonical correlations have also been investigated in [3], where they have been called *circularity coefficients* and shown to play a key role in independent component analysis of complex signals. Among the many plausible functions of these canonical correlations, one stands out because it relates the entropy of an improper Gaussian random vector to its proper version. We demonstrate that this function can be interpreted as a measure of the eigenvalue spread of the augmented covariance matrix, thus validating the insights of [1].

Our key contribution is the development of lower and upper bounds on this measure when the eigenvalues of the augmented covariance matrix are known. There are several reasons why we are interested in these bounds.

Firstly, computing an eigenvalue decomposition (EVD) is computationally much less expensive than a canonical correlation decomposition. Thus, computing bounds on the degree of impropriety is cheaper than computing the actual degree in the reduced-rank case.

Secondly, when canonical correlations are estimated from sample covariance matrices, they can be numerically unstable. The eigenvalues of the augmented covariance matrix, on the other hand, are numerically stable. Thus, in the reduced-rank sample case, it may be more meaningful to work with bounds in terms of the estimated eigenvalues of the augmented covariance matrix than with estimates of the canonical correlations.

Most importantly, the bounds show how to maximize or minimize entropy under widely unitary transformations, which leave the eigenvalues of the augmented covariance matrix unchanged.

Our program for this letter is as follows. In Section II, we present some background material on improper complex random vectors. In Section III, we look at the canonical correlations between $\mathbf{s}$ and $\mathbf{s}^*$, and we define measures for the degree of impropriety. Section IV discusses bounds on the degree of impropriety for a random vector with given spectrum.

II. Preliminaries

Let $\mathbf{s} \in \mathbb{C}^n$ denote a zero-mean complex random vector with covariance matrix $\mathbf{R} = E\mathbf{s}\mathbf{s}^H$ and complementary covariance (or pseudo-covariance) matrix $\widetilde{\mathbf{R}} = E\mathbf{s}\mathbf{s}^T$. If $\widetilde{\mathbf{R}} = \mathbf{0}$, $\mathbf{s}$ is called *proper*, otherwise *improper*. It is convenient to work with an augmented vector $\overline{\mathbf{s}} = [\mathbf{s}^T, \mathbf{s}^H]^T$ [4]–[6] whose covariance matrix

$$\overline{\mathbf{R}} = E\overline{\mathbf{s}}\overline{\mathbf{s}}^H = \begin{bmatrix} \mathbf{R} & \widetilde{\mathbf{R}} \\ \widetilde{\mathbf{R}}^* & \mathbf{R}^* \end{bmatrix} \tag{1}$$

is called the *augmented covariance matrix* of $\mathbf{s}$ [4]. The advantage of working with augmented vectors and augmented covariance matrices is that it allows us to exploit the vast number of results on 2×2 block matrices.

A transformation of the form $\mathbf{s}' = \overline{\mathbf{A}}_1\mathbf{s} + \overline{\mathbf{A}}_2\mathbf{s}^*$ is called widely linear [7] (or linear-conjugate linear). It may be represented in augmented form as

$$\overline{\mathbf{s}}' = \begin{bmatrix} \mathbf{s}' \\ \mathbf{s}'^* \end{bmatrix} = \overline{\mathbf{A}}\overline{\mathbf{s}} = \begin{bmatrix} \overline{\mathbf{A}}_1 & \overline{\mathbf{A}}_2 \\ \overline{\mathbf{A}}_2^* & \overline{\mathbf{A}}_1^* \end{bmatrix} \begin{bmatrix} \mathbf{s} \\ \mathbf{s}^* \end{bmatrix}. \tag{2}$$

Note that in this augmented representation, $\overline{\mathbf{R}}$ and $\overline{\mathbf{A}} \in \mathbb{C}^{2n \times 2n}$ satisfy a particular block structure where the northwest is the conjugate of the southeast block, and the northeast is the conjugate of the southwest block. Throughout this letter, matrices that satisfy this block pattern are overlined.

