Spectral Relations for Multidimensional Complex Improper Stationary and (Almost) Cyclostationary Processes

Patrik Wahlberg, *Member, IEEE*, and Peter J. Schreier, *Member, IEEE*

***Abstract*—We study continuous-time multidimensional wide-sense stationary (WSS) and (almost) cyclostationary processes in the frequency domain. Under the assumption that the correlation function is uniformly continuous, we prove the existence of a unique sequence of spectral measures, which coincide with the restrictions to certain subdiagonals of the spectral measure in the strongly harmonizable case. Moreover, the off-diagonal measures are absolutely continuous with respect to the diagonal measure. As a consequence, for strongly harmonizable scalar improper almost cyclostationary processes, we obtain representation formulas for the components of the complementary spectral measure and the off-diagonal components of the spectral measure, in terms of the diagonal component of the spectral measure. We apply these results to analytic signals, which produces sufficient conditions for propriety for almost cyclostationary analytic signals.**

***Index Terms*—(Almost) cyclostationary processes, analytic signals, complementary correlation, harmonizable processes, improper processes, multidimensional processes, wide-sense stationary (WSS) processes.**

I. INTRODUCTION

THE second-order statistics of a zero-mean scalar complex stochastic process $s(t)$ are characterized by the correlation function $r(t,u)=E(s(t)s^*(u))$ and the complementary correlation function $\widetilde{r}(t,u)=E(s(t)s(u))$, which is sometimes also called *pseudo-correlation* [1]. Processes with vanishing complementary correlation are called *proper* [1]. Proper processes are circular (or circularly symmetric) in the sense that $s(t)$ and a rotated version $s(t)e^{j\alpha}$ have the same second-order statistics for all real angles α [2]. Complex-valued stochastic processes may arise as equivalent descriptions of underlying real signals. The two most prominent examples are the analytic signal and the equivalent baseband signal in communications. Analytic signals are important in several areas of signal analysis [3], in particular time–frequency analysis [4]. The analytic signal enables a decomposition of a real signal into amplitude and phase. The equivalent baseband signal describes a real bandpass signal independently of the carrier frequency [5].

Historically, the focus has been on proper signals. The main reason, besides the simplicity of description, is that a complex signal is known to be proper if it is either the analytic signal or the equivalent baseband representation of a real wide-sense stationary (WSS) bandpass signal [6]. However, in other situations, propriety is not guaranteed. Recently, there has been an increased interest in the study of improper complex signals because they can arise in modern signal processing and communications applications, e.g., multiuser detection [7], [8], equalization [9], and array processing [10]. Correctly accounting for the improper nature of signals can lead to significant performance gains [11].

In this paper, we consider zero-mean continuous-time second-order multidimensional complex (vector-valued) improper stochastic processes. In order of increasing generality, we treat WSS, cyclostationary, and almost cyclostationary processes. Cyclostationary processes are periodically correlated and serve as models of phenomena occurring in science and technology, including communications, physics, meteorology, oceanography, climatology, astronomy, biology, and economics [12]–[19]. The concept has a natural extension to the class of almost cyclostationary processes [20], [21], which is of interest both from a theoretical and a practical point of view. Almost cyclostationary processes are interesting as mathematical objects in a way similar to almost periodic functions, which also comprise nonperiodic functions such as $\sin(x)+\sin(\pi x)$. They are also important as models of physical signals that are close to being periodic but have nonharmonically related frequency content, e.g., in climatology [22].

Cyclostationary and almost cyclostationary processes have been given much attention in the engineering and mathematical literature [12]. One of the pioneers was Gladyshev [15], [16], who found a criterion for a function to be the correlation function of an almost cyclostationary process. This criterion reveals a connection between almost cyclostationary processes and vector-valued WSS processes, which has been further explored by Gardner, among others. Hurd, Gardner, Dehay, and others have made important contributions to the spectral theory of (almost) cyclostationary processes [13], [14], [18]–[21], [23]. In [24], a more general class of Hilbert space-valued processes, called asymptotically stationary and defined on locally compact Abelian groups, is discussed. The paper [25] treats estimation of the frequencies of discrete-time almost cyclostationary processes.

For strongly harmonizable processes, there exists a two-dimensional spectral measure, which is the Fourier transform of

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The authors are with the School of Electrical Engineering and Computer Science, The University of Newcastle, Callaghan, NSW 2308 Australia (e-mail Patrik.Wahlberg@newcastle.edu.au; peter@peter-schreier.com).

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the correlation function, and another two-dimensional *complementary* spectral measure, which is the Fourier transform of the complementary correlation function. The spectral measure of a strongly harmonizable cyclostationary process has support along diagonals whose intersection with a coordinate axis constitutes a frequency lattice. For almost cyclostationary processes, the describing set of frequencies is still countable but may be incommensurate, i.e., not supported by a lattice. This means that the spectral measure may be described by a sequence of measures defined on $\mathbb{R}$. There exists such a sequence of spectral measures under weaker assumptions than strong harmonizability. For scalar processes, Hurd [21] showed that uniform continuity of the correlation function is sufficient for the existence of a sequence of measures. Later, Dehay [20] showed that the off-diagonal measures are absolutely continuous with respect to the diagonal measure under the same assumption. In this paper, we extend these findings to infinite-dimensional processes. As a consequence, we obtain results on the spectral measure and complementary spectral measure in the scalar strongly harmonizable case, using the two-dimensional process $(s(t)\ s^*(t))^T$.

Our paper is constructed as follows. In Section II, we give a characterization for a pair of matrices to be the correlation and complementary correlation matrices of a finite-dimensional stochastic vector. This generalizes a well-known result for invertible correlation matrices [26] to the singular case. In Section III, we review the framework of strongly harmonizable multidimensional complex processes. Section IV treats WSS processes. In the scalar case, we obtain a characterization of the spectral measure and the complementary spectral measure, which generalizes a result by Picinbono [6] to signals whose spectral measure and complementary spectral measure are not necessarily absolutely continuous with respect to Lebesgue measure.

Sections V and VI deal with strongly harmonizable scalar cyclostationary and multidimensional almost cyclostationary processes, respectively. First, we show that, for scalar processes, the assumption of strong harmonizability leads to an extension of Dehay's result on absolute continuity of the off-diagonal measures with respect to the diagonal measure. This result gives a stronger conclusion under a stronger hypothesis, and the proof is based on a simple decomposition technique and the Cauchy–Schwarz inequality. Then, in Section VI, we relax the assumption of strong harmonizability to uniform continuity of the correlation function. We give a generalization of [20, Theorem 2.3] to *multidimensional* almost cyclostationary processes. This result says that there exists a sequence of spectral measures supported on subdiagonals, and the off-diagonal components are absolutely continuous with respect to the diagonal component. As a special case, this gives representation formulas for the complementary spectral measures and the off-diagonal spectral measures of a *scalar* improper strongly harmonizable almost cyclostationary process, in terms of the diagonal spectral measure. The latter result is similar to, but slightly weaker than, the corresponding result for cyclostationary strongly harmonizable processes in Section V.

Intuitively, the results we obtain for almost cyclostationary processes might be anticipated as a consequence of the Cauchy–Schwarz inequality in the frequency domain. This inequality bounds the off-diagonal values in terms of the diagonal. However, since both the diagonal and off-diagonal spectral measures and complementary spectral measures may contain Dirac measures, some measure-theoretic care must be taken in the application of the Cauchy–Schwarz inequality. Even if the spectral measures do not contain Diracs, they are themselves "Dirac ridges" along certain subdiagonals.

Finally, in Section VII, we discuss some implications of our results for analytic signals. Using our results, we obtain information on how the Hilbert transform affects the complementary spectral measure for (almost) cyclostationary processes. We give sufficient conditions, in the spectral domain, for an analytic almost cyclostationary signal to be proper.

A. Notations and Definitions

We let $L_0^2(\Omega) = L_0^2(\Omega, \mathcal{B}, P)$ denote a given space of second-order zero-mean complex-valued random variables, and $L_0^2(\Omega, \mathbb{C}^d)$ a space of zero-mean second-order stochastic variables with values in $\mathbb{C}^d$. When $d = \infty$, we study $L_0^2(\Omega, l^2)$, which is the space of random variables that take values in the square-summable sequence space $l^2 = l^2(\mathbb{Z})$. We will use the term nonnegative definite (NND) for matrices, functions, and measures. Thus, the term must be interpreted in slightly different ways, depending on the context. They all resemble the NND definition for matrices: $\boldsymbol{A} \in \mathbb{C}^{d\times d}$ is NND if and only if $\boldsymbol{x}^H\boldsymbol{A}\boldsymbol{x} \geq 0$ for all $\boldsymbol{x} \in \mathbb{C}^d$. Let $\mathcal{R}(\boldsymbol{R})$ denote the range space of a matrix $\boldsymbol{R} \in \mathbb{C}^{d\times d}$, and $\mathcal{N}(\boldsymbol{R})$ its null space. The pseudoinverse, or Moore–Penrose generalized inverse [27], of a possibly singular matrix $\boldsymbol{R}$ is $\boldsymbol{R}^+$. When $\boldsymbol{R}$ is Hermitian, we have $\boldsymbol{R}\boldsymbol{R}^+ = \boldsymbol{R}^+\boldsymbol{R} = I_{\mathcal{R}(\boldsymbol{R})}$, which is the matrix that leaves $\mathcal{R}(\boldsymbol{R})$ invariant and is zero on $\mathcal{R}(\boldsymbol{R})^\perp = \mathcal{N}(\boldsymbol{R})$. We define the translation operator for measures and functions as $(\tau_a\mu)(x) = \mu(x-a)$, and for sets as $\tau_a A = A + a$, $A \subset \mathbb{R}$, $a \in \mathbb{R}$. Note that $(\tau_a\mu)(A) = \mu(\tau_{-a}A)$.

We will utilize some measure theory [28]. The domain of a complex-valued measure μ on $\mathbb{R}$ is a class of subsets of $\mathbb{R}$, which is closed under countable unions and complement, and it includes the empty set. The Borel σ-algebra, denoted $\mathcal{B}(\mathbb{R})$, will be used as domain for measures. A measure fulfills $\mu(\emptyset) = 0$. It is countably additive in the sense of $\mu(\bigcup_1^\infty A_k) = \sum_1^\infty \mu(A_k)$ when $\{A_k\}_1^\infty$ are pairwise disjoint.

The Dirac measure is defined by $\delta_0(A) = 1$ if $0 \in A$ and $\delta_0(A) = 0$ if $0 \notin A$. The measure μ on $\mathbb{R}^2$, defined by $\mu(x,y) = \delta_0(x-y)$, fulfills $\mu(A\times B) = \delta_0(A\cap B)$, $A, B \in \mathcal{B}(\mathbb{R})$. The tensor product $\otimes$ is defined as $(f\otimes g)(x,y) = f(x)g(y)$ for functions or measures f, g. Composition of a measure μ and a transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is denoted $(\mu\circ T)(x) = \mu(T(x))$. The *total variation* $|\mu|$ of a measure μ is

$$|\mu|(A) = \sup_{\{A_k\}_1^\infty} \sum_1^\infty |\mu(A_k)|, \qquad A \in \mathcal{B}(\mathbb{R})$$

where the supremum is taken over all $\{A_k\}_1^\infty$ such that $\bigcup_1^\infty A_k = A$ and $\{A_k\}_1^\infty$ are pairwise disjoint. The total variation $|\mu|$ is a finite measure if μ is a finite measure. If μ is a nonnegative measure and ν is a measure, ν is said to be absolutely continuous with respect to μ if $\mu(A) = 0$ implies

$\nu(A) = 0$ for all $A \in \mathcal{B}(\mathbb{R})$. Then the Radon–Nikodym theorem says that there exists a unique function $f \in L^1(\mu)$ such that

$$\nu(A) = \int_A f(x)\mu(dx).$$

We denote the set of bounded linear operators between two Hilbert spaces H_1, H_2 by $B(H_1, H_2)$ and its norm by $\|\cdot\|$. The set of trace-class operators [29] from H_1 to H_2 is denoted $T(H_1, H_2)$ and $T(H_1)$ if $H_2 = H_1$. It consists of all $T \in B(H_1, H_2)$ such that the norm

$$\|T\|_\tau = \sup \sum_{n\geq 0} |(Tf_n, g_n)_{H_2}|$$

is finite, where the supremum is taken over all orthonormal sequences $\{f_n\}_{n\geq 0} \subset H_1$ and $\{g_n\}_{n\geq 0} \subset H_2$. The subset of nonnegative definite trace-class operators is denoted $T(H_1)^+$. If $A \in B(H_2, H_1)$, $B \in B(H_1, H_2)$, and $T \in T(H_2)$ then $ATB \in T(H_1)$ and

$$\|ATB\|_{\tau(H_1)} \leq \|A\|\|B\|\|T\|_{\tau(H_2)}. \tag{1}$$

II. CHARACTERIZATION OF CORRELATION MATRICES

We first consider finite-dimensional, zero-mean stochastic vectors $\boldsymbol{s} = (s_1 \cdots s_d)^T$, and characterize the pair of matrices $\boldsymbol{R}$, $\widetilde{\boldsymbol{R}} \in \mathbb{C}^{d\times d}$ such that $E(\boldsymbol{s}\boldsymbol{s}^H) = \boldsymbol{R}$ and $E(\boldsymbol{s}\boldsymbol{s}^T) = \widetilde{\boldsymbol{R}}$. There is a well-known characterization when $\boldsymbol{R}$ is invertible [26], which says that $\boldsymbol{R}$, $\widetilde{\boldsymbol{R}}$ are a pair of correlation and complementary correlation matrices if and only if $\boldsymbol{R}$ is NND, $\widetilde{\boldsymbol{R}}$ is symmetric, and $\widetilde{\boldsymbol{R}}^* - \widetilde{\boldsymbol{R}}^*\boldsymbol{R}^{-1}\widetilde{\boldsymbol{R}}$ is NND. The following result is a generalization of this theorem when $\boldsymbol{R}$ is singular.

