NONLINEAR CONTROL TOOLS FOR LOW THRUST ORBITAL TRANSFER

Christopher M. Kellett * **Laurent Praly** **

* *Department of Electrical and Electronic Engineering, University of Melbourne, Victoria 3052, Australia*
** *Centre Automatique et Systèmes, École des Mines de Paris, 35 rue St. Honoré, 77305 Fontainebleau, France*

Abstract: We consider the problem of orbital transfer under propulsion constraints. The solution is arrived at by appealing to several recent nonlinear control design methods including backstepping, forwarding, and a modified form of Jurdjevic-Quinn control.

Keywords: Orbital Transfer, Backstepping, Forwarding, Passivity

1. INTRODUCTION

The advent of ion propulsion engines, such as demonstrated on NASA's Deep Space 1 spacecraft, have made the consideration of propulsion under severe control constraints a problem of great interest. Such engines are particularly attractive due to the reduced weight and space dedicated to fuel. This in turn leads to significantly reduced launch costs.

In particular, we will look at the problem of transfering a satellite from one orbit to another while respecting a priori fixed constraints on the control. Previously, such transfers have generally been accomplished by finding open-loop optimal trajectories and then fine-tuning the orbit. However, when considering the problem under low thrust, this solution is computationally expensive and may be difficult to implement.

In Chang et al. (2002), the problem of orbital transfer under low thrust was posed as a stabilization problem for the target orbit. There, the injection point and final time are unspecified. We take a similar approach in that the final time is not specified, but we do asymptotically stabilize a point on the target orbit.

This problem provides us with an opportunity to make use of several recent nonlinear control techniques such as those espoused in Sepulchre et al. (1997). Of particular interest is that these tools can be used to achieve the goal of asymptotically stabilizing a point on the target orbit while respecting the control constraints. In Section 2 we review certain nonlinear control tools. Section 3 presents the mathematical model used for the orbital transfer problem. Sections 4-6 are devoted to deriving a bounded state feedback controller.

2. TOOLS

For a control-affine system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

a control Lyapunov function is a positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$L_g V(x) = 0 \, , \; x \neq 0 \quad \Longrightarrow \quad L_f V(x) < 0 \tag{2}$$

where $L_g V(x)$ is the Lie derivative of V along the vector field g. In what follows, we use the phrase "$L_g V$ term" for the quantity multiplying the control in the time derivative of a Lyapunov function. We will similarly use the phrase "$L_f V$ term" for those terms not multiplied by the control.

2.1 Backstepping

The backstepping technique (see Krstić et al. (1995) and Sepulchre et al. (1997)) applies to systems in strict feedback form such as

$$\begin{aligned} \dot{z} &= h(z,x) \\ \dot{x} &= f(z,x) + g(z,x)u, \end{aligned} \tag{3}$$

where $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $g : \mathbb{R}^{n+1} \to \mathbb{R}$, and $h : \mathbb{R}^{n+1} \to \mathbb{R}^n$ are locally Lipschitz and $g(\cdot,\cdot)$ is never zero. One considers x as a virtual control, $\phi(z)$, for the z-subsystem and obtains a Lyapunov function for that subsystem when $x = \phi(z)$; i.e., a continuously differentiable, positive definite, radially unbounded function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$L_h V|_{x=\phi(z)} < 0. \tag{4}$$

We assume that $\phi(\cdot)$ is continuously differentiable and $\phi(0) = 0$. One may then "step back" to the full system. A typical control Lyapunov function is then given by

$$W(x,z) = V(z) + \frac{1}{2}\left(x - \phi(z)\right)^2 \tag{5}$$

which can then be used to design a globally asymptotically stabilizing feedback.