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In factorizations of $\overline{\mathbf{R}}$, all factors must also have this block structure. This is why the EVD of $\mathbf{s}$ takes on the unfamiliar form [4, Prop. 1]

$$\overline{\mathbf{R}} = \overline{\mathbf{U}}\overline{\mathbf{\Lambda}}\overline{\mathbf{U}}^H \tag{3}$$

with

$$\overline{\mathbf{\Lambda}} = \frac{1}{2}\begin{bmatrix} \mathbf{\Lambda}^{(1)} + \mathbf{\Lambda}^{(2)} & \mathbf{\Lambda}^{(1)} - \mathbf{\Lambda}^{(2)} \\ \mathbf{\Lambda}^{(1)} - \mathbf{\Lambda}^{(2)} & \mathbf{\Lambda}^{(1)} + \mathbf{\Lambda}^{(2)} \end{bmatrix} \tag{4}$$

$$\mathbf{\Lambda}^{(1)} = \mathrm{diag}(\lambda_1, \lambda_3, \ldots, \lambda_{2n-1}) \tag{5}$$

$$\mathbf{\Lambda}^{(2)} = \mathrm{diag}(\lambda_2, \lambda_4, \ldots, \lambda_{2n}). \tag{6}$$

The eigenvalues of $\overline{\mathbf{R}}$ are arranged in nonincreasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n}$. The *widely unitary* transform $\overline{\mathbf{U}} \in \mathbb{C}^{2n\times 2n}$ has the pattern of (2) and satisfies $\overline{\mathbf{U}}\,\overline{\mathbf{U}}^H = \overline{\mathbf{U}}^H\overline{\mathbf{U}} = \mathbf{I}$. The EVD of $\mathbf{s}$ can be determined as follows [4]. Let $\mathbf{s} = 1/\sqrt{2}(\mathbf{x} + j\mathbf{y})$, $\mathbf{R}_{xx} = E\mathbf{x}\mathbf{x}^T$, $\mathbf{R}_{yy} = E\mathbf{y}\mathbf{y}^T$, $\mathbf{R}_{xy} = E\mathbf{x}\mathbf{y}^T$, and

$$\check{\mathbf{R}} = E\begin{bmatrix}\mathbf{x}\\ \mathbf{y}\end{bmatrix}\begin{bmatrix}\mathbf{x}^T & \mathbf{y}^T\end{bmatrix} = \begin{bmatrix}\mathbf{R}_{xx} & \mathbf{R}_{xy}\\ \mathbf{R}_{xy}^T & \mathbf{R}_{yy}\end{bmatrix}. \tag{7}$$

Furthermore, define the unitary matrix

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{bmatrix}\mathbf{I} & j\mathbf{I}\\ \mathbf{I} & -j\mathbf{I}\end{bmatrix} \tag{8}$$

so that $\overline{\mathbf{R}} = \mathbf{T}\check{\mathbf{R}}\mathbf{T}^H$. Hence, $\overline{\mathbf{R}}$ and $\check{\mathbf{R}}$ have the same eigenvalues. From the real EVD

$$\check{\mathbf{R}} = \check{\mathbf{U}}\begin{bmatrix}\mathbf{\Lambda}^{(1)} & \mathbf{0}\\ \mathbf{0} & \mathbf{\Lambda}^{(2)}\end{bmatrix}\check{\mathbf{U}}^T \tag{9}$$

where $\mathbf{\Lambda}^{(1)}$ and $\mathbf{\Lambda}^{(2)}$ are defined in (5) and (6), we obtain $\overline{\mathbf{U}} = \mathbf{T}\check{\mathbf{U}}\mathbf{T}^H$.

We emphasize that $\overline{\mathbf{\Lambda}}$ is diagonal if and only if all eigenvalues of $\overline{\mathbf{R}}$ have even multiplicity [4, Prop. 1]. In general, $\overline{\mathbf{\Lambda}}$ is not diagonal but has diagonal blocks instead. This is a consequence of the fact that, in the latent description $\mathbf{s}'' = \overline{\mathbf{U}}_1^H\mathbf{s} + \overline{\mathbf{U}}_2^T\mathbf{s}^*$, it is not generally possible to decorrelate s_i'' and $s_i''^*$ using widely unitary transformations. The latent description $\mathbf{s}''$ always has *uncorrelated* components, i.e., $Es_i''s_j''^* = Es_i''s_j'' = 0$ for $i \neq j$. However, if the eigenvalues do not have even multiplicity, then $\mathbf{s}''$ is *improper* because $Es_i''^2 = (\lambda_{2i-1} - \lambda_{2i})/2$.