Theorem 1: Let $\boldsymbol{R}$, $\widetilde{\boldsymbol{R}} \in \mathbb{C}^{d\times d}$. There exists a stochastic vector $\boldsymbol{s}$ with correlation matrix $\boldsymbol{R}$ and complementary correlation matrix $\widetilde{\boldsymbol{R}}$ if and only if $\boldsymbol{R}$ is NND, $\widetilde{\boldsymbol{R}}$ is symmetric, $\mathcal{N}(\boldsymbol{R}) \subseteq \mathcal{N}(\widetilde{\boldsymbol{R}}^*)$, and $\widetilde{\boldsymbol{R}}^* - \widetilde{\boldsymbol{R}}^*\boldsymbol{R}^+\widetilde{\boldsymbol{R}}$ is NND.

Proof: Suppose there exists a vector $\boldsymbol{s}$ such that $E(\boldsymbol{s}\boldsymbol{s}^H) = \boldsymbol{R}$ and $E(\boldsymbol{s}\boldsymbol{s}^T) = \widetilde{\boldsymbol{R}}$. Then $\boldsymbol{R}$ is NND and $\widetilde{\boldsymbol{R}}$ is symmetric. The matrix

$$\overline{\boldsymbol{R}} = \begin{pmatrix} \boldsymbol{R} & \widetilde{\boldsymbol{R}} \\ \widetilde{\boldsymbol{R}}^* & \boldsymbol{R}^* \end{pmatrix} \in \mathbb{C}^{2d\times 2d} \tag{2}$$

is NND since it is the correlation matrix of the vector $(\boldsymbol{s}^T\ \boldsymbol{s}^H)^T$. The identity $\mathcal{N}(\boldsymbol{A}) = \mathcal{R}(\boldsymbol{A}^H)^\perp$, valid for any matrix $\boldsymbol{A} \in \mathbb{C}^{d\times d}$, gives $\mathcal{N}(\widetilde{\boldsymbol{R}}^*) = \mathcal{R}(\widetilde{\boldsymbol{R}})^\perp$. Suppose $\mathcal{N}(\boldsymbol{R}) \nsubseteq \mathcal{N}(\widetilde{\boldsymbol{R}}^*)$. Then there exist $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^d \setminus \{\boldsymbol{0}\}$ such that $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{0}$ and $\boldsymbol{x}^H\widetilde{\boldsymbol{R}}\boldsymbol{y} \neq \boldsymbol{0}$. Without loss of generality, we can choose $\boldsymbol{y}$ such that $\boldsymbol{x}^H\widetilde{\boldsymbol{R}}\boldsymbol{y} = b < 0$ and $\|\boldsymbol{y}\| = 1$. Let λ_1 denote the largest eigenvalue of $\boldsymbol{R}$, which is positive (because otherwise we have the trivial case $\boldsymbol{s} = \boldsymbol{0}$). Then, if $a \in \mathbb{R}$, we get

$$\begin{aligned}
&(a\boldsymbol{x}^H\ \boldsymbol{y}^H)\overline{\boldsymbol{R}}\begin{pmatrix} a\boldsymbol{x} \\ \boldsymbol{y} \end{pmatrix} \\
&\quad = a^2\boldsymbol{x}^H\boldsymbol{R}\boldsymbol{x} + \boldsymbol{y}^H\boldsymbol{R}^*\boldsymbol{y} + a\boldsymbol{x}^H\widetilde{\boldsymbol{R}}\boldsymbol{y} + a\boldsymbol{y}^H\widetilde{\boldsymbol{R}}^*\boldsymbol{x} \\
&\quad = \boldsymbol{y}^H\boldsymbol{R}^*\boldsymbol{y} + 2a\mathrm{Re}(\boldsymbol{x}^H\widetilde{\boldsymbol{R}}\boldsymbol{y}) \\
&\quad \leq \lambda_1 + 2ab < 0
\end{aligned}$$

if a is chosen large enough. This contradicts the assumption that $\overline{\boldsymbol{R}}$ is NND, and thus $\mathcal{N}(\boldsymbol{R}) \subseteq \mathcal{N}(\widetilde{\boldsymbol{R}}^*)$.

We now show that $\overline{\boldsymbol{R}}$ can be factored as

$$\begin{aligned}
\overline{\boldsymbol{R}} &= \begin{pmatrix} \boldsymbol{I} & \boldsymbol{0} \\ \widetilde{\boldsymbol{R}}^H\boldsymbol{R}^+ & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}^* - \widetilde{\boldsymbol{R}}^*\boldsymbol{R}^+\widetilde{\boldsymbol{R}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \boldsymbol{I} & \boldsymbol{R}^+\widetilde{\boldsymbol{R}} \\ \boldsymbol{0} & \boldsymbol{I} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{R} & \boldsymbol{R}\boldsymbol{R}^+\widetilde{\boldsymbol{R}} \\ \widetilde{\boldsymbol{R}}^*\boldsymbol{R}^+\boldsymbol{R} & \boldsymbol{R}^* \end{pmatrix}.
\end{aligned} \tag{3}$$

Since $\boldsymbol{R} = \boldsymbol{R}^H$ and $\widetilde{\boldsymbol{R}} = \widetilde{\boldsymbol{R}}^T$, we have $\mathcal{N}(\boldsymbol{R}) = \mathcal{R}(\boldsymbol{R})^\perp$ and $\mathcal{N}(\widetilde{\boldsymbol{R}}^*) = \mathcal{R}(\widetilde{\boldsymbol{R}})^\perp$. Thus, $\mathcal{N}(\boldsymbol{R}) \subseteq \mathcal{N}(\widetilde{\boldsymbol{R}}^*)$ is equivalent to $\mathcal{R}(\widetilde{\boldsymbol{R}}) \subseteq \mathcal{R}(\boldsymbol{R})$. It follows that $\boldsymbol{R}\boldsymbol{R}^+\widetilde{\boldsymbol{R}} = \widetilde{\boldsymbol{R}}$, since $\boldsymbol{R}\boldsymbol{R}^+$ restricted to $\mathcal{R}(\boldsymbol{R})$ acts like the identity matrix. Now let $\boldsymbol{x} \in \mathbb{C}^d$ be arbitrary, and decompose it into $\boldsymbol{x} = \boldsymbol{x}_n + \boldsymbol{x}_r$ where $\boldsymbol{x}_n \in \mathcal{N}(\boldsymbol{R})$ and $\boldsymbol{x}_r \in \mathcal{R}(\boldsymbol{R})$. Then $\widetilde{\boldsymbol{R}}^*\boldsymbol{R}^+\boldsymbol{R}\boldsymbol{x} = \widetilde{\boldsymbol{R}}^*\boldsymbol{R}^+\boldsymbol{R}\boldsymbol{x}_r = \widetilde{\boldsymbol{R}}^*\boldsymbol{x}_r = \widetilde{\boldsymbol{R}}^*\boldsymbol{x}$, since $\mathcal{N}(\boldsymbol{R}) \subseteq \mathcal{N}(\widetilde{\boldsymbol{R}}^*)$. This proves the decomposition (3), which implies that $\boldsymbol{R}^* - \widetilde{\boldsymbol{R}}^*\boldsymbol{R}^+\widetilde{\boldsymbol{R}}$ is NND. Hence, the four conditions in the statement of the theorem have been established.

We now prove the sufficiency of the four conditions. Assume that $\boldsymbol{R}$ is NND, $\widetilde{\boldsymbol{R}}$ is symmetric, $\mathcal{N}(\boldsymbol{R}) \subseteq \mathcal{N}(\widetilde{\boldsymbol{R}}^*)$ and $\boldsymbol{R}^* - \widetilde{\boldsymbol{R}}^*\boldsymbol{R}^+\widetilde{\boldsymbol{R}}$ is NND. By the same arguments as above, decomposition (3) holds. Because $\boldsymbol{R}$ and $\boldsymbol{R}^* - \widetilde{\boldsymbol{R}}^*\boldsymbol{R}^+\widetilde{\boldsymbol{R}}$ are NND, $\overline{\boldsymbol{R}}$ is NND. If we define the unitary matrix [30]

$$\boldsymbol{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} \boldsymbol{I} & j\boldsymbol{I} \\ \boldsymbol{I} & -j\boldsymbol{I} \end{pmatrix} \in \mathbb{C}^{2d\times 2d}$$

we obtain

$$\boldsymbol{T}^H\overline{\boldsymbol{R}}\boldsymbol{T} = \begin{pmatrix} \mathrm{Re}(\boldsymbol{R} + \widetilde{\boldsymbol{R}}) & -\mathrm{Im}(\boldsymbol{R} - \widetilde{\boldsymbol{R}}) \\ \mathrm{Im}(\boldsymbol{R} + \widetilde{\boldsymbol{R}}) & \mathrm{Re}(\boldsymbol{R} - \widetilde{\boldsymbol{R}}) \end{pmatrix}.$$

Since $\boldsymbol{T}^H\overline{\boldsymbol{R}}\boldsymbol{T}$ is NND, there exists a real-valued stochastic vector $(\boldsymbol{x}^T\ \boldsymbol{y}^T)^T$, $\boldsymbol{x} \in \mathbb{R}^d$, and $\boldsymbol{y} \in \mathbb{R}^d$, such that the correlation matrix of $(\boldsymbol{x}^T\ \boldsymbol{y}^T)^T$ is $\boldsymbol{T}^H\overline{\boldsymbol{R}}\boldsymbol{T}$. If we define $\boldsymbol{s} = 1/\sqrt{2}(\boldsymbol{x} + j\boldsymbol{y})$, it follows that $E(\boldsymbol{s}\boldsymbol{s}^H) = \boldsymbol{R}$ and $E(\boldsymbol{s}\boldsymbol{s}^T) = \widetilde{\boldsymbol{R}}$ [30]. ☐

III. STRONGLY HARMONIZABLE PROCESSES

A. Scalar-Valued Processes

Let $s(t)$ denote a zero-mean complex-valued second-order stochastic process defined on $\mathbb{R}$. Its correlation function is

$$r(t, u) = E(s(t)s^*(u))$$

and the complementary correlation function is defined by

$$\widetilde{r}(t, u) = E(s(t)s(u)).$$

If $\widetilde{r}(t, u) = 0$ for all $t, u \in \mathbb{R}$, the process is said to be *proper* (or circularly symmetric), otherwise, improper [1]. The Cauchy–Schwarz inequality gives the bound

$$|\widetilde{r}(t, u)|^2 \leq r(t, t)r(u, u), \qquad t, u \in \mathbb{R}.$$

We assume that a scalar process $s(t)$ is strongly harmonizable [17], [31]. This means that it can be represented as

$$s(t) = \int_{\mathbb{R}} e^{jt\xi} S(d\xi) \tag{4}$$

where S is an $L_0^2(\Omega)$-valued measure, called spectral process [17], [31], such that the correlation function can be written as

$$r(t,u) = \iint_{\mathbb{R}^2} e^{j(t\xi - u\eta)} m(d\xi, d\eta) \tag{5}$$

where

$$m(A,B) = E(S(A)S^*(B)), \qquad A,B \in \mathcal{B}(\mathbb{R}). \tag{6}$$

By definition, m is assumed to be a measure on $\mathbb{R}^2$ of bounded variation [17], [31]. The measure m is called the spectral measure. It is NND, which means

$$\sum_{i,k=1}^{n} x_i x_k^* m(A_i, A_k) \geq 0 \quad \forall \{x_i\}_{i=1}^n \subset \mathbb{C},$$
$$\{A_i\}_{i=1}^n \subset \mathcal{B}(\mathbb{R}) \text{ pairwise disjoint}, \quad n > 0.$$

It follows from (4) that the complementary correlation function can be represented as

$$\widetilde{r}(t,u) = \iint_{\mathbb{R}^2} e^{j(t\xi - u\eta)} \widetilde{m}(d\xi, d\eta) \tag{7}$$

where

$$\widetilde{m}(A,B) = E(S(A)S(-B)), \qquad A,B \in \mathcal{B}(\mathbb{R}) \tag{8}$$

is called the complementary spectral measure. From the assumption that m is a measure it follows that $\widetilde{m}$ is a *bimeasure*, i.e., $\widetilde{m}(A,\cdot)$ and $\widetilde{m}(\cdot,B)$ are measures of bounded variation on $\mathbb{R}$ for all $A,B \in \mathcal{B}(\mathbb{R})$ [17], [31]. The set of bimeasures is larger than the set of measures. We will assume that $\widetilde{m}$ is not only a bimeasure, but a measure of bounded variation on $\mathbb{R}^2$.