To allow for more design freedom, following Praly et al. (1991), we propose the control Lyapunov function

$$W(x,z) = V(z) + \int_{\phi(z)}^{x} \psi(z,\xi)d\xi \tag{6}$$

where $\psi(\cdot,\cdot)$ is a continuous function satisfying

$$x \neq \phi(z) \quad \Longrightarrow \quad [x - \phi(z)]\psi(z,x) > 0 \tag{7}$$

such that W is radially unbounded in x and there exist $\eta > 0$ and $\varepsilon > 0$ satisfying

$$|x| + |z| \leq \varepsilon \quad \Longrightarrow \quad |\psi(z,x)| \geq \eta|x - \phi(z)|. \tag{8}$$

To see that (6) is indeed a control Lyapunov function we take the time derivative and obtain

$$\begin{aligned} \dot{W}(x,z) = L_h V(z,x) + \psi(z,x)[f(z,x) + g(z,x)u] \\ + \frac{\partial}{\partial z}\left[\int_{\phi(z)}^{x} \psi(z,\xi)d\xi\right] h(z,x). \end{aligned} \tag{9}$$

We see that the $L_g V$ term is $\psi(z,x)g(z,x)$. Since $g(\cdot,\cdot)$ is never zero, if the $L_g V$ term is zero then $\psi(z,x) = 0$. From (7), we then see that this implies $x = \phi(z)$. Therefore, with $\psi(z,x) = 0$ and $x = \phi(z)$, we see that (9) becomes

$$\dot{W}(x,z) = L_h V|_{x=\phi(z)} < 0; \tag{10}$$

that is, the $L_f V$ term is strictly negative.

2.2 Forwarding

The forwarding technique (see Sepulchre et al. (1997) or Praly (2001)), by contrast, applies to systems in strict feedforward form. In particular, we are interested in a specifically structured feedforward form

$$\begin{aligned} \dot{z} &= h(x) \\ \dot{x} &= f(x) + g(x)u \end{aligned} \tag{11}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, and $h : \mathbb{R}^n \to \mathbb{R}$ are locally Lipschitz, $g(0) \neq 0$ and the origin is globally asymptotically stable for $\dot{x} = f(x)$. Therefore, there exists a continuously differentiable, positive definite, radially unbounded function $V_2 : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that $L_f V_2(\cdot)$ is negative definite. According to the forwarding technique, we then search for a continuously differentiable function $\mathcal{M}(x)$ that solves the partial differential equation

$$L_f \mathcal{M}(x) = h(x), \quad \mathcal{M}(0) = 0. \tag{12}$$

The function

$$W(x,z) = V_2(x) + \frac{1}{2}\left(z - \mathcal{M}(x)\right)^2 \tag{13}$$

then serves as a control Lyapunov function. With the constraint (12), this is obvious from the time derivative of W:

$$\begin{aligned} \dot{W}(x,z) &= L_f V_2(x) \\ &\quad + \left[L_g V_2(x) - (z - \mathcal{M}(x))L_g\mathcal{M}(x)\right]u. \end{aligned} \tag{14}$$

The technique known as forwarding modulo $L_g V$ (see Praly et al. (2001)) adds a free parameter to the PDE (12) which may make solving the PDE simpler. With $k : \mathbb{R}^n \to \mathbb{R}$ a continuous function, satisfying $k(0) \neq 0$, the modified PDE is

$$L_f \mathcal{M}(x) = h(x) + k(x)L_g V_2(x). \tag{15}$$

We see that (13) still serves as a control Lyapunov function as

$$\begin{aligned} \dot{W}(x,z) &= -(z - \mathcal{M}(x))L_g\mathcal{M}(x)u + L_f V_2(x) \\ &\quad + L_g V_2(x)\left[u - k(x)(z - \mathcal{M}(x))\right]. \end{aligned} \tag{16}$$

In some cases, including the problem considered herein, one may choose the function $\mathcal{M}(x)$ independant of variables directly affected by the control. In this case, $L_g\mathcal{M}(x) \equiv 0$, allowing us to rewrite (16) as

$$\begin{aligned} \dot{W}(x,z) &= L_f V_2(x) \\ &\quad + L_g V_2(x)\left[u - k(x)(z - \mathcal{M}(x))\right]. \end{aligned}$$