III. Measures of Impropriety

Propriety is preserved under *linear* (but not widely linear) transformation, which includes rotation and scaling of $\mathbf{s}$. A maximal set of invariants for the augmented covariance matrix $\overline{\mathbf{R}}$ under nonsingular linear transformation is the set of *canonical correlations* [8], [9] between $\mathbf{s}$ and $\mathbf{s}^*$ [2]. This means that any function of $\overline{\mathbf{R}}$ that is invariant under linear transformation must be a function of these canonical correlations. In order to determine them, we begin with the *coherence matrix* [9]

$$\mathbf{M} = \mathbf{R}^{-1/2}\widetilde{\mathbf{R}}\mathbf{R}^{-T/2}. \tag{10}$$

Since $\mathbf{M}$ is complex symmetric, $\mathbf{M} = \mathbf{M}^T$, there exists a special singular value decomposition (SVD), called *Takagi's factorization*, which is

$$\mathbf{M} = \mathbf{F}\mathbf{K}\mathbf{F}^T. \tag{11}$$

It is shown in [10, Sec. 4.4] how to compute the Takagi factorization. The complex matrix $\mathbf{F}$ is unitary, and $\mathbf{K} = \mathrm{diag}(k_1, k_2, \ldots, k_n)$ contains the canonical correlations $1 \geq k_1 \geq k_2 \geq \cdots \geq k_n \geq 0$ on its diagonal. The latent description $\mathbf{s}' = \mathbf{F}^H\mathbf{R}^{-1/2}\mathbf{s}$ is said to be given in *canonical coordinates*. The canonical coordinates are uncorrelated, i.e., $Es_i's_j'^* = Es_i's_j' = 0$ for $i \neq j$, and they have unit variance, $E|s_i'|^2 = 1$. However, they are generally improper as $Es_i'^2 = k_i$. In [3], vectors that are uncorrelated with unit variance, but possibly improper, are called *strongly uncorrelated*, and the transformation $\mathbf{F}^H\mathbf{R}^{-1/2}$, which transforms $\mathbf{s}$ into canonical coordinates $\mathbf{s}'$, is called the *strong uncorrelating transform*. The canonical correlations $\{k_i\}_{i=1}^n$ are referred to as the *circularity spectrum* of $\mathbf{s}$ in [3]. A more thorough discussion of properties of $\{k_i\}$ is contained in [3] and [2]. The insight that the $\{k_i\}$ are canonical correlations is critical as it enables us to utilize the many results on this topic in the literature.

The canonical coordinates are determined as $\mathbf{s}' = \mathbf{F}^H\mathbf{R}^{-1/2}\mathbf{s}$, so that their complementary covariance matrix is $E\mathbf{s}'\mathbf{s}'^T = \mathbf{F}^H\mathbf{R}^{-1/2}\widetilde{\mathbf{R}}\mathbf{R}^{-T/2}\mathbf{F}^* = \mathbf{F}^H\mathbf{M}\mathbf{F}^*$. In order to make this matrix diagonal, Takagi's factorization (11) rather than the "regular" SVD $\mathbf{M} = \mathbf{U}\mathbf{K}\mathbf{V}^H$ (that generally yields $\mathbf{U} \neq \mathbf{V}^*$) must be employed. However, if only the canonical correlations $\{k_i\}$ need to be computed and the canonical coordinates $\mathbf{s}'$ are not required, $\mathbf{M}$ may be resolved by a "regular" SVD as it yields the same canonical correlations.