When they exist, the derivatives of m and $\widetilde{m}$ with respect to Lebesgue measure are called the spectral correlation and complementary spectral correlation function, respectively [32]. From (6), (8), and the Cauchy–Schwarz inequality we obtain the basic symmetries and bounds

$$\begin{aligned} m(A,B) &= m^*(B,A) \\ \widetilde{m}(A,B) &= \widetilde{m}(-B,-A) \\ |m(A,B)|^2 &\leq m(A,A)m(B,B) \\ |\widetilde{m}(A,B)|^2 &\leq m(A,A)m(-B,-B), \quad A,B \in \mathcal{B}(\mathbb{R}). \end{aligned} \tag{9}$$

B. Vector-Valued Processes

We will also work with vector-valued (multidimensional) processes $\boldsymbol{s}(t) = (s_1(t) \cdots s_d(t))^T$ [17], [33]. A vector-valued process can be seen as a map $\mathbb{R} \mapsto L_0^2(\Omega, \mathbb{C}^d)$ if $d < \infty$, or a map $\mathbb{R} \mapsto L_0^2(\Omega, l^2)$ if $d = \infty$ [17]. When $d < \infty$, the correlation function $\boldsymbol{R}$ of a multidimensional process $\boldsymbol{s}(t)$ is the matrix-valued function

$$\boldsymbol{R}(t,u) = E\left(\boldsymbol{s}(t)\boldsymbol{s}^H(u)\right) \in \mathbb{C}^{d\times d}, \qquad t,u \in \mathbb{R} \tag{10}$$

and the complementary correlation function is

$$\widetilde{\boldsymbol{R}}(t,u) = E\left(\boldsymbol{s}(t)\boldsymbol{s}^T(u)\right).$$

We have $\boldsymbol{R}^H(t,u) = \boldsymbol{R}(u,t)$ and $\widetilde{\boldsymbol{R}}^T(t,u) = \widetilde{\boldsymbol{R}}(u,t)$. When $d = \infty$, the correlation function is defined as an operator on l^2 by

$$(\boldsymbol{R}(t,u)\boldsymbol{x},\boldsymbol{y})_{l^2} = E((\boldsymbol{s}(t),\boldsymbol{y})_{l^2}(\boldsymbol{x},\boldsymbol{s}(u))_{l^2}), \qquad \boldsymbol{x},\boldsymbol{y} \in l^2.$$

It follows that $\boldsymbol{R}(t,u) \in T(l^2)$ [17], i.e., $\boldsymbol{R}$ is trace-class operator valued, and the same is true for $\widetilde{\boldsymbol{R}}$.

As in the scalar-valued case, a multidimensional process is said to be proper if $E(\boldsymbol{s}(t)\boldsymbol{s}^T(u)) = \boldsymbol{0}$ for all $t,u \in \mathbb{R}$. This means that each process $s_i(t)$, $1 \leq i \leq d$, is proper. Furthermore, all processes are jointly proper, i.e., $E(s_i(t)s_k(u)) = 0$ for all $t,u \in \mathbb{R}$ and all $1 \leq i \neq k \leq d$.

The theory of strongly harmonizable processes extends to the vector-valued case, with some subtle modifications when $d = \infty$ [17]. A vector-valued process is said to be strongly harmonizable if we can write

$$\begin{aligned} \boldsymbol{R}(t,u) &= \iint_{\mathbb{R}^2} e^{j(t\xi - u\eta)} \boldsymbol{M}(d\xi, d\eta) \\ \widetilde{\boldsymbol{R}}(t,u) &= \iint_{\mathbb{R}^2} e^{j(t\xi - u\eta)} \widetilde{\boldsymbol{M}}(d\xi, d\eta) \end{aligned} \tag{11}$$

where $\boldsymbol{M}$ and $\widetilde{\boldsymbol{M}}$ are $\mathbb{C}^{d\times d}$-valued bounded measures on $\mathbb{R}^2$. When $d = \infty$, $\boldsymbol{M}$ and $\widetilde{\boldsymbol{M}}$ are $T(l^2)$-valued, which implies that each entry of the matrices $\boldsymbol{M}$ and $\widetilde{\boldsymbol{M}}$ is a bounded measure on $\mathbb{R}^2$.

As in the scalar case, the assumption that $\boldsymbol{M}$ has bounded variation implies that $\widetilde{\boldsymbol{M}}$ is a bimeasure [17], but we will again assume the stronger property that $\widetilde{\boldsymbol{M}}$ has bounded variation.

The spectral measure $\boldsymbol{M}$ is NND according to the definition

$$\begin{aligned} &\sum_{i,k=1}^{n} \boldsymbol{x}_i^H \boldsymbol{M}(A_i, A_k)\boldsymbol{x}_k \geq 0, \quad \forall \{\boldsymbol{x}_i\}_{i=1}^n \subset \mathbb{C}^d, \\ &\{A_i\}_{i=1}^n \subset \mathcal{B}(\mathbb{R}) \text{ pairwise disjoint}, \quad n > 0. \end{aligned} \tag{12}$$

Corresponding to (4), we have the spectral representation

$$\boldsymbol{s}(t) = \int_{\mathbb{R}} e^{jt\xi} \boldsymbol{S}(d\xi),$$

where $\boldsymbol{S}$ is a d-dimensional spectral process that satisfies $\boldsymbol{M}(A,B) = E(\boldsymbol{S}(A)\boldsymbol{S}^H(B))$, $A,B \in \mathcal{B}(\mathbb{R})$, and $\widetilde{\boldsymbol{M}}(A,B) = E(\boldsymbol{S}(A)\boldsymbol{S}^T(-B))$.

We will need the following characterization of $\mathbb{C}^{d\times d}$-valued functions on $\mathbb{R}^2$ that are correlation functions. It is a special case of [17, Proposition IV.1.2].

Lemma 1: If $\boldsymbol{s}(t)$ is a $\mathbb{C}^d$-valued process, then its correlation function satisfies

$$\begin{aligned} &\sum_{i,k=1}^{n} \boldsymbol{x}_i^H \boldsymbol{R}(t_i, t_k)\boldsymbol{x}_k \geq 0, \quad \forall \{\boldsymbol{x}_i\}_{i=1}^n \subset \mathbb{C}^d, \\ &\{t_i\}_{i=1}^n \subset \mathbb{R}, \quad n > 0. \end{aligned} \tag{13}$$

Conversely, for every $\boldsymbol{R} : \mathbb{R}^2 \mapsto \mathbb{C}^{d\times d}$ that fulfills (13), there exists a proper process $\boldsymbol{s}(t)$ with correlation function $\boldsymbol{R}(t,u)$, and $\widetilde{\boldsymbol{R}}(t,u) = \boldsymbol{0}$ for all $t,u \in \mathbb{R}$.

Results for the complementary spectral measure of a scalar-valued process $s(t)$ can be obtained by setting $d = 2$ and considering vectors with the special structure $\overline{\boldsymbol{s}}(t) = (s(t)\ s^*(t))^T$. Their correlation function is

$$\overline{\boldsymbol{R}}(t,u) = \begin{pmatrix} r(t,u) & \widetilde{r}(t,u) \\ \widetilde{r}^*(t,u) & r^*(t,u) \end{pmatrix} \in \mathbb{C}^{2\times 2} \tag{14}$$

and the corresponding spectral measure is

$$\overline{\boldsymbol{M}}(A,B) = \begin{pmatrix} m(A,B) & \widetilde{m}(A,B) \\ \widetilde{m}^*(-A,-B) & m^*(-A,-B) \end{pmatrix}, \qquad A,B \in \mathcal{B}(\mathbb{R}). \tag{15}$$

Both $\overline{\boldsymbol{R}}$ and $\overline{\boldsymbol{M}}$ are NND, i.e., they satisfy (13) and (12), respectively. However, it is not easy to interpret (12) as a relation between m and $\widetilde{m}$. Nevertheless, as we will see, a simple characterization of the set of possible m and $\widetilde{m}$ exists in the case of scalar WSS processes.

IV. WSS PROCESSES

In this section, we assume that $d < \infty$. A zero mean d-dimensional process $\boldsymbol{s}(t)$ is said to be WSS if there exist single-variable matrix-valued functions $\boldsymbol{\rho}$ and $\widetilde{\boldsymbol{\rho}}$ such that the correlation and complementary correlation functions satisfy $\boldsymbol{R}(t,u) = \boldsymbol{\rho}(t-u)$ and $\widetilde{\boldsymbol{R}}(t,u) = \widetilde{\boldsymbol{\rho}}(t-u)$. If $\boldsymbol{\rho}$ is continuous then it automatically has a spectral representation with a measure supported on the diagonal only, i.e., $\boldsymbol{M}(A,B) = \boldsymbol{\mu}(A\cap B)$ [17]. Our definition of strong harmonizability includes that $\widetilde{\boldsymbol{M}}$ is a $\mathbb{C}^{d\times d}$-valued measure, and $\widetilde{\boldsymbol{M}}(A,B) = \widetilde{\boldsymbol{\mu}}(A\cap B)$ follows from $\widetilde{\boldsymbol{R}}(t,u) = \widetilde{\boldsymbol{\rho}}(t-u)$. Hence, we have

$$\begin{aligned} \boldsymbol{\rho}(t) &= \int_{\mathbb{R}} e^{jt\xi}\boldsymbol{\mu}(d\xi) \\ \widetilde{\boldsymbol{\rho}}(t) &= \int_{\mathbb{R}} e^{jt\xi}\widetilde{\boldsymbol{\mu}}(d\xi) \end{aligned} \tag{16}$$

where $\boldsymbol{\mu}(A) \in \mathbb{C}^{d\times d}$ is a NND matrix for all $A \in \mathcal{B}(\mathbb{R})$ [17]. Moreover, $\widetilde{\boldsymbol{\rho}}^T(u-t) = \widetilde{\boldsymbol{R}}^T(u,t) = \widetilde{\boldsymbol{R}}(t,u) = \widetilde{\boldsymbol{\rho}}(t-u)$ implies $\widetilde{\boldsymbol{\mu}}(A) = \widetilde{\boldsymbol{\mu}}^T(-A)$.

Next, we prove a characterization of the measure pairs $(\boldsymbol{\mu}, \widetilde{\boldsymbol{\mu}})$ such that $\boldsymbol{\rho}$ and $\widetilde{\boldsymbol{\rho}}$ are the correlation and the complementary correlation functions of a d-dimensional WSS process. For this purpose, we define the $2d$-dimensional WSS process $\overline{\boldsymbol{s}}(t) = (\boldsymbol{s}^T(t)\ \ \boldsymbol{s}^H(t))^T$, which has correlation function

$$\overline{\boldsymbol{\rho}}(t) = \begin{pmatrix} \boldsymbol{\rho}(t) & \widetilde{\boldsymbol{\rho}}(t) \\ \widetilde{\boldsymbol{\rho}}^*(t) & \boldsymbol{\rho}^*(t) \end{pmatrix} \in \mathbb{C}^{2d\times 2d}. \tag{17}$$

The corresponding spectral process is $\overline{\boldsymbol{S}}(A) = (\boldsymbol{S}^T(A)\ \boldsymbol{S}^H(-A))^T$, and

$$\begin{aligned} \overline{\boldsymbol{\mu}}(A) &= E(\overline{\boldsymbol{S}}(A)\overline{\boldsymbol{S}}^H(A)) = \begin{pmatrix} \boldsymbol{\mu}(A) & \widetilde{\boldsymbol{\mu}}(A) \\ \widetilde{\boldsymbol{\mu}}^*(-A) & \boldsymbol{\mu}^*(-A) \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{\mu}(A) & \widetilde{\boldsymbol{\mu}}(A) \\ \widetilde{\boldsymbol{\mu}}^H(A) & \boldsymbol{\mu}^*(-A) \end{pmatrix}, \qquad A \in \mathcal{B}(\mathbb{R}). \end{aligned} \tag{18}$$

Theorem 2: Let $\boldsymbol{\rho}$ and $\widetilde{\boldsymbol{\rho}}$ be the inverse Fourier transforms of the measures $\boldsymbol{\mu}$ and $\widetilde{\boldsymbol{\mu}}$, respectively, as specified by (16). There exists a d-dimensional WSS process $\boldsymbol{s}(t)$ with correlation function $\boldsymbol{\rho}$ and complementary correlation function $\widetilde{\boldsymbol{\rho}}$ if and only if, for each $A \in \mathcal{B}(\mathbb{R})$, $\widetilde{\boldsymbol{\mu}}^T(-A) = \widetilde{\boldsymbol{\mu}}(A)$, $\boldsymbol{\mu}(A)$ is NND, and $\overline{\boldsymbol{\mu}}(A)$ is NND.