Furthermore, we may obtain an additional degree of design freedom by defining our control Lyapunov function as

$$W(x,z) = \gamma(V_2(x)) + V_1\left(z - \mathcal{M}(x)\right) \tag{17}$$

where $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuously differentiable with a strictly positive derivative and $V_1 : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is continuously differentiable, positive definite, and such that its derivative only vanishes at zero. Clearly the second term in (13) satisfies the requirements of V_1. With W so defined, and

$L_g\mathcal{M}(z) \equiv 0$, we see that the time derivative expression for W is

$$\dot{W}(x,z) = \gamma'(V_2(x))L_fV_2(x) + L_gV_2(x)\left[\gamma'(V_2(x))u - k(x)V_1'(z-\mathcal{M}(x))\right].$$

An obvious control choice in this case is then

$$u = -\frac{L_gV_2(x) + k(x)V_1'(z-\mathcal{M}(x))}{\gamma'(V_2(x))}. \quad (18)$$

The fact that this feedback asymptotically stabilizes $(x,z)=0$ follows from the fact that V_1' only vanishes at 0 and $k(0)\neq 0$ and $g(0)\neq 0$.

2.3 Passivity

The control affine system (1) is said to be $\mathcal{C}^1$ *dissipative* if there exists a continuously differentiable Lyapunov function $V(x)$ such that $L_fV(x)\leq 0$ for all $x\in\mathbb{R}^n$. Suppose we append an output function

$$y = h(x,u) \quad (19)$$

to the control affine system (1). Given a subset $\mathcal{S}$ of $\mathbb{R}^n$, the system is then said to be $\mathcal{S}$ *observable* if the solutions of the system with $u\equiv 0$ satisfying

$$h(x(t),0) = 0, \quad \forall t\geq 0 \quad (20)$$

are in $\mathcal{S}$.

Within the context of C^1 dissipativeness and $\mathcal{S}$ observability we can design appropriate control laws satisfying magnitude limitations.

Theorem 1. Assume (1) is $\mathcal{C}^1$ dissipative and $\mathcal{S}$ observable with output function

$$x \mapsto (L_fV(x) \;\; L_gV(x)).$$

Then, for any real number $\bar{u}\in(0,+\infty]$, there exists a continuous control law, strictly bounded in norm by $\bar{u}$, which makes the largest invariant set of $\dot{x}=f(x)$ contained in $\mathcal{S}$ globally attractive.

One such control law is given by

$$u(x) = -\min\left\{\frac{\bar{u}}{|L_gV(x)|},\, 1\right\}L_gV(x). \quad (21)$$

3. ORBITAL EQUATIONS

Let (a,e,ω,Ω,i,f) denote the orbital parameters of the space vehicle. The variable a is the semi-major axis of the orbital ellipse and e is the eccentricity, while f is the true anomaly. The three quantities (ω,Ω,i) are the Euler angles and are the longitude of the ascending node, the angle of inclination, and the argument of perihelion, respectively. Let

$$\begin{cases} p &= a\,(1-e^2)\,, \\ e_x &= e\,\cos(\omega+\Omega)\,, \\ e_y &= e\,\sin(\omega+\Omega)\,, \\ h_x &= \tan(i/2)\,\cos(\Omega)\,, \\ h_y &= \tan(i/2)\,\sin(\Omega)\,, \\ L &= \omega+\Omega+f\,. \end{cases}$$

Note that p is the semilatus rectum of the orbital ellipse and is sometimes referred to as the parameter, while L is often referred to as the true longitude.