Combining our results so far and proceeding along the lines of [9], we may factor $\overline{\mathbf{R}}$ as

$$\begin{bmatrix}\mathbf{R} & \widetilde{\mathbf{R}}\\ \widetilde{\mathbf{R}}^* & \mathbf{R}^*\end{bmatrix} = \begin{bmatrix}\mathbf{R}^{1/2} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}^{*/2}\end{bmatrix}\begin{bmatrix}\mathbf{F} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}^*\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{K}\\ \mathbf{K} & \mathbf{I}\end{bmatrix} \times \begin{bmatrix}\mathbf{F}^H & \mathbf{0}\\ \mathbf{0} & \mathbf{F}^T\end{bmatrix}\begin{bmatrix}\mathbf{R}^{H/2} & \mathbf{0}\\ \mathbf{0} & \mathbf{R}^{T/2}\end{bmatrix}. \tag{12}$$

Note that each factor satisfies the pattern of (2). The factorization (12) establishes, similarly to [11]

$$\det\overline{\mathbf{R}} = \det{}^2\mathbf{R}\det(\mathbf{I} - \mathbf{K}^2) = \det{}^2\mathbf{R}\prod_{i=1}^n\left(1 - k_i^2\right). \tag{13}$$

This allows us to derive the connection between the entropy of an improper Gaussian random vector with augmented covariance matrix $\overline{\mathbf{R}}$ and the corresponding proper Gaussian random vector with covariance matrix $\mathbf{R}$ (see [3, Cor. 2]) in the bargain as

$$\begin{aligned} H_{\text{improper}} &= \frac{1}{2}\log\left[(\pi e)^{2n}\det\overline{\mathbf{R}}\right] \\ &= \underbrace{\log\left[(\pi e)^n\det\mathbf{R}\right]}_{H_{\text{proper}}} + \frac{1}{2}\log\prod_{i=1}^n\left(1 - k_i^2\right). \end{aligned} \tag{14}$$

This again shows the classic result that $H_{\text{improper}} \leq H_{\text{proper}}$.

We would now like to introduce a measure for the degree of impropriety. If that measure is supposed to be invariant under linear transformation, it must be a function of the canonical correlations $\{k_i\}$, but several functions seem plausible. For instance

$$d_1 = 1 - \prod_{i=1}^n\left(1 - k_i^2\right) = 1 - \det\overline{\mathbf{R}}\det{}^{-2}\mathbf{R} \tag{15}$$

$$d_2 = \prod_{i=1}^n k_i^2 = \det(\widetilde{\mathbf{R}}\mathbf{R}^{-*}\widetilde{\mathbf{R}}^*)\det{}^{-1}\mathbf{R} \tag{16}$$

$$d_3 = \frac{1}{n}\sum_{i=1}^n k_i^2 = \frac{1}{n}\mathrm{tr}(\mathbf{R}^{-1}\widetilde{\mathbf{R}}\mathbf{R}^{-*}\widetilde{\mathbf{R}}^*) \tag{17}$$

are all conceivable as measures of impropriety. These functions have been discussed in [12] as measures of multivariate association between an arbitrary pair of real vectors. They satisfy $0 \leq d_i \leq 1$, and they can be defined for reduced rank $r < n$, considering only the r largest canonical correlations in the computation. We consider d_1 the most compelling measure mainly for two reasons. Firstly, as shown above in (14), d_1 connects the entropy of the proper and improper cases. Secondly, it is a measure of the linear dependence between $\mathbf{s}$ and $\mathbf{s}^*$ and, as such, can be used to design a generalized likelihood ratio test for impropriety [11], [2]. In the remainder of this letter, we will discuss $d = d_1$.

In the case of a scalar random variable s with covariance $R = E|s|^2$ and complementary covariance $\widetilde{R} = Es^2$, we have $d = |\widetilde{R}|^2/R^2$. For example, M-PSK, $M \geq 4$, and QAM symbols are proper, $d = 0$, whereas BPSK and PAM symbols are maximally improper, $d = 1$. This reflects the fact that M-PSK and QAM are rotationally symmetric and thus maximize entropy, whereas BPSK and PAM are maximally statistically redundant and thus minimize entropy. For QPSK with I/Q imbalance characterized by gain imbalance (factor) $G > 0$ and quadrature skew ϕ, it is easy to show that

$$d = \frac{G^4 - 2G^2 \cos 2\phi + 1}{(1+G^2)^2}. \quad (18)$$

Clearly, QPSK with perfect I/Q balance has $G = 1$, $\phi = 0$, and thus $d = 0$, whereas the worst possible I/Q imbalance $\phi = \pi/2$ results in $d = 1$.