Proof: Suppose there exists a process $\boldsymbol{s}(t)$ with correlation function $\boldsymbol{\rho}$ and complementary correlation function $\widetilde{\boldsymbol{\rho}}$. By a generalization of Bochner's theorem [17, Theorem II.5.5], $\boldsymbol{\mu}$ is a $\mathbb{C}^{d\times d}$-valued measure such that $\boldsymbol{\mu}(A)$ is NND for all $A \in \mathcal{B}(\mathbb{R})$. Since $\widetilde{\boldsymbol{\rho}}^T(-t) = \widetilde{\boldsymbol{R}}^T(0,t) = \widetilde{\boldsymbol{R}}(t,0) = \widetilde{\boldsymbol{\rho}}(t)$, we have $\widetilde{\boldsymbol{\mu}}(A) = \widetilde{\boldsymbol{\mu}}^T(-A)$ for all $A \in \mathcal{B}(\mathbb{R})$. Finally, the fact that $\overline{\boldsymbol{\mu}}(A)$ is NND for all $A \in \mathcal{B}(\mathbb{R})$ follows from (18). Thus, the three conditions are necessary.

To prove sufficiency, suppose that the three conditions are satisfied. Define $\overline{\boldsymbol{M}}(A,B) = \overline{\boldsymbol{\mu}}(A\cap B)$ and let $\boldsymbol{x}_i \in \mathbb{C}^{2d}$, $A_i \in \mathcal{B}(\mathbb{R})$ (pairwise disjoint), for $1 \le i \le n$, and $n > 0$ be arbitrary. The diagonal support of $\overline{\boldsymbol{M}}$ gives

$$\sum_{i,k=1}^{n} \boldsymbol{x}_i^H \overline{\boldsymbol{M}}(A_i, A_k)\boldsymbol{x}_k = \sum_{i=1}^{n} \boldsymbol{x}_i^H \overline{\boldsymbol{\mu}}(A_i)\boldsymbol{x}_i \ge 0 \tag{19}$$

since $\overline{\boldsymbol{\mu}}(A_i)$ is NND for all $1 \le i \le n$. If we define $\overline{\boldsymbol{R}}$ by the first line of (11) with $\boldsymbol{M} = \overline{\boldsymbol{M}}$, it follows from (19) that $\overline{\boldsymbol{R}}$ is NND. Furthermore, the diagonal support of $\overline{\boldsymbol{M}}$ implies that $\overline{\boldsymbol{R}}$ is the correlation function of a WSS process, and we have $\overline{\boldsymbol{R}}(t,u) = \overline{\boldsymbol{\rho}}(t-u)$, defined by (17). Lemma 1 implies that there exists a proper WSS process $\boldsymbol{w}(t) = (\boldsymbol{w}_1^T(t)\ \boldsymbol{w}_2^T(t))^T$, where $\boldsymbol{w}_1(t)$ and $\boldsymbol{w}_2(t)$ are d-dimensional processes whose correlation function is $E(\boldsymbol{w}(t)\boldsymbol{w}^H(u)) = \overline{\boldsymbol{R}}(t,u)$, and $E(\boldsymbol{w}(t)\boldsymbol{w}^T(u)) = \boldsymbol{0}$ for all $t,u \in \mathbb{R}$. If we now define $\boldsymbol{s}(t) = 1/\sqrt{2}(\boldsymbol{w}_1(t) + \boldsymbol{w}_2^*(t))$, we get

$$\begin{aligned} E(\boldsymbol{s}(t)\boldsymbol{s}^H(u)) &= \frac{1}{2}\left(E(\boldsymbol{w}_1(t)\boldsymbol{w}_1^H(u)) + E(\boldsymbol{w}_2^*(t)\boldsymbol{w}_2^T(u))\right) \\ &= \boldsymbol{\rho}(t-u), \\ E(\boldsymbol{s}(t)\boldsymbol{s}^T(u)) &= \frac{1}{2}\left(E(\boldsymbol{w}_1(t)\boldsymbol{w}_2^H(u)) + E(\boldsymbol{w}_2^*(t)\boldsymbol{w}_1^T(u))\right) \\ &= \widetilde{\boldsymbol{\rho}}(t-u). \end{aligned} \qquad \square$$

If we choose any pair of indices $1 \le i,k \le d$ and take the determinant of the 2×2 submatrix of the NND matrix (18) with row and column indices i and $k+d$, we obtain $|\widetilde{\boldsymbol{\mu}}_{i,k}(A)|^2 \le \boldsymbol{\mu}_{i,i}(A)\boldsymbol{\mu}_{k,k}(-A)$, for all $A \in \mathcal{B}(\mathbb{R})$, since $\boldsymbol{\mu}_{i,i}(A) \ge 0$ for all $1 \le i \le d$. This is a necessary but not sufficient condition for a WSS process $\boldsymbol{s}(t)$ to exist, since a matrix is not necessarily NND even if all its 2×2 submatrices involving the diagonal are NND.

In the scalar case, however, Theorem 2 gives a simple characterization of all μ and $\widetilde{\mu}$ such that μ and $\widetilde{\mu}$ are the spectral measure and complementary spectral measure of a WSS scalar-valued process $s(t)$. Similar results are discussed in [6], [30]. The following result is more general since it allows spectral measures μ and $\widetilde{\mu}$ that are not necessarily absolutely continuous with respect to the Lebesgue measure, as it is assumed in [6]. (The paper [6] treats the discrete-time case.)

Corollary 1: Let the scalar functions ρ and $\widetilde{\rho}$ be the inverse Fourier transforms of the scalar measures μ and $\widetilde{\mu}$ as specified by (16). There exists a scalar WSS process $s(t)$ with correlation function ρ and complementary correlation function $\widetilde{\rho}$ if and only if, for each $A \in \mathcal{B}(\mathbb{R})$, $\widetilde{\mu}(-A) = \widetilde{\mu}(A)$, $\mu(A) \ge 0$, and $|\widetilde{\mu}(A)|^2 \le \mu(A)\mu(-A)$.

The third condition in Corollary 1 implies that $\widetilde{\mu}$ is absolutely continuous with respect to μ. Hence, by the Radon–Nikodym theorem, there exists a unique function $f \in L^1(\mu)$ such that

$$\widetilde{\mu}(A) = \int_A f(\xi)\mu(d\xi), \quad A \in \mathcal{B}(\mathbb{R}).$$

V. SCALAR CYCLOSTATIONARY PROCESSES

In this section, we consider strongly harmonizable cyclostationary scalar processes $s(t)$. Multidimensional cyclostationary processes are included in the more general analysis of Section VI. A process is called cyclostationary if there exists a $T > 0$ such that r and $\widetilde{r}$ are T-periodic along the diagonals, i.e., $r(t+T, u+T) = r(t,u)$ and $\widetilde{r}(t+T, u+T) = \widetilde{r}(t,u)$ for all $t,\ u \in \mathbb{R}$. Alternatively, we can define $b(u,t) = r(u+t,u)$, $\widetilde{b}(u,t) = \widetilde{r}(u+t,u)$, and require periodicity in the first variable

$$\begin{aligned} b(u+T,t) &= b(u,t), \\ \widetilde{b}(u+T,t) &= \widetilde{b}(u,t), \qquad \forall u,t \in \mathbb{R}. \end{aligned} \tag{20}$$

The periodicity in u motivates the definition of Fourier coefficients in the first variable

$$\begin{aligned} b_k(t) &= \frac{1}{T}\int_0^T b(u,t)e^{-j2\pi uk/T}du, \\ \widetilde{b}_k(t) &= \frac{1}{T}\int_0^T \widetilde{b}(u,t)e^{-j2\pi uk/T}du, \qquad k \in \mathbb{Z}. \end{aligned} \tag{21}$$

The assumption that r is strongly harmonizable is sufficient for pointwise convergence of the Fourier series [19], [21] according to

$$\begin{aligned} b(u,t) &= \sum_{k\in\mathbb{Z}} b_k(t)e^{j2\pi uk/T} \\ \widetilde{b}(u,t) &= \sum_{k\in\mathbb{Z}} \widetilde{b}_k(t)e^{j2\pi uk/T}. \end{aligned} \tag{22}$$

Moreover, the Fourier coefficients $b_k(t)$ and $\widetilde{b}_k(t)$ can be written as [19]

$$\begin{aligned} b_k(t) &= \int_{\mathbb{R}} e^{j\xi t}\mu_k(d\xi) \\ \widetilde{b}_k(t) &= \int_{\mathbb{R}} e^{j\xi t}\widetilde{\mu}_k(d\xi) \end{aligned} \tag{23}$$

where μ_k and $\widetilde{\mu}_k$ are measures, with the symmetries

$$\begin{aligned} \mu_k(\xi) &= \mu^*_{-k}(\xi - 2\pi k/T) \\ \widetilde{\mu}_k(\xi) &= \widetilde{\mu}_k(-\xi + 2\pi k/T), \qquad k \in \mathbb{Z}. \end{aligned} \tag{24}$$

Insertion of (23) into (22) gives

$$\begin{aligned} r(u+t,u) &= b(u,t) = \sum_{k\in\mathbb{Z}} e^{j2\pi uk/T}\int_{\mathbb{R}} e^{j\xi t}\mu_k(d\xi) \\ \widetilde{r}(u+t,u) &= \widetilde{b}(u,t) = \sum_{k\in\mathbb{Z}} e^{j2\pi uk/T}\int_{\mathbb{R}} e^{j\xi t}\widetilde{\mu}_k(d\xi). \end{aligned} \tag{25}$$

To see the relation between m and μ_k, and between $\widetilde{m}$ and $\widetilde{\mu}_k$, we use the coordinate transformation $\kappa(x,y) = (x, x-y) = \kappa^{-1}(x,y)$, and rewrite the formula (5) as

$$r(u+t,u) = \iint_{\mathbb{R}^2} e^{j(t\xi+u\eta)} m\circ\kappa(d\xi, d\eta). \tag{26}$$

Comparison of (25) and (26) yields

$$m = \sum_{k=-\infty}^{\infty} \mu_k \otimes \delta_{2\pi k/T} \circ \kappa.$$

Hence, we have

$$\begin{aligned} m &= \sum_{k=-\infty}^{\infty} \mu_k \otimes \delta_0 \circ \tau_{(0,-2\pi k/T)} \circ \kappa \\ &= \sum_{k=-\infty}^{\infty} \mu_k \otimes \delta_0 \circ \kappa \circ \tau_{(0,2\pi k/T)}. \end{aligned} \tag{27}$$

From (27), we obtain

$$m(A,B) = \sum_{k=-\infty}^{\infty} \mu_k(A \cap (B + 2\pi k/T)). \tag{28}$$

The relation

$$\widetilde{m}(A,B) = \sum_{k=-\infty}^{\infty} \widetilde{\mu}_k(A \cap (B + 2\pi k/T)) \tag{29}$$

is derived analogously.

It is known that μ_k is absolutely continuous with respect to μ_0 for all $k \in \mathbb{Z}$ [20]. The proof in [20] is designed for almost cyclostationary processes (see Section VI) and the more general situation where the process is not assumed to be strongly harmonizable. However, in the more restrictive strongly harmonizable cyclostationary case this fact can alternatively be proved in a more straightforward way using (9), as shown by the following theorem. While our assumptions are more restrictive, the conclusion is stronger, since only the first of the identities (30) was derived in [20].