With a_r, a_θ, and a_h denoting the accelerations the propulsion is able to provide, the Gauss equations in the local polar coordinate system are described by the dynamics (Battin, 1987, pg. 488)

$$\begin{aligned} \frac{d}{dt}p &= 2\,k\,p\,a_\theta \\ \frac{d}{dt}e_x &= k\left[Z\sin(L)a_r + Aa_\theta - e_yYa_h\right] \\ \frac{d}{dt}e_y &= k\left[-Z\cos(L)a_r + Ba_\theta + e_xYa_h\right] \\ \frac{d}{dt}L &= \sqrt{\frac{\mu}{p^3}}\,Z^2 + kYa_h \\ \frac{d}{dt}h_x &= \frac{k}{2}X\,\cos(L)\,a_h \\ \frac{d}{dt}h_y &= \frac{k}{2}X\,\sin(L)\,a_h \end{aligned} \quad (22)$$

where

$$k = \sqrt{\frac{p}{\mu}}\,\frac{1}{Z}\,, \quad Z = 1+e\,\cos(f)\,,$$

$$\begin{aligned} A &= e_x + (1+Z)\cos(L)\,, \\ B &= e_y + (1+Z)\sin(L)\,, \\ X &= 1 + h_x^2 + h_y^2\,, \\ Y &= h_x\sin(L) - h_y\cos(L)\,. \end{aligned}$$

Our problem then is to design control laws for the accelerations so that the space vehicle reaches an orbit with parameters $p=p_0$, $e_x=e_y=h_x=h_y=0$ (this is a circular orbit in the equatorial plane) with a true longitude given by

$$L_0(t) = \sqrt{\frac{\mu}{p_0^3}}\,t \quad (\mathrm{mod}\, 2\pi)\,. \quad (23)$$

Furthermore, we wish to achieve the target orbit using limited accelerations. To ease our presentation, we write these limitations as

$$|a_r|\leq A_r\,, \quad |a_\theta|\leq A_\theta\,, \quad |a_h|\leq A_h\,. \quad (24)$$

In order to solve this problem, we rewrite the dynamics in a form amenable to the application of the tools summarized in the previous Section. Since we have a sixth order system and three controls, the ideal situation would be to have three decoupled second order systems with one acceleration driving each second order system. Unfortunately this is impossible. The best we have been able to obtain is a decomposition into a second order system, a first order system and third order system. Each of them drives the following and is driven by one acceleration. This leads us to apply our tools in a hierarchical way.

3.1 Structuring the system

We observe that setting $a_h \equiv 0$ makes h_x and h_y constant as well as removing the effect of these coordinates on the other dynamics. So it will be sufficient to deal with the (h_x, h_y) subsystem only at the end with a_h dedicated to it.

In what follows, we will use $j^2 = -1$. With $a_h \equiv 0$, the dynamics for the remaining four coordinates (p, e_x, e_y, L) simplify to

$$\frac{d}{dt}\, p = 2kpa_\theta \ , \quad \frac{d}{dt}\, L = \sqrt{\frac{\mu}{p^3}}\, Z^2 \ ,$$

$$\frac{d}{dt}\ (e_x - je_y) \ = \ j\sqrt{\frac{p}{\mu}}\ \exp(-jL)\, a_r + \sqrt{\frac{p}{\mu}}\,\frac{1}{Z}\left[(e_x - je_y) + (1+Z)\exp(-jL)\right] a_\theta \ .$$

Define $\bar{x}_2 = e\cos(f)$, and $\bar{x}_3 = e\sin(f)$. Note that $\bar{x}_2 + j\bar{x}_3 = (e_x - je_y)\exp(jL)$ and $Z = 1 + \bar{x}_2$. Consequently,

$$\begin{aligned} \frac{d}{dt}\ (\bar{x}_2 + j\bar{x}_3) \ &= \ j\sqrt{\frac{p}{\mu}}\, a_r \\ &\quad + \ j\sqrt{\frac{\mu}{p^3}}\,(1+\bar{x}_2)^2\,(\bar{x}_2 + j\bar{x}_3) \\ &\qquad + \sqrt{\frac{p}{\mu}}\left[2 + j\frac{\bar{x}_3}{1+\bar{x}_2}\right] a_\theta \ . \end{aligned}$$