IV. Bounds on Impropriety

With specified covariance matrix $\mathbf{R}$, it follows from (10) and (11) that all valid complementary covariance matrices are of the form

$$\widetilde{\mathbf{R}} = \mathbf{R}^{1/2}\mathbf{F}\mathbf{K}\mathbf{F}^T\mathbf{R}^{T/2} \quad (19)$$

where $\mathbf{F}$ is an arbitrary unitary matrix, and $\mathbf{K}$ is a matrix of arbitrary canonical correlations. Hence, besides the trivial bounds $0 \leq d \leq 1$, nothing more can be inferred about the degree of impropriety. In this letter, we are interested in the case where the eigenvalues of $\overline{\mathbf{R}}$, denoted by $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_{2n}]^T$ with ordering $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{2n} \geq 0$, are specified. In this scenario, there exist both nontrivial upper and lower bounds on the achievable degree of impropriety.

A. Upper and Lower Bounds

Proposition 1: The degree of impropriety d of a vector $\mathbf{s}$ with prescribed eigenvalues $\boldsymbol{\lambda}$ of the augmented covariance matrix $\overline{\mathbf{R}}$ is upper bounded by

$$d = 1 - \prod_{i=1}^{r}\left(1-k_i^2\right) \leq 1 - \prod_{i=1}^{r}\frac{4\lambda_i\lambda_{2n+1-i}}{(\lambda_i + \lambda_{2n+1-i})^2}, \quad r = 1, \ldots, n.$$

This upper bound is attained when $\mathbf{R} = 1/2\,\mathrm{diag}(\lambda_1 + \lambda_{2n}, \lambda_2 + \lambda_{2n-1}, \ldots, \lambda_n + \lambda_{n+1})$ and $\widetilde{\mathbf{R}} = 1/2\,\mathrm{diag}(\lambda_1 - \lambda_{2n}, \lambda_2 - \lambda_{2n-1}, \ldots, \lambda_n - \lambda_{n+1})$.

Proof: This bound has been derived in [13] for canonical correlations between arbitrary pairs of random vectors $(\mathbf{s}, \mathbf{r})$, and it holds *a forteriori* for the canonical correlations between $\mathbf{s}$ and $\mathbf{s}^*$. It is easy to see that $\overline{\mathbf{R}}$, with $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ as specified in the Proposition, has spectrum $\boldsymbol{\lambda}$ and attains the bound. □

There is no nontrivial lower bound on the canonical correlations between arbitrary pairs of random vectors $(\mathbf{s}, \mathbf{r})$. It is always possible to choose, for instance, $E\mathbf{s}\mathbf{s}^H = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $E\mathbf{r}\mathbf{r}^H = \mathrm{diag}(\lambda_{n+1}, \ldots, \lambda_{2n})$, and $E\mathbf{s}\mathbf{r}^H = \mathbf{0}$, which has the required spectrum $\boldsymbol{\lambda}$ and all-zero canonical correlations $\mathbf{K} = \mathbf{0}$. That there is a lower bound for the canonical correlations between $\mathbf{s}$ and $\mathbf{s}^*$ stems from the special structure of $\overline{\mathbf{R}}$, where the northwest and southeast blocks must be complex conjugates.

Proposition 2: The degree of impropriety d of a vector $\mathbf{s}$ with prescribed eigenvalues $\boldsymbol{\lambda}$ of the augmented covariance matrix $\overline{\mathbf{R}}$ is lower bounded by

$$d = 1 - \prod_{i=1}^{n}\left(1-k_i^2\right) \geq 1 - \prod_{i=1}^{n}\frac{4\lambda_{2i-1}\lambda_{2i}}{(\lambda_{2i-1} + \lambda_{2i})^2}. \quad (20)$$

This lower bound is attained when $\mathbf{R} = 1/2(\boldsymbol{\Lambda}^{(1)} + \boldsymbol{\Lambda}^{(2)})$ and $\widetilde{\mathbf{R}} = 1/2(\boldsymbol{\Lambda}^{(1)} - \boldsymbol{\Lambda}^{(2)})$, with $\boldsymbol{\Lambda}^{(1)}$ and $\boldsymbol{\Lambda}^{(2)}$ as in (5) and (6).