Theorem 3: If $s(t)$ is a scalar strongly harmonizable cyclostationary process, there exist unique functions $f_k,\ \widetilde{f}_k \in L^1(\mu_0)$, $k \in \mathbb{Z}$, such that

$$\mu_k(A) = \int_A f_k(\xi)\mu_0(d\xi) = \int_{A-2\pi k/T} f^*_{-k}(\xi)\mu_0(d\xi) \tag{30}$$

$$\widetilde{\mu}_k(A) = \int_A \widetilde{f}_k(\xi)\mu_0(d\xi) = \int_{-A+2\pi k/T} \widetilde{f}_k(\xi)\mu_0(d\xi). \tag{31}$$

Proof: Suppose $\mu_0(A) = 0$. Let $\{B_q\}_{q\in\mathbb{Z}}$ be an arbitrary partition of A, i.e., the $\{B_q\}_{q\in\mathbb{Z}}$ are pairwise disjoint and $\bigcup_{q\in\mathbb{Z}} B_q = A$. Set $B_{q,l} = B_q \bigcap [2\pi l/T, 2\pi(l+1)/T)$ for $l \in \mathbb{Z}$, which means that $\{B_{q,l}\}_{l\in\mathbb{Z}}$ are pairwise disjoint, $B_q = \bigcup_{l\in\mathbb{Z}} B_{q,l}$, and $B_{q,l}$ is contained in an interval of length $2\pi/T$ for each l. By assumption, $\mu_0(B_{q,l}) = 0$ for all $q, l \in \mathbb{Z}$. From (28) and (29) we obtain

$$\begin{aligned} m(B_{q,l}, B_{q,l} - 2\pi p/T) &= \mu_p(B_{q,l}) \\ m(B_{q,l} + 2\pi p/T, B_{q,l}) &= (\tau_{-2\pi p/T}\mu_p)(B_{q,l}) \\ \widetilde{m}(B_{q,l}, B_{q,l} - 2\pi p/T) &= \widetilde{\mu}_p(B_{q,l}). \end{aligned} \tag{32}$$

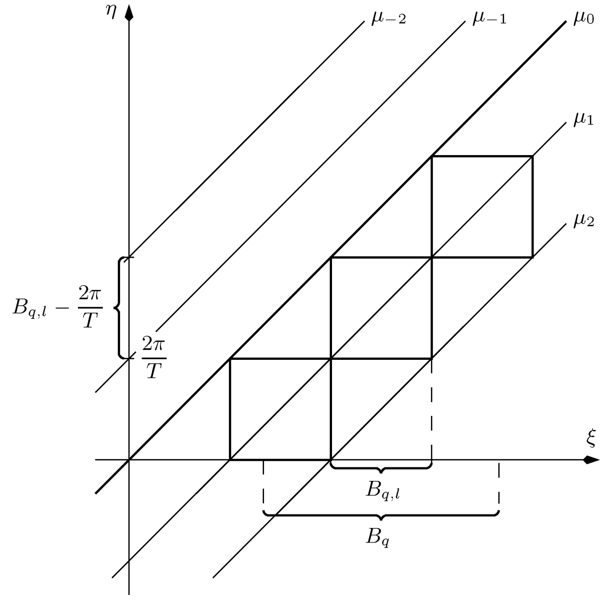

Fig. 1. An illustration of the decomposition technique used in the proof of Theorem 3. Since $\mu_1(B_{q,l}) = m(B_{q,l}, B_{q,l} - 2\pi/T)$, the assumption $m(B_{q,l}, B_{q,l}) = \mu_0(B_{q,l}) = 0$ and the Cauchy–Schwarz inequality (9) gives $\mu_1(B_{q,l}) = 0$.

Likewise, (28) shows that $m(B_{q,l}, B_{q,l}) = \mu_0(B_{q,l}) = 0$. Thus $\mu_p(B_{q,l}) = m(B_{q,l}, B_{q,l} - 2\pi p/T)$ and (9) give

$$|\mu_p(B_{q,l})|^2 \leq m(B_{q,l}, B_{q,l}) m(B_{q,l} - 2\pi p/T, B_{q,l} - 2\pi p/T) = 0$$

and we may conclude that

$$|\mu_p(B_{q,l})| = |\widetilde{\mu}_p(B_{q,l})| = |(\tau_{-2\pi p/T}\mu_p)(B_{q,l})| = 0$$

for all $q, l \in \mathbb{Z}$. Hence $\sum_{l \in \mathbb{Z}} |\mu_p(B_{q,l})| = 0$ for all $q \in \mathbb{Z}$, and

$$|\mu_p|(A) = \sup_{\{B_q\}} \sum_{q \in \mathbb{Z}} |\mu_p(B_q)| \leq \sup_{\{B_q\}} \sum_{q,l \in \mathbb{Z}} |\mu_p(B_{q,l})| = 0.$$

Similarly, it follows that $|\widetilde{\mu}_p|(A) = |\tau_{-2\pi p/T}\mu_p|(A) = 0$. This proves that μ_p, $\widetilde{\mu}_p$, and $\tau_{-2\pi p/T}\mu_p$ are all absolutely continuous with respect to μ_0. The Radon–Nikodym theorem [28] now yields the representations

$$\mu_k(A) = \int_A f_k(\xi)\mu_0(d\xi) = \int_{A-2\pi k/T} g_k(\xi)\mu_0(d\xi)$$
$$\widetilde{\mu}_k(A) = \int_A \widetilde{f}_k(\xi)\mu_0(d\xi) = \int_{-A+2\pi k/T} \widetilde{f}_k(\xi)\mu_0(d\xi)$$

for unique functions $f_k, g_k, \widetilde{f}_k \in L^1(\mu_0)$, where the second equality for $\widetilde{\mu}_k$ follows from the symmetry $\widetilde{\mu}_k(A) = \widetilde{\mu}_k(-A + 2\pi k/T)$ in (24). The symmetries (24) also imply that $g_k = f_{-k}^*$. ☐

The proof of Theorem 3 is based on the fact that the frequencies of a cyclostationary process are equidistant. This admits the employed decomposition procedure, where a set is split into parts of length not greater than the frequency distance, as shown in Fig. 1. The result then follows from the Cauchy–Schwarz inequality (9).

As we will see in the next section, this idea breaks down for almost cyclostationary processes. For such processes, the spectral measure and the complementary spectral measure still have support along subdiagonals as in the cyclostationary case, but the spectral components are no longer equidistant. In fact, there may even be no positive distance between them, which is why we cannot use the decomposition technique used in the proof of Theorem 3.

Nevertheless, we will present a result corresponding to Theorem 3 for strongly harmonizable almost cyclostationary processes (Theorem 5), with a different proof. In fact, Theorem 5 is a corollary of a result for multidimensional almost cyclostationary processes presented in the next section.

VI. ALMOST CYCLOSTATIONARY PROCESSES

In many applications, it is natural to relax the requirement of T-periodicity (20) to the weaker requirement that $b(\cdot, t)$ and $\widetilde{b}(\cdot, t)$ are *almost periodic* in the sense of Bohr [34], [35]. The process $s(t)$ is then said to be almost cyclostationary or almost periodically correlated [12], [18], [20], [21]. As for periodic functions, the spectrum of an almost periodic function is countable. However, the frequencies may not be contained in a lattice of the form $\{2\pi k/T\}_{k \in \mathbb{Z}}$, and may even come arbitrarily close to each other [18], [20], [21].

In this section, we work with vector-valued almost cyclostationary processes $\boldsymbol{s}(t) = (s_1(t) \cdots s_d(t))^T$, where each $s_i(t)$ is a complex-valued almost cyclostationary process. We assume that the dimension d is infinite, which contains the finite-dimensional case. We will prove a result on the existence of the off-diagonal spectral measures and their dependence on the diagonal measure. We relax the assumption of strong harmonizability and assume only that the correlation function is uniformly continuous. This result is a multidimensional generalization of Hurd's result [21, Proposition 3] and Dehay's result [20, Theorem 2.3]. Section VI-B specializes these findings to the scalar strongly harmonizable case, where we give a result corresponding to Theorem 3 for almost cyclostationary processes.

We work again with the coordinate-transformed correlation function $\boldsymbol{B}(u,t) = \boldsymbol{R}(u+t, u)$, which is $T(l^2)$-valued and assumed to be uniformly continuous. The assumption that $\boldsymbol{B}(\cdot, t)$ is almost periodic implies that the limit

$$\boldsymbol{B}_\lambda(t) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T+a}^{T+a} \boldsymbol{B}(u,t) e^{-j\lambda u} du \tag{33}$$

exists for all frequencies $\lambda \in \mathbb{R}$, independently of $a \in \mathbb{R}$. The uniform continuity of $\boldsymbol{B}$ implies that the set $\Lambda = \{\lambda \in \mathbb{R} : \exists t \in \mathbb{R} : \boldsymbol{B}_\lambda(t) \neq 0\}$ is countable [21, Proposition 4]. Therefore, we can assume that Λ is indexed by $\mathbb{Z}$ according to $\Lambda = \{\lambda_k\}_{k \in \mathbb{Z}}$. We will also assume that $\lambda_0 = 0$ and $k > 0 \Leftrightarrow \lambda_k > 0$. We may assume that Λ is closed under all finite linear combinations with rational coefficients, as in [16]. The set Λ is symmetric in the sense $\lambda \in \Lambda \Rightarrow -\lambda \in \Lambda$ [21]. Thus, we may assume that $-\lambda_k = \lambda_{-k}$. The functions $\{\boldsymbol{B}_{\lambda_k}(t)\}_{k \in \mathbb{Z}}$ are the Fourier coefficients for $\boldsymbol{B}(\cdot, t)$.

For $d = 1$, Gladyshev [16] proved that a uniformly continuous function $\boldsymbol{R}(t,u)$ that is almost periodic along the subdiagonals (i.e., satisfies the almost cyclostationarity criterion) is a correlation function of a scalar process if and only if

$$\boldsymbol{B}_{\lambda_k,\lambda_i}(t) = \boldsymbol{B}_{\lambda_i-\lambda_k}(t)e^{j\lambda_k t} \tag{34}$$

is the correlation function of a multidimensional WSS process. This result is the foundation for much of the subsequent development on (almost) cyclostationary processes [12]. For multidimensional cyclostationary processes, the function (34) has been defined in [17]. Note that $\boldsymbol{B}_{\lambda_k,\lambda_i}(t)$ is defined in terms of the Fourier coefficients (33), $\lambda_i - \lambda_k \in \Lambda$, and $\boldsymbol{B}_{\lambda_k,\lambda_i}(t)$ is indexed by $(k,i) \in \mathbb{Z}^2$. Hence, $\boldsymbol{B}_{\lambda_k,\lambda_i}(t)$ can be considered an infinite-dimensional matrix, indexed by $(k,i) \in \mathbb{Z}^2$, of $T(l^2)$-valued functions of $t \in \mathbb{R}$.

The following lemma is an extension of Gladyshev's result to multidimensional almost cyclostationary processes (see also [17, Theorem V.5.9]).

Lemma 2: Suppose $\boldsymbol{R} : \mathbb{R}^2 \mapsto T(l^2)$ is uniformly continuous. Let $\boldsymbol{B}(u,t) = \boldsymbol{R}(u+t,u)$ and suppose $\boldsymbol{B}(\cdot,t)$ is almost periodic for all $t \in \mathbb{R}$. Then $\boldsymbol{R}$ is the correlation function of a l^2-valued process if and only if

$$\sum_{i,k=1}^{n} (\boldsymbol{B}_{\lambda_{n_i},\lambda_{n_k}}(t_i - t_k)\boldsymbol{x}_k, \boldsymbol{x}_i)_{l^2} \ge 0, \quad \forall\{\boldsymbol{x}_i\}_{i=1}^{n} \subset l^2,\ \{n_i\}_{i=1}^{n} \subset \mathbb{Z},\ \{t_i\}_{i=1}^{n} \subset \mathbb{R},\ n>0. \tag{35}$$

Proof: Suppose $\boldsymbol{R}$ is the correlation function of an l^2-valued process. By Lemma 1 we have

$$\sum_{i,k=1}^{n} (\boldsymbol{R}(t_i,t_k)\boldsymbol{x}_k, \boldsymbol{x}_i)_{l^2} \ge 0, \quad \forall\{\boldsymbol{x}_i\}_{i=1}^{n} \subset l^2,\ \{t_i\}_{i=1}^{n} \subset \mathbb{R},\ n > 0. \tag{36}$$

Let $n > 0$, $\{\boldsymbol{x}_i\}_{i=1}^{n} \subset l^2$, $\{n_i\}_{i=1}^{n} \subset \mathbb{Z}$, and $\{t_i\}_{i=1}^{n} \subset \mathbb{R}$ be arbitrary. Using (33) with $a = t_k$ and $\boldsymbol{B}(u+t_k, t_i - t_k) = \boldsymbol{R}(u+t_i, u+t_k)$ we obtain

$$\begin{aligned}
&\sum_{i,k=1}^{n} (\boldsymbol{B}_{\lambda_{n_i},\lambda_{n_k}}(t_i - t_k)\boldsymbol{x}_k, \boldsymbol{x}_i)_{l^2} \\
&\quad = \sum_{i,k=1}^{n} (\boldsymbol{B}_{\lambda_{n_k}-\lambda_{n_i}}(t_i - t_k)e^{j\lambda_{n_i}(t_i - t_k)}\boldsymbol{x}_k, \boldsymbol{x}_i)_{l^2} \\
&\quad = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sum_{i,k=1}^{n} (\boldsymbol{B}(u+t_k, t_i - t_k) \\
&\qquad \times e^{j[\lambda_{n_i}(t_i-t_k) - (\lambda_{n_k}-\lambda_{n_i})(u+t_k)]}\boldsymbol{x}_k, \boldsymbol{x}_i)_{l^2}\,du \\
&\quad = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \sum_{i,k=1}^{n} (\boldsymbol{R}(u+t_i, u+t_k)\boldsymbol{x}_k \\
&\qquad \times e^{-j\lambda_{n_k}(t_k+u)}, \boldsymbol{x}_i e^{-j\lambda_{n_i}(t_i+u)})_{l^2}\,du \ge 0
\end{aligned}$$

where the inequality follows from (36).