Furthermore, let $\bar{x}_1 = L - L_0$, where L_0 is the reference true longitude (23). This then yields,

$$\frac{d}{dt}\ \bar{x}_1 = \sqrt{\frac{\mu}{p^3}}(1+\bar{x}_2)^2 \ - \ \sqrt{\frac{\mu}{p_0^3}} \ . \qquad (25)$$

Therefore, our system in $(p, \bar{x}_1, \bar{x}_2, \bar{x}_3)$ coordinates is:

$$\begin{aligned} \frac{d}{dt}\ p \ &= \ 2\sqrt{\frac{p^3}{\mu}}\,\frac{1}{1+\bar{x}_2}\, a_\theta \ , \\ \frac{d}{dt}\ \bar{x}_1 \ &= \ \sqrt{\frac{\mu}{p^3}}\left[(1+\bar{x}_2)^2 - 1\right] \\ &\quad + \sqrt{\mu}\left[\sqrt{\frac{1}{p^3}} - \sqrt{\frac{1}{p_0^3}}\right] , \\ \frac{d}{dt}\ \bar{x}_2 \ &= \ -\sqrt{\frac{\mu}{p^3}}(1+\bar{x}_2)^2\bar{x}_3 + 2\sqrt{\frac{p}{\mu}}\, a_\theta \ , \\ \frac{d}{dt}\ \bar{x}_3 \ &= \ \sqrt{\frac{\mu}{p^3}}\,(1+\bar{x}_2)^2\,\bar{x}_2 \ + \ \sqrt{\frac{p}{\mu}}\, a_r \\ &\quad + \sqrt{\frac{p}{\mu}}\,\frac{\bar{x}_3}{1+\bar{x}_2}\, a_\theta \ , \end{aligned} \qquad (26)$$

which is (22) with $a_h = 0$ and a change of coordinates.

The appearance of the control a_θ in the expressions for $\frac{d}{dt}\ \bar{x}_2$ and $\frac{d}{dt}\ \bar{x}_3$ is inconvenient for the purposes of applying our tools. Consequently, we propose the change of coordinates:

$$x_1 = \bar{x}_1, \quad x_2 = \frac{p_0}{p}(1+\bar{x}_2) - 1, \quad x_3 = \sqrt{\frac{p_0}{p}}\,\bar{x}_3.$$

Our system in (p, x_1, x_2, x_3) coordinates is then

$$\begin{aligned} \frac{d}{dt}\ p \ &= \ 2p_0\sqrt{\frac{p}{\mu}}\,\frac{1}{1+x_2}\, a_\theta \\ \frac{d}{dt}\ x_1 \ &= \ \sqrt{\frac{\mu p}{p_0^4}}(1+x_2)^2 - \sqrt{\frac{\mu}{p_0^3}} \\ \frac{d}{dt}\ x_2 \ &= \ -\sqrt{\frac{\mu}{p_0^3}}(1+x_2)^2 x_3 \\ \frac{d}{dt}\ x_3 \ &= \ \sqrt{\frac{\mu}{p_0^3}}(1+x_2)^2\left(\frac{p}{p_0}(1+x_2) - 1\right) \\ &\quad + \sqrt{\frac{p_0}{\mu}}\, a_r \end{aligned} \qquad (27)$$