Proof: Refer to the definitions in (7). From [4], we know that $\mathbf{R} = 1/2(\mathbf{R}_{xx} + \mathbf{R}_{yy} + j(\mathbf{R}_{xy}^T - \mathbf{R}_{xy}))$ and $\widetilde{\mathbf{R}} = 1/2(\mathbf{R}_{xx} - \mathbf{R}_{yy} + j(\mathbf{R}_{xy}^T + \mathbf{R}_{xy}))$. In (13), $\det \overline{\mathbf{R}} = \prod_{i=1}^{2n} \lambda_i$ is fixed. Hence, the minimum d is achieved when $\det \mathbf{R}$ is minimized. We can assume without loss of generality that $\mathbf{R}$ is diagonal. If it is not, it can be made diagonal with a strictly unitary transform that leaves $\det \mathbf{R}$ and the eigenvalues $\boldsymbol{\lambda}$ unchanged. Thus, denoting $\mathbf{R}_{ii}$ the ith diagonal element of $\mathbf{R}$, we have

$$\min \det \mathbf{R} = \min \prod_{i=1}^{n} \mathbf{R}_{ii} = \min \prod_{i=1}^{n} \frac{1}{2}(\mathbf{R}_{xx} + \mathbf{R}_{yy})_{ii}. \quad (21)$$

Now let q_i be the ith largest diagonal element of $\mathbf{R}_{xx} + \mathbf{R}_{yy}$, and r_i the ith largest diagonal element of $\check{\mathbf{R}}$. Then

$$\sum_{i=1}^{r} q_i \leq \sum_{i=1}^{r} r_{2i-1} + r_{2i} \leq \sum_{i=1}^{r} \lambda_{2i-1} + \lambda_{2i} \quad (22)$$

for $1 \leq r \leq n$, with equality for $r = n$. We have the second inequality because the diagonal elements of $\check{\mathbf{R}}$ are majorized by the eigenvalues of $\check{\mathbf{R}}$ [14, Thm. 9.B.1]. Since $\prod q_i$ is a Schur-concave function [14, Ch. 3], a consequence of (22) is the following variant of Hadamard's inequality:

$$\prod_{i=1}^{n} (\mathbf{R}_{xx} + \mathbf{R}_{yy})_{ii} \geq \prod_{i=1}^{n} \lambda_{2i-1} + \lambda_{2i} \quad (23)$$

and therefore

$$\min \det \mathbf{R} = \prod_{i=1}^{n} \frac{\lambda_{2i-1} + \lambda_{2i}}{2}. \quad (24)$$

Using this result in (13), we obtain

$$\prod_{i=1}^{n}\left(1-k_i^2\right) \leq \frac{\prod_{i=1}^{2n}\lambda_i}{\prod_{i=1}^{n}\frac{(\lambda_{2i-1}+\lambda_{2i})^2}{4}} \quad (25)$$

from which the lower bound on d follows. The lower bound is attained by choosing $\mathbf{R}_{xx} = \boldsymbol{\Lambda}^{(1)}$, $\mathbf{R}_{yy} = \boldsymbol{\Lambda}^{(2)}$, and $\mathbf{R}_{xy} = \mathbf{0}$, or $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ as stated in the Proposition. □

Given a random vector $\mathbf{s}$, we can produce a *least improper analog* $\overline{\mathbf{s}}'' = \overline{\mathbf{U}}^H\overline{\mathbf{s}}$ with the same spectrum $\boldsymbol{\lambda}$, utilizing the eigenvalue decomposition $\overline{\mathbf{R}} = \overline{\mathbf{U}}\,\overline{\boldsymbol{\Lambda}}\,\overline{\mathbf{U}}^H$. The least improper analog $\mathbf{s}''$ has the same power as $\mathbf{s}$, but it maximizes entropy for a given spectrum $\boldsymbol{\lambda}$. We note that a least improper analog is not unique, as any strictly unitary transform will leave both spectrum $\boldsymbol{\lambda}$ and canonical correlations $\{k_i\}$ unchanged.