The converse implication is proved by showing that (35) implies (36), and then applying Lemma 1. The proof that (35) $\Longrightarrow$ (36) is entirely analogous to the proof of [16, Theorem 2], and rather technical. We omit it. ☐

Remark 1: The condition (35) is equivalent to the existence of a vector-valued WSS process $\boldsymbol{z}(t) = \{\boldsymbol{z}_k(t)\}_{k\in\mathbb{Z}}$, whose components $\boldsymbol{z}_k(t)$ are l^2-valued processes such that $E(\boldsymbol{z}_k(t)\boldsymbol{z}_i^H(u)) = \boldsymbol{B}_{\lambda_k,\lambda_i}(t-u)$. Under certain circumstances, such a process can in fact be constructed from the original process [13], [21], [36], using independent time shifts. In particular, if the process $\boldsymbol{s}(t)$ is cyclostationary with period T, the construction is very simple: If θ is a random variable with uniform distribution over $[0,T]$, independent of $\boldsymbol{s}(t)$, then

$$\boldsymbol{z}_k(t) = \boldsymbol{s}(t+\theta)e^{j2\pi k(t+\theta)/T}$$

will meet the requirement. For almost cyclostationary processes, the construction of $\boldsymbol{z}_k(t)$ is more complicated and extra conditions are required [18], [21], [36].

A. A Multidimensional Generalization of Results by Hurd and Dehay

In the following proposition, we use Lemma 2 to prove that the assumption that $\boldsymbol{R}$ is uniformly continuous implies that each Fourier coefficient $\boldsymbol{B}_{\lambda_k}(t)$ is the Fourier transform of a $T(l^2)$-valued measure $\boldsymbol{\mu}_{\lambda_k}$, $k \in \mathbb{Z}$. The requirement that $\boldsymbol{R}$ is uniformly continuous is weaker than $\boldsymbol{R}$ being a Fourier transform of a measure (i.e., $\boldsymbol{s}$ strongly harmonizable). We also show that $\boldsymbol{\mu}_{\lambda_k}$ are absolutely continuous with respect to $\operatorname{tr}\boldsymbol{\mu}_0$, which is a nonnegative measure. In the case when $\boldsymbol{s}$ is strongly harmonizable, the measures $\boldsymbol{\mu}_{\lambda_k}$ are the restrictions of the spectral measure $\boldsymbol{M}$ to the subdiagonals as

$$\boldsymbol{M}(A,B) = \sum_{k=-\infty}^{\infty} \boldsymbol{\mu}_{\lambda_k}(A \cap (B + \lambda_k)) \tag{37}$$

which is a generalization of the formula for the scalar cyclostationary case (28).

Proposition 1: Suppose $\boldsymbol{R} : \mathbb{R}^2 \mapsto T(l^2)$ is uniformly continuous and the correlation function of an almost cyclostationary l^2-valued process, let $\boldsymbol{B}(u,t) = \boldsymbol{R}(u+t,u)$ and $\boldsymbol{B}_{\lambda_k}(t)$ be defined by (33). Then there exist unique measures $\boldsymbol{\mu}_{\lambda_k} : \mathcal{B}(\mathbb{R}) \mapsto T(l^2)$ such that

$$\boldsymbol{B}_{\lambda_k}(t) = \int_{\mathbb{R}} e^{jt\xi}\boldsymbol{\mu}_{\lambda_k}(d\xi), \qquad k \in \mathbb{Z} \tag{38}$$

and $\boldsymbol{\mu}_0$ is $T(l^2)^+$-valued. We have

$$\begin{aligned}
\|\boldsymbol{\mu}_{\lambda_k}(A)\|_\tau^2 &\le \operatorname{tr}(\boldsymbol{\mu}_0(A))\operatorname{tr}(\boldsymbol{\mu}_0(A - \lambda_k)) \\
&\le \operatorname{tr}(\boldsymbol{\mu}_0(A))\operatorname{tr}(\boldsymbol{B}_0(0)), \\
&\quad A \in \mathcal{B}(\mathbb{R}),\ k \in \mathbb{Z}\setminus\{0\}.
\end{aligned} \tag{39}$$

Proof: The idea of the proof is to create, for each integer $N \ge 0$, a $T(H)$-valued map $\boldsymbol{B}^{(N)}(t)$, using $\boldsymbol{B}_{\lambda_i,\lambda_k}(t)$ for a finite number of indices i,k. Here H is a Hilbert space direct sum consisting of $2N+1$ copies of l^2. If the dimension $d < \infty$, this construction means that $\boldsymbol{B}^{(N)}(t)$ is a block matrix with blocks $\{\boldsymbol{B}_{\lambda_i,\lambda_k}(t)\}_{|i|,|k|\le N}$. The map $\boldsymbol{B}^{(N)}$ will turn out to be the correlation function of an infinite-dimensional WSS process. Then we may use the spectral representation of infinite-dimensional

WSS correlation functions [17], which is a generalization of Bochner's theorem to Hilbert space-valued processes.

Let $N \geq 0$ be an arbitrary integer, set $K = 2N+1$ and define the Hilbert space $H = l^2 \oplus \cdots \oplus l^2$, the direct sum of K copies of l^2, with inner product

$$(\boldsymbol{x}, \boldsymbol{y})_H = \sum_{m=-N}^{N} (\boldsymbol{x}_k, \boldsymbol{y}_k)_{l^2}, \qquad \boldsymbol{x} = (\boldsymbol{x}_{-N}, \ldots, \boldsymbol{x}_N), \boldsymbol{y} \in H.$$

Define the projection operator $P_n : H \mapsto l^2$ by $P_n \boldsymbol{x} = \boldsymbol{x}_n$, let $P_n^* : l^2 \mapsto H$ denote its adjoint $P_n^* \boldsymbol{x} = (0, \ldots, 0, \boldsymbol{x}, 0, \ldots, 0)$ ($\boldsymbol{x}$ at position n), and set

$$\begin{aligned} \boldsymbol{B}^{(N)}(t) = & \sum_{m,n=-N}^{N} P_m^* \boldsymbol{B}_{\lambda_m, \lambda_n}(t) P_n \\ & \Longleftrightarrow \\ \boldsymbol{B}_{\lambda_m, \lambda_n}(t) = & P_m \boldsymbol{B}^{(N)}(t) P_n^*, \quad |n|, |m| \leq N. \end{aligned}$$

Since $\boldsymbol{B}_{\lambda_m, \lambda_n}(t) \in T(l^2)$, the inequality (1) implies that $\boldsymbol{B}^{(N)}(t) \in T(H)$ for all $t \in \mathbb{R}$. Let $M > 0$, $\{t_i\}_{i=1}^M \subset \mathbb{R}$, and $\{\boldsymbol{y}_i\}_{i=1}^M \subset H$ be arbitrary, and denote $\boldsymbol{y}_i = (\boldsymbol{x}_{i,-N} \cdots \boldsymbol{x}_{i,N})$ where $\boldsymbol{x}_{i,m} \in l^2$ for $-N \leq m \leq N$. Using Lemma 2 we obtain

$$\begin{aligned} & \sum_{i,k=1}^{M} (\boldsymbol{B}^{(N)}(t_i - t_k) \boldsymbol{y}_k, \boldsymbol{y}_i)_H \\ & \quad = \sum_{i=1}^{M} \sum_{m=-N}^{N} \sum_{k=1}^{M} \sum_{n=-N}^{N} (\boldsymbol{B}_{\lambda_m, \lambda_n}(t_i - t_k) \boldsymbol{x}_{k,n}, \boldsymbol{x}_{i,m})_{l^2} \geq 0 \end{aligned}$$

due to (35). Thus, by Lemma 1, $\boldsymbol{B}^{(N)}(t)$ is the correlation function of an H-valued WSS process. Furthermore, the assumption that $\boldsymbol{R}$ is uniformly continuous implies that $\boldsymbol{B}_{\lambda_k}(t)$ is uniformly continuous due to (33). Therefore, $\boldsymbol{B}^{(N)}(t)$ is a uniformly continuous $T(H)$-valued function. Now [17, Theorem II.5.5] (which is a generalization of Bochner's theorem of spectral representation of positive-definite $\mathbb{C}$-valued functions to $T(H)$-valued functions) says that we have the spectral representation

$$\boldsymbol{B}^{(N)}(t) = \int_{\mathbb{R}} e^{jt\xi} \boldsymbol{M}^{(N)}(d\xi) \tag{40}$$

where $\boldsymbol{M}^{(N)}$ is a unique $T(H)^+$-valued measure, i.e.,

$$\begin{aligned} & \sum_{m,n=-N}^{N} (P_m \boldsymbol{M}^{(N)}(A) P_n^* \boldsymbol{x}_n, \boldsymbol{x}_m)_{l^2} \geq 0, \\ & \qquad \boldsymbol{x}_n \in l^2, \quad |n| \leq N, \quad A \in \mathcal{B}(\mathbb{R}). \end{aligned} \tag{41}$$

Since $\boldsymbol{B}_{\lambda_k}(t) = \boldsymbol{B}_{0,\lambda_k}(t) = P_0 \boldsymbol{B}^{(N)}(t) P_k^*$ for $|k| \leq N$, we have the representation

$$\boldsymbol{B}_{\lambda_k}(t) = \int_{\mathbb{R}} e^{jt\xi} \boldsymbol{\mu}_{\lambda_k}(d\xi)$$

where

$$\boldsymbol{\mu}_{\lambda_k} = P_0 \boldsymbol{M}^{(N)} P_k^* \tag{42}$$

is $T(l^2)$-valued, which follows from (1) and the fact that $\boldsymbol{M}^{(N)}$ is $T(H)$-valued. The measure $\boldsymbol{\mu}_{\lambda_k}$ is unique since $\boldsymbol{M}^{(N)}$ is unique. The definition (42) is independent of N if $N \geq |k|$, since $\boldsymbol{B}_{\lambda_k}(t) = P_0 \boldsymbol{B}^{(N)}(t) P_k^*$ holds for all $N \geq |k|$. The NND property (41) implies that $\boldsymbol{\mu}_0$ is a $T(l^2)^+$-valued measure.