We observe that, similar to our treatment of a_h, by setting $a_\theta \equiv 0$, p is constant. However, in this case, p is still present in the dynamics of the remaining three coordinates; i.e., the first order p subsystem drives the third order (x_1, x_2, x_3) subsystem. To deal with the latter first, and in a simplified way, for the moment we consider the control $a_\theta \equiv 0$ and the coordinate $p \equiv p_0$. More precisely, consider the system

$$\left\{ \begin{aligned} \frac{d}{dt}\ x_1 \ &= \ \sqrt{\frac{\mu}{p_0^3}}\left[(1+x_2)^2 - 1\right] \ , \\ \frac{d}{dt}\ x_2 \ &= \ -\sqrt{\frac{\mu}{p_0^3}}\,(1+x_2)^2\, x_3 \ , \\ \frac{d}{dt}\ x_3 \ &= \ \sqrt{\frac{\mu}{p_0^3}}\,(1+x_2)^2\, x_2 \ + \ \sqrt{\frac{p_0}{\mu}}\, a_r \ . \end{aligned} \right. \qquad (28)$$

To ease the presentation we rescale the time variable and redefine the control as

$$\sqrt{\frac{\mu}{p_0^3}}\, dt \ = \ d\tau \ , \quad \text{and} \quad u \ = \ \frac{p_0^2}{\mu}\, a_r \ .$$

The reduced dynamics can then be written (in the τ time scale) as

$$\left\{ \begin{aligned} \frac{d}{d\tau}\ x_1 \ &= \ (2+x_2)\, x_2 \\ \frac{d}{d\tau}\ x_2 \ &= \ -(1+x_2)^2\, x_3 \\ \frac{d}{d\tau}\ x_3 \ &= \ (1+x_2)^2\, x_2 \ + \ u \end{aligned} \right. \qquad (29)$$

Our objective for this third order subsystem, then, is to asymptotically stabilize

$$\cos(x_1) \ = \ 1 \ , \quad x_2 \ = \ x_3 \ = \ 0 \qquad (30)$$

while respecting the constraint $|u| \le c$.

4. THIRD ORDER SYSTEM

In equation (29), we observe that the (x_2, x_3) subsystem is in strict feedback form. Therefore, from (6), the backstepping technique furnishes a control-Lyapunov function of the form

$$V_{23}(x_2, x_3) \;=\; V_2(x_2) \;+\; \int_{\phi_2(x_2)}^{x_3} \psi(x_2, x)dx \quad (31)$$

where the sign of $\phi_2(\cdot)$ is the same as that of its argument and $\psi(\cdot,\cdot)$ satisfies the conditions put forth in Section 2.1. Note that $\phi_2(x_2)$ is the virtual control for the x_2 subsystem. Consequently, V_{23} is a control-Lyapunov function for the (x_2, x_3) subsystem. One may therefore construct a control ϕ_3 which renders the derivative of V_{23} negative definite. We do not concern ourselves with the precise form of either ϕ_2 or ϕ_3 for the moment.

We next observe that the $(x_1,(x_2, x_3))$ system is in strict feedforward form. Consequently, we require a change of coordinates modulo $L_g V$; i.e., we search for a continuous function, $\mathcal{M}(\cdot)$, dependant solely on x_2 (since the control does not act on x_2) such that $\mathcal{M}(0) = 0$ and

$$\begin{aligned} -\,\mathcal{M}'(x_2)\,(1+x_2)^2\,x_3 \;&=\; (2+x_2)\,x_2 \\ &-\; k(x_2,x_3)\,L_g V_{23}(x_2,x_3) \quad (32) \end{aligned}$$

where k is a continuous function that satisfies $k(0, 0) \neq 0$. Clearly, in this case,

$$L_g V_{23}(x_2, x_3) = \frac{\partial V_{23}}{\partial x_3}(x_2, x_3) = \psi(x_2, x_3) \quad (33)$$

Note that, from (32), if $L_g V_{23}(x_2, x_3) = 0$ then (unless x_2 is zero) we necessarily have

$$x_3 \;=\; -\frac{(2+x_2)\,x_2}{(1+x_2)^2\mathcal{M}'(x_2)}\;. \quad (34)$$

Therefore, from (33) and (7), we have the identity

$$\phi_2(x_2) \;=\; -\frac{(2+x_2)\,x_2}{(1+x_2)^2\mathcal{M}'(x_2)}\;. \quad (35)$$