B. Discussion

Both the lower and the upper bound are attained when $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ are diagonal matrices. For diagonal $\mathbf{R}$ and $\widetilde{\mathbf{R}}$

$$\prod_{i=1}^{n}\left(1-k_i^2\right)=\prod_{i=1}^{n}\left(1-\frac{|\widetilde{\mathbf{R}}_{ii}|^2}{\mathbf{R}_{ii}^2}\right). \tag{26}$$

Since $\overline{\mathbf{R}}$, with diagonal $\mathbf{R}$ and $\widetilde{\mathbf{R}}$, has eigenvalues $\{\mathbf{R}_{ii} \pm |\widetilde{\mathbf{R}}_{ii}|\}_{i=1}^{n}$, this gives

$$\prod_{i=1}^{n}\left(1-k_i^2\right)=\prod_{i=1}^{n}\left(1-\frac{(a_i-b_i)^2}{(a_i+b_i)^2}\right)=\prod_{i=1}^{n}\frac{4a_ib_i}{(a_i+b_i)^2} \tag{27}$$

where $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ are two disjoint subsets of $\boldsymbol{\lambda}$. Each factor $4a_ib_i/(a_i+b_i)^2$ is the squared ratio of geometric and arithmetic mean of a_i and b_i. Hence, it is 1 if $a_i = b_i$, and 0 if a_i or b_i are 0, and thus measures the spread between a_i and b_i. We can minimize or maximize (27) by choosing the subsets $\{a_i\}$ and $\{b_i\}$ from the spectrum $\boldsymbol{\lambda}$ using a combinatorial argument [15]. In order to minimize (27), we need maximum spread between the two sets $\{a_i\}$ and $\{b_i\}$, which is achieved by $a_i = \lambda_i$, $b_i = \lambda_{2n-i}$. To maximize (27), we need minimum spread between $\{a_i\}$ and $\{b_i\}$, which is achieved by $a_i = \lambda_{2i-1}$, $b_i = \lambda_{2i}$. This validates the insight of [1] that the degree of impropriety relates to the eigenvalue spread of $\overline{\mathbf{R}}$.

While Proposition 1 holds for $r = 1, \ldots, n$, Proposition 2 has only been established for $r = n$. A natural conjecture is to assume that the diagonal $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ given in Proposition 2 also attain the lower bound for $r < n$. Let c_i be the ith largest of the factors

$$\frac{4\lambda_{2j-1}\lambda_{2j}}{(\lambda_{2j-1}+\lambda_{2j})^2}, \quad j = 1, \ldots, n.$$

Then the conjecture may be written as

$$d = 1-\prod_{i=1}^{r}\left(1-k_i^2\right) \geq 1-\prod_{i=n-r+1}^{n} c_i, \quad r = 1, \ldots, n. \tag{28}$$

C. Example

Consider the spectrum $\boldsymbol{\lambda} = [100, 80, 80, 2]$. It follows from Propositions 1 and 2 that the degree of impropriety of any complex random vector with this spectrum must satisfy $0.906 \leq d \leq 0.923$. The lower bound is achieved for

$$\mathbf{R} = \mathrm{diag}(90, 41) \text{ and } \widetilde{\mathbf{R}} = \mathrm{diag}(10, 39) \tag{29}$$

with $k_1 = 39/41$ and $k_2 = 1/9$. The upper bound is achieved for

$$\mathbf{R} = \mathrm{diag}(51, 80) \text{ and } \widetilde{\mathbf{R}} = \mathrm{diag}(49, 0) \tag{30}$$

which gives $k_1 = 49/51$ and $k_2 = 0$.

We note that $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ in (29) do not necessarily give lower bounds for other measures of impropriety, for instance, d_2. A random vector with $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ given by (29) has $d_2 = 0.011$. Yet a random vector with $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ given by (30)—which is the most improper case according to measure d—actually has a lower degree of impropriety, $d_2 = 0$.

V. Conclusions

We have presented a measure for the degree of impropriety of a complex random vector. If the eigenvalues of its augmented covariance matrix are known, it is possible to give tight upper and lower bounds on the degree of impropriety. We point out that the matrices $\mathbf{R}$ and $\widetilde{\mathbf{R}}$ that achieve the upper and lower bounds in Propositions 1 and 2 do not necessarily give upper and lower bounds for other measures [16], such as d_2 and d_3 in (16) and (17). In [16], an upper bound for d_2 is proved, and an upper bound for d_3 is conjectured. Lower bounds for d_2 and d_3 are open problems.

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