We deduce from (40) and (34), with $k \in \mathbb{Z}$ arbitrary, that

$$\begin{aligned} P_k \boldsymbol{B}^{(|k|)}(t) P_k^* & = \boldsymbol{B}_{\lambda_k, \lambda_k}(t) = \boldsymbol{B}_0(t) e^{j\lambda_k t} = \boldsymbol{B}_{0,0}(t) e^{j\lambda_k t} \\ & = P_0 \boldsymbol{B}^{(|k|)}(t) P_0^* e^{j\lambda_k t} \\ & = \int_{\mathbb{R}} e^{jt(\xi + \lambda_k)} P_0 \boldsymbol{M}^{(|k|)} P_0^*(d\xi) \\ & = \int_{\mathbb{R}} e^{jt\xi} P_0 \boldsymbol{M}^{(|k|)} P_0^*(d\xi - \lambda_k) \end{aligned}$$

i.e.,

$$P_k \boldsymbol{M}^{(|k|)} P_k^*(A) = P_0 \boldsymbol{M}^{(|k|)} P_0^*(A - \lambda_k), \quad A \in \mathcal{B}(\mathbb{R}). \tag{43}$$

Next we fix $k \in \mathbb{Z} \setminus \{0\}$. Let $\{f_n\}_{n \geq 0}$ and $\{g_n\}_{n \geq 0}$ be orthonormal sequences in l^2. For $n \geq 0$ fixed, we set $\boldsymbol{x}_0 = a_1 g_n$, $\boldsymbol{x}_k = a_2 f_n$ where $a_1, a_2 \in \mathbb{C}$, and we also set $\boldsymbol{x}_m = 0$ for $|m| \leq |k|$, $m \neq 0$, and $m \neq k$. We insert these $\{\boldsymbol{x}_m\}_{m=-N}^{N}$ into (41) with $N = |k|$, and conclude that the 2×2 matrix

$$\begin{pmatrix} (P_0 \boldsymbol{M}^{(|k|)}(A) P_0^* g_n, g_n)_{l^2} & (P_0 \boldsymbol{M}^{(|k|)}(A) P_k^* f_n, g_n)_{l^2} \\ (P_k \boldsymbol{M}^{(|k|)}(A) P_0^* g_n, f_n)_{l^2} & (P_k \boldsymbol{M}^{(|k|)}(A) P_k^* f_n, f_n)_{l^2} \end{pmatrix}$$

is NND for any $A \in \mathcal{B}(\mathbb{R})$. Hence, using (42) and (43), we have the inequality

$$\begin{aligned} |(\boldsymbol{\mu}_{\lambda_k}(A) f_n, g_n)_{l^2}|^2 \leq (\boldsymbol{\mu}_0(A) g_n, g_n)_{l^2} (\boldsymbol{\mu}_0(A - \lambda_k) f_n, f_n)_{l^2}, \\ A \in \mathcal{B}(\mathbb{R}). \end{aligned}$$

This inequality holds for all $n \geq 0$, which finally gives

$$\begin{aligned} & \|\boldsymbol{\mu}_{\lambda_k}(A)\|_\tau^2 \\ & = \sup \Big(\sum_{n \geq 0} |(\boldsymbol{\mu}_{\lambda_k}(A) f_n, g_n)_{l^2}| \Big)^2 \\ & \leq \sup \Big(\sum_{n \geq 0} (\boldsymbol{\mu}_0(A) g_n, g_n)_{l^2}^{1/2} (\boldsymbol{\mu}_0(A - \lambda_k) f_n, f_n)_{l^2}^{1/2} \Big)^2 \\ & \leq \sup \sum_{n \geq 0} (\boldsymbol{\mu}_0(A) g_n, g_n)_{l^2} \sum_{n \geq 0} (\boldsymbol{\mu}_0(A - \lambda_k) f_n, f_n)_{l^2} \\ & \leq \operatorname{tr}(\boldsymbol{\mu}_0(A)) \operatorname{tr}(\boldsymbol{\mu}_0(A - \lambda_k)), \quad A \in \mathcal{B}(\mathbb{R}). \end{aligned}$$

The inequality $\operatorname{tr}(\boldsymbol{\mu}_0(A - \lambda_k)) \leq \operatorname{tr}(\boldsymbol{\mu}_0(\mathbb{R})) = \operatorname{tr}(\boldsymbol{B}_0(0))$ follows from the fact that $\boldsymbol{\mu}_0$ is $T(l^2)^+$-valued. ☐

From (39) we may conclude that $\boldsymbol{\mu}_{\lambda_k}$ is absolutely continuous with respect to $\operatorname{tr}(\boldsymbol{\mu}_0)$. This is included in the following generalization of a theorem of Dehay (see [20, Theorem 2.3]) to multidimensional processes.

Theorem 4: Suppose $\boldsymbol{R} : \mathbb{R}^2 \mapsto T(l^2)$ is uniformly continuous and the correlation function of an almost cyclostationary l^2-valued process. Let $\boldsymbol{B}(u,t) = \boldsymbol{R}(u+t, u)$ and $\boldsymbol{B}_{\lambda_k}(t)$ be defined by (33), and let $\boldsymbol{\mu}_{\lambda_k}$ be the $T(l^2)$-valued measures that satisfy (38). Then

$$\begin{aligned} \left\| \int_{\mathbb{R}} f(\xi) \boldsymbol{\mu}_{\lambda_k}(d\xi) \right\|_\tau^2 \leq \operatorname{tr}(\boldsymbol{B}_0(0)) \int_{\mathbb{R}} |f(\xi)|^2 \operatorname{tr}(\boldsymbol{\mu}_0)(d\xi), \\ \forall k \in \mathbb{Z} \setminus \{0\}, \quad f \in L^\infty(\mathbb{R}). \end{aligned} \tag{44}$$

Proof: Let f be a simple function, i.e.,

$$f(\xi) = \sum_{i=1}^{n} a_i \chi_{A_i}(\xi)$$

where $a_i \in \mathbb{C}$, $A_i \in \mathcal{B}(\mathbb{R})$ for $1 \le i \le n$, $\{A_i\}_{i=1}^{n}$ are pairwise disjoint, and χ_{A_i} denotes the indicator function of the set A_i. The sets $\{A_i - \lambda_k\}_{i=1}^{n}$ are also pairwise disjoint, and hence

$$\begin{aligned}\sum_{i=1}^{n} \operatorname{tr}(\boldsymbol{\mu}_0(A_i - \lambda_k)) &= \operatorname{tr}(\boldsymbol{\mu}_0(\cup_{i=1}^{n}(A_i - \lambda_k)) \\ &\le \operatorname{tr}(\boldsymbol{\mu}_0(\mathbb{R})) = \operatorname{tr}(\boldsymbol{B}_0(0)).\end{aligned}$$

Using (39) we obtain

$$\begin{aligned}&\left\| \int_{\mathbb{R}} f(\xi) \boldsymbol{\mu}_{\lambda_k}(d\xi) \right\|_{\tau}^{2} \\ &= \left\| \sum_{i=1}^{n} a_i \boldsymbol{\mu}_{\lambda_k}(A_i) \right\|_{\tau}^{2} \\ &\le \left(\sum_{i=1}^{n} |a_i| \operatorname{tr}^{1/2}(\boldsymbol{\mu}_0(A_i)) \operatorname{tr}^{1/2}(\boldsymbol{\mu}_0(A_i - \lambda_k)) \right)^{2} \\ &\le \sum_{i=1}^{n} |a_i|^2 \operatorname{tr}(\boldsymbol{\mu}_0(A_i)) \sum_{i=1}^{n} \operatorname{tr}(\boldsymbol{\mu}_0(A_i - \lambda_k)) \\ &\le \operatorname{tr}(\boldsymbol{B}_0(0)) \sum_{i=1}^{n} |a_i|^2 tr(\boldsymbol{\mu}_0(A_i)) \\ &= \operatorname{tr}(\boldsymbol{B}_0(0)) \int_{\mathbb{R}} |f(\xi)|^2 \operatorname{tr}(\boldsymbol{\mu}_0)(d\xi).\end{aligned}$$

The extension from simple functions to $f \in L^{\infty}(\mathbb{R})$ can be obtained from Lebesgue's dominated convergence theorem [28], [37]. ☐

If we assume that $\boldsymbol{R}$ is strongly harmonizable then the Fourier series with coefficients $\boldsymbol{B}_\lambda(t)$ converges pointwise according to [17], [21]

$$\boldsymbol{B}(u,t) = \sum_{\lambda \in \Lambda} \boldsymbol{B}_\lambda(t) e^{j\lambda u}.$$

The same assumption also allows the representation

$$\boldsymbol{B}_{\lambda_k}(t) = \int_{\mathbb{R}} e^{jt\xi} \boldsymbol{\mu}_{\lambda_k}(d\xi), \qquad k \in \mathbb{Z}$$

where each $\boldsymbol{\mu}_{\lambda_k}$ is a $T(l^2)$-valued measure [17], [21], which, according to (38), are identical to the measures (42). Since $\boldsymbol{R}(t,u) = \boldsymbol{B}(u, t-u)$ we have

$$\boldsymbol{R}(t,u) = \sum_{\lambda \in \Lambda} e^{j\lambda u} \int_{\mathbb{R}} e^{j\xi(t-u)} \boldsymbol{\mu}_\lambda(d\xi) \tag{45}$$

similar to (25).

B. Application to Strongly Harmonizable Scalar-Valued Processes

We now specialize to the case $d = 2$ and $\overline{\boldsymbol{s}}(t) = (s(t)\ s^*(t))^T$ where $s(t)$ is a scalar strongly harmonizable process. Then $\overline{\boldsymbol{s}}(t)$ has correlation function $\overline{\boldsymbol{R}}(t,u)$ defined by (14). We denote the spectral measures for $\overline{\boldsymbol{R}}(t,u)$ in (45) by $\overline{\boldsymbol{\mu}}_\lambda$. Hence, using a generalization of (25) from the cyclostationary case to the almost cyclostationary case [20], [21], (14), and (45), it follows that

$$\overline{\boldsymbol{\mu}}_\lambda(A) = \begin{pmatrix} \mu_\lambda(A) & \widetilde{\mu}_\lambda(A) \\ \widetilde{\mu}_{-\lambda}^*(-A) & \mu_{-\lambda}^*(-A) \end{pmatrix}, \quad \lambda \in \Lambda, \ \ A \in \mathcal{B}(\mathbb{R}). \tag{46}$$

Similarly to (24), the symmetries

$$\begin{aligned}\mu_{\lambda_k}(\xi) &= \mu_{\lambda_{-k}}^*(\xi - \lambda_k) \\ \widetilde{\mu}_{\lambda_k}(\xi) &= \widetilde{\mu}_{\lambda_k}(-\xi + \lambda_k), \qquad k \in \mathbb{Z}\end{aligned} \tag{47}$$

hold for almost cyclostationary processes [20]. Thus, $\widetilde{\mu}_0(-A) = \widetilde{\mu}_0(A)$. Since $\overline{\boldsymbol{\mu}}_0(A) = P_0 \boldsymbol{M}^{(0)}(A) P_0^* \in \mathbb{C}^{2\times 2}$ is NND by (41), and because $\mu_0(A) \ge 0$, we have

$$|\widetilde{\mu}_0(A)|^2 \le \mu_0(A)\mu_0(-A), \qquad A \in \mathcal{B}(\mathbb{R}). \tag{48}$$

This implies that $\widetilde{\mu}_0$ is absolutely continuous with respect to μ_0. Moreover, Proposition 1 and the Radon–Nikodym theorem yield the following theorem.

Theorem 5: If $s(t)$ is a scalar strongly harmonizable almost cyclostationary process there exist unique functions f_λ, $\widetilde{f}_\lambda \in L^1(\mu_0)$, $\lambda \in \Lambda$, such that for all $A \in \mathcal{B}(\mathbb{R})$

$$\mu_\lambda(A) = \int_A f_\lambda(\xi)\mu_0(d\xi) = \int_{A-\lambda} f_{-\lambda}^*(\xi)\mu_0(d\xi), \quad \lambda \in \Lambda \tag{49}$$

$$\widetilde{\mu}_0(A) = \int_A \widetilde{f}_0(\xi)\mu_0(d\xi) = \int_{-A} \widetilde{f}_0(\xi)\mu_0(d\xi) \tag{50}$$

$$\begin{aligned}\widetilde{\mu}_\lambda(A) &= \int_A \widetilde{f}_\lambda(\xi)(\mu_0 + \check{\mu}_0)(d\xi) \\ &= \int_{-A+\lambda} \widetilde{f}_\lambda(\xi)(\mu_0 + \check{\mu}_0)(d\xi), \quad \lambda \in \Lambda \setminus \{0\}.\end{aligned} \tag{51}$$

Proof: Formula (49) follows from Proposition 1 with $d = 1$, the Radon–Nikodym theorem, and (47). Alternatively, it follows from [20, Theorem 2.3]. Formula (50) follows from (48) and (47). To prove (51), suppose $A \in \mathcal{B}(\mathbb{R})$ and $(\mu_0 + \check{\mu}_0)(A) = \mu_0(A) + \mu_0(-A) = 0$. Then $\operatorname{tr}(\overline{\boldsymbol{\mu}}_0(A)) = 0$ by (46). By Proposition 1, $\overline{\boldsymbol{\mu}}_\lambda(A) = 0$ for all $\lambda \in \Lambda \setminus \{0\}$, that is, $\mu_\lambda(A) = \mu_{-\lambda}(-A) = \widetilde{\mu}_\lambda(A) = \widetilde{\mu}_{-\lambda}(-A) = 0$. Hence, $\widetilde{\mu}_\lambda$ is absolutely continuous with respect to the measure $\mu_0 + \check{\mu}_0$, and therefore the Radon–Nikodym theorem implies that there exist unique functions $\widetilde{f}_\lambda \in L^1(\mu_0)$, $\lambda \in \Lambda \setminus \{0\}$, such that

$$\widetilde{\mu}_\lambda(A) = \int_A \widetilde{f}_\lambda(\xi)(\mu_0 + \check{\mu}_0)(d\xi).$$

Thus, the first identity of (51) has been proved. The second identity follows from (47). ☐

We may conclude from this theorem that for almost cyclostationary processes, similarly to cyclostationary processes, the measures μ_λ and $\widetilde{\mu}_\lambda$, $\lambda \in \Lambda$, all depend on the diagonal spectral measure μ_0. While the statement in Theorem 5 resembles Theorem 3, the conclusion for $\widetilde{\mu}_\lambda$, $\lambda \in \Lambda \setminus \{0\}$, is weaker since $\widetilde{\mu}_\lambda$ is absolutely continuous with respect to $\mu_0 + \check{\mu}_0$, $\check{\mu}_0(\xi) = \mu_0(-\xi)$, instead of μ_0 in Theorem 3. If a measure is absolutely continuous with respect to μ_0, then it is also absolutely continuous with respect to $\mu_0 + \check{\mu}_0$, so Theorem 3 has a stronger conclusion than Theorem 5. We have to integrate with respect to the measure $\mu_0 + \check{\mu}_0$ instead of μ_0 in (51).