Note that $\mathcal{M}(-1)$ is not well-defined. In fact, with the ϕ_2 we define later, as $x_2 \to -1$, $\mathcal{M}'(x_2) \to -\infty$. In the final step of the analysis, we demonstrate that solutions never leave the region $x_2 > -1$. With the relation (35), once we define the virtual control ϕ_2, we may calculate $\mathcal{M}(x_2)$. Therefore, from Section 2.2 we have a control-Lyapunov function of the form

$$V_{123}(x) = V_1(x_1 - \mathcal{M}(x_2)) + \gamma\,(V_{23}(x_2, x_3)) \quad (36)$$

where we leave the choice of γ until later.

4.1 Control Design

The problem which remains is to deal with the constraint $|u| \leq c$. In order to do this, we will make use of the CLF derived in the previous section and then tailor the two available degrees of freedom ($\mathcal{M}$ and γ) to satisfy the constraint. We make the following choices (others are certainly possible, and perhaps more useful):

$$\begin{aligned} V_1(x_1) \;&=\; \bar{\varepsilon}\int_0^{x_1} \mathrm{sat}\left(\frac{\bar{\lambda}}{\bar{\varepsilon}}\sin(s)\right)ds, \\ V_2(x_2) \;&=\; \frac{1}{2}\,x_2^2, \\ \psi(x_2,x_3) \;&=\; x_3 \;-\; \phi_2(x_2)\frac{1+\alpha\phi_2^2(x_2)}{1+\alpha x_3^2}\;. \end{aligned} \quad (37)$$

Let $\Upsilon(x_2, x_3) := \frac{\phi_2(x_2)(1+\alpha\phi_2^2(x_2))}{1+\alpha x_3^2}$, and $\Delta\mathrm{atg} := \arctan(\sqrt{\alpha}x_3) - \arctan(\sqrt{\alpha}\phi_2(x_2))$. We see that

$$\begin{aligned} V_{23}(x_2,x_3) &= \frac{1}{2}\,x_2^2 + \frac{1}{2}x_3^2 \\ &- \frac{\phi_2(x_2)(1+\alpha\phi_2^2(x_2))}{\sqrt{\alpha}}\Delta\mathrm{atg} - \frac{1}{2}\phi_2^2(x_2) \end{aligned} \quad (38)$$

With these choices and the relation in (35) we therefore obtain

$$\begin{aligned} &\frac{d}{d\tau}\,V_{123}(x_1,x_2,x_3) = \gamma'(V_{23})\frac{d}{d\tau}\,V_{23}(x_2,x_3) \\ &+ V_1'(x_1 - \mathcal{M}(x_2))\left[(2+x_2)x_2 - \frac{(2+x_2)x_2x_3}{\phi_2(x_2)}\right]. \end{aligned}$$

More calculations yield $\frac{\partial V_{23}}{\partial \phi_2} = -\frac{1+3\alpha\phi_2^2}{\sqrt{\alpha}}\Delta\mathrm{atg}$ and, consequently,

$$\begin{aligned} \frac{d}{d\tau}\,V_{23}(x_2,x_3) &= -(1+x_2)^2x_2x_3 \\ &+ \phi_2'(x_2)\left(\frac{1+3\alpha\phi_2^2(x_2)}{\sqrt{\alpha}}\right)\Delta\mathrm{atg}(1+x_2)^2x_3 \\ &+ (x_3 - \Upsilon(x_2,x_3))\,((1+x_2)^2x_2 + u). \end{aligned} \quad (39)$$

Collecting terms in (39) we obtain

$$\begin{aligned} \frac{d}{d\tau}\,V_{23}(x_2,x_3) &= -\Upsilon(x_2,x_3)(1+x_2)^2x_2 \\ &+ \psi(x_2,x_3)\,[\Gamma(x_2,x_3) + u] \end{aligned}$$

where $\Gamma(x_2, x_3) := \frac{\phi_2'(x_2)(1+x_2)^2x_3\left(\frac{1+3\alpha\phi_2^2(x_2)}{\sqrt{\alpha}}\right)\Delta\mathrm{atg}}{\psi(x_2,x_3)}$.