VII. APPLICATION TO ANALYTIC SIGNALS

Let $s(t)$ be a complex-valued strongly harmonizable process with spectral process S. The Hilbert transform [3] of $s(t)$ is defined by

$$(Hs)(t) = -j\int_{\mathbb{R}} e^{jt\xi}\operatorname{sgn}(\xi)S(d\xi)$$

and the analytic signal is defined by

$$z(t) = s(t) + j(Hs)(t) = 2\int_0^\infty e^{jt\xi}S(d\xi).$$

Let r and $\widetilde{r}$ denote the correlation and complementary correlation function of $z(t)$, respectively. The spectral measure of $z(t)$ has support in the first quadrant

$$r(t,u) = \int_0^\infty\int_0^\infty e^{j(t\xi-u\eta)}m(d\xi,d\eta)$$

whereas the spectral complementary measure of $z(t)$ has support in the fourth quadrant

$$\widetilde{r}(t,u) = \int_0^\infty\int_{-\infty}^0 e^{j(t\xi-u\eta)}\widetilde{m}(d\xi,d\eta). \tag{52}$$

From Corollary 1, it is clear that an analytic WSS process such that $\mu(\{0\}) = 0$ has $\widetilde{\mu}\equiv 0$, i.e., it is proper. This is well known in the literature [1]–[3], [6], [30], [32]. We would like to investigate the implications of our results for the complementary spectral measure $\widetilde{m}$ for (almost) cyclostationary processes. Related results are discussed in [38]. According to (52), the support of $\widetilde{m}$ is contained in the fourth quadrant. The version of (29) for almost cyclostationary processes [20], is thus

$$\begin{aligned}\widetilde{m}(A,B) &= \widetilde{m}(A\cap[0,\infty), B\cap(-\infty,0])\\ &= \sum_{k=-\infty}^{\infty}\widetilde{\mu}_{\lambda_k}(A\cap(B+\lambda_k)\cap[0,\infty)\cap(-\infty,\lambda_k])\\ &= \sum_{k=0}^{\infty}\widetilde{\mu}_{\lambda_k}(A\cap(B+\lambda_k)\cap[0,\lambda_k]).\end{aligned} \tag{53}$$

Since $\widetilde{\mu}_{\lambda_k}$ is identically zero when $k<0$, we can restrict our attention to $\widetilde{\mu}_{\lambda_k}$ for $k\ge 0$. We consider cyclostationary processes and almost cyclostationary processes separately.

A. Cyclostationary Analytic Processes

Let $z(t)$ be a complex-valued strongly harmonizable analytic cyclostationary process with period T, and let the components of its spectral measure and complementary spectral measure be denoted by μ_k and $\widetilde{\mu}_k$, $k\in\mathbb{Z}$, as in Section V. From Theorem 3 we obtain the following result.

Corollary 2:

(i) If $p\in\mathbb{Z}$, $p>0$, and

$$\mu_0([0,\pi p/T]) = 0 \tag{54}$$

then $\widetilde{\mu}_k\equiv 0$ for $k\le p$.

(ii) If $p\in\mathbb{Z}$, $p>0$, and

$$\mu_0([\pi p/T,\infty)) = 0 \tag{55}$$

then $\widetilde{\mu}_k\equiv 0$ for $k\ge p$.

Proof:

(i) Let $0\le k\le p$. From Theorem 3 we obtain $\widetilde{\mu}_k([0,\pi k/T])=0$, and from (24) we have $\widetilde{\mu}_k(\pi k/T+\xi) = \widetilde{\mu}_k(\pi k/T-\xi)$, which implies that $\widetilde{\mu}_k([0,2\pi k/T]) = 0$. Since $\operatorname{supp}(\widetilde{\mu}_k)\subseteq[0,2\pi k/T]$ according to (53), we have $\widetilde{\mu}_k\equiv 0$ for $k\le p$.

(ii) Let $k\ge p$. From Theorem 3 we again obtain $\widetilde{\mu}_k([\pi k/T, 2\pi k/T]) = 0$. As above, the symmetry (24) implies $\widetilde{\mu}_k\equiv 0$. □

The next result shows another consequence of Theorem 3.

Corollary 3: Let $p\in\mathbb{Z}$ and $p>0$. For each integer $n>0$, define the covering of the interval $[0,2\pi p/T]$ consisting of $[0,2\pi p/T] = \bigcup_{l=0}^{2n-1} I_l$, where $I_l = 2\pi p/T[l/(2n),(l+1)/(2n)]$ for $0\le l\le 2n-1$. Suppose that there exists $n>0$ such that

$$\begin{aligned}&\text{(i)}\quad \mu_0(I_{2l}) = 0,\quad 0\le l\le n-1\\ &\text{or}\\ &\text{(ii)}\quad \mu_0(I_{2l+1}) = 0,\quad 0\le l\le n-1.\end{aligned} \tag{56}$$

Then $\widetilde{\mu}_p\equiv 0$.

Proof: For $0\le l\le 2n-1$ we have

$$\begin{aligned}-I_l + 2\pi p/T &= 2\pi p/T[(2n-l-1)/(2n),(2n-l)/(2n)]\\ &= I_{2n-l-1}.\end{aligned}$$

From (31) we infer that $\widetilde{\mu}_p(I_l) = 0$ if $\mu_0(I_l) = 0$ or $\mu_0(-I_l+2\pi p/T) = \mu_0(I_{2n-l-1}) = 0$. Thus, if (i) is true, then $\widetilde{\mu}_p(I_{2l}) = 0$ and $\widetilde{\mu}_p(I_{2l+1}) = 0$ for $0\le l\le n-1$, i.e., $\widetilde{\mu}_p\equiv 0$. Likewise, if (ii) is true, then $\widetilde{\mu}_p(I_{2l+1}) = 0$ and $\widetilde{\mu}_p(I_{2l}) = 0$ for $0\le l\le n-1$, i.e., $\widetilde{\mu}_p\equiv 0$. □

We may interpret Corollary 2 and 3 as follows. If μ_0 is high-pass in the sense of (54), then the impropriety of the process is restricted in the sense that $\widetilde{\mu}_k\equiv 0$ for $k\le p$. If μ_0 is low-pass in the sense of (55), then the impropriety of the process is restricted in the sense that $\widetilde{\mu}_k\equiv 0$ for $k\ge p$. If μ_0 fulfills the multiple band-stop criterion (56), then the impropriety of the process is restricted in the sense of $\widetilde{\mu}_p\equiv 0$ for a given $p>0$.

B. Almost Cyclostationary Analytic Processes

Let $z(t)$ be an analytic scalar almost cyclostationary process, and let the components of its spectral measure and complementary spectral measure be denoted by μ_λ and $\widetilde{\mu}_\lambda$, $\lambda\in\Lambda$, $\Lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, as in Section VI. Recall the assumption that $\lambda_0 = 0$ and $k>0\Leftrightarrow\lambda_k>0$. Since $\operatorname{supp}(\mu_0)\subseteq[0,\infty)$, (51) gives for $\lambda\in\Lambda\setminus\{0\}$

$$\widetilde{\mu}_\lambda(A) = \int_A\widetilde{f}_\lambda(\xi)\mu_0(d\xi) + \widetilde{f}_\lambda(0)\mu_0(-A\cap\{0\}),\ A\subseteq[0,\infty).$$

With arguments similar to the proof of Corollary 2 one can prove the following result.

Corollary 4:

(i) If $p\in\mathbb{Z}$, $p\ge 0$ and $\mu_0([0,\lambda_p/2]) = 0$, then $\widetilde{\mu}_\lambda\equiv 0$ for $\Lambda\ni\lambda\le\lambda_p$.

(ii) If $p\in\mathbb{Z}$, $p\ge 0$ and $\mu_0([\lambda_p/2,\infty)) = 0$, then $\widetilde{\mu}_\lambda\equiv 0$ for $\Lambda\ni\lambda\ge\lambda_p$.

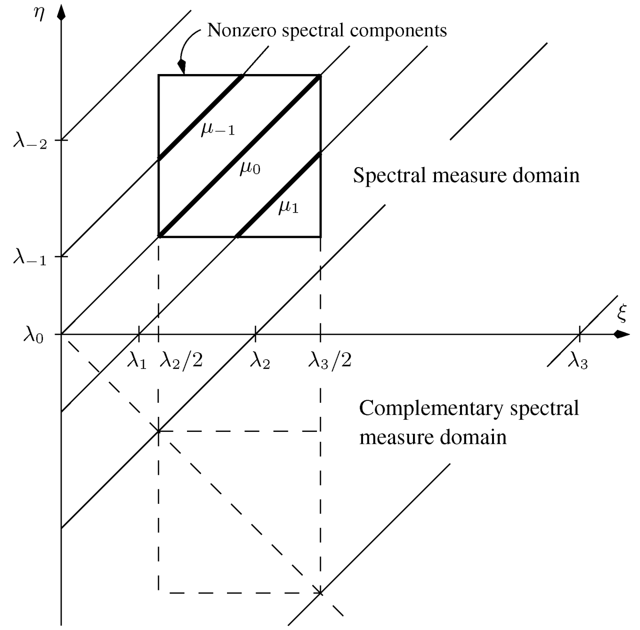

Fig. 2. An example of the support of the spectral measure $m(\xi, \eta)$ of an analytic, proper, almost cyclostationary, but not WSS signal.

The final result gives a sufficient condition for an analytic almost cyclostationary process to be proper.

Corollary 5: Suppose there exist $p, q \in \mathbb{Z}$, $p, q \geq 0$, such that $\lambda_p < \lambda_q$, $\Lambda \cap [\lambda_p, \lambda_q] = \emptyset$, $\mu_0([0, \lambda_p/2]) = 0$, and $\mu_0([\lambda_q/2, \infty)) = 0$. Then $\widetilde{\mu}_{\lambda_k} \equiv 0$ for all $k \in \mathbb{Z}$, i.e., $z(t)$ is proper.

Note that the process is not necessarily WSS, since there may exist frequencies that are not forced to be zero by the assumptions on the support of μ_0 and the Cauchy–Schwarz inequality. This is illustrated in Fig. 2.

VIII. CONCLUSION

We have discussed three topics concerning the pair of correlation and complementary correlation functions of complex-valued processes and vectors. First, we have generalized a characterization of a pair of matrices to be the correlation matrix and the complementary correlation matrix of a finite-dimensional stochastic vector. Second, for multidimensional WSS processes, we have characterized the pair of spectral measure and complementary spectral measure. For scalar signals, this reduces to a simple criterion. Third, and most importantly, we have studied cyclostationary and almost cyclostationary processes. In the multidimensional case, we have proved the existence of spectral measures $\boldsymbol{\mu}_\lambda$, $\lambda \in \Lambda$, assuming a uniformly continuous correlation function. We have shown that the off-diagonal spectral measure components $\boldsymbol{\mu}_\lambda$, $\lambda \in \Lambda \setminus \{0\}$, are absolutely continuous with respect to the diagonal component $\boldsymbol{\mu}_0$ under the same assumption. In the scalar strongly harmonizable almost cyclostationary case, this gives representation formulas for the off-diagonal spectral measure components μ_λ, $\lambda \in \Lambda \setminus \{0\}$, and the complementary spectral measure components $\widetilde{\mu}_\lambda$, $\lambda \in \Lambda$, in terms of the diagonal measure μ_0. In the cyclostationary case, the positive distance between the frequencies allows a stronger result without the use of multidimensional techniques. As an application of these results, we have studied spectral relations for analytic almost cyclostationary signals.

Finally, we remark that there are corresponding results for discrete-time signals, with the modification that the domain of the spectral measure is $[0, 2\pi] \times [0, 2\pi]$. The treatment of discrete-time cyclostationary processes is simpler than in the continuous-time case, because the set of frequencies is finite.

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