The τ-derivative of $V_{123}(\cdot)$ is then

$$\begin{aligned} \frac{d}{d\tau}\,V_{123}(x) &= -\gamma'(V_{23})\Upsilon(x_2,x_3)(1+x_2)^2x_2 \\ &+\gamma'(V_{23})\psi(x_2,x_3)\Bigg[-\bar{\varepsilon}\mathrm{sat}\left(\frac{\bar{\lambda}}{\bar{\varepsilon}}\sin(x_1-\mathcal{M}(x_2))\right) \\ &\left(\frac{(2+x_2)x_2}{\gamma'(V_{23})\phi_2(x_2)}\;\;\frac{x_3-\phi_2(x_2)}{\psi(x_2,x_3)}\right) \\ &+\Gamma(x_2,x_3)+u\Bigg]\;. \end{aligned} \quad (40)$$

4.2 Defining $\mathcal{M}$ (via ϕ_2) and γ

We make the two following definitions in hopes of bounding u: $\gamma'(s) = c_1 + c_2 s$ and

$$\phi_2(x_2) = \varepsilon \arctan\left(\frac{\lambda}{\varepsilon} x_2\right), \tag{41}$$

where λ, ε, c_1, and c_2 are free (positive) parameters. Note that $\gamma(s) = c_1 s + \frac{c_2}{2} s^2$. The selection of (41) is made to account for the ϕ_2^2 term in (40). From (38) we see that

$$\gamma'(V_{23}(x_2, x_3)) = c_1 + c_2 V_{23}(x_2, x_3) \geq c_1 + \frac{c_2}{2} x_2^2.$$

This allows us to account for the $(2 + x_2)x_2$ term in (40). We observe that

$$\begin{aligned}\psi(x_2, x_3) &= x_3 - \phi_2(x_2) \frac{1 + \alpha \phi_2^2(x_2)}{1 + \alpha x_3^2} \\ &= (x_3 - \phi_2(x_2)) \frac{1 + \alpha(x_3^2 + x_3\phi_2(x_2) + \phi_2^2(x_2))}{1 + \alpha x_3^2}.\end{aligned}$$

Consequently,

$$\frac{x_3 - \phi_2(x_2)}{\psi(x_2, x_3)} \leq \frac{1 + \alpha x_3^2}{1 + \frac{\alpha}{2} x_3^2} \leq 2. \tag{42}$$

4.3 Control Expression

By choosing the control as :

$$\frac{p_0^2}{\mu} a_r = u = u_1 + u_2 + u_3 \tag{43}$$

where

$$u_1 = V_1'(x_1 - \mathcal{M}(x_2)) \left(\frac{(2 + x_2)x_2}{\gamma'(V_{23}(x_2, x_3))\phi_2(x_2)}\right) \left(\frac{x_3 - \phi_2(x_2)}{\psi(x_2, x_3)}\right) \tag{44}$$

$$u_2 = -\Gamma(x_2, x_3) \tag{45}$$

$$u_3 = -\kappa_b \operatorname{sat}\left(\frac{\kappa_\ell}{\kappa_b} \psi(x_2, x_3)\right), \tag{46}$$

the time derivative of the closed-loop Lyapunov function becomes

$$\begin{aligned}-U_{123}(x) &:= \frac{d}{d\tau} V_{123}(x) = \\ &- (c_1 + c_2 V_{23})\ x_2 \Upsilon(x_2, x_3)\ (1 + x_2)^2 \\ &- (c_1 + c_2 V_{23})\ \kappa_b\ \psi(x_2, x_3) \operatorname{sat}\left(\frac{\kappa_\ell}{\kappa_b}\psi(x_2, x_3)\right).\end{aligned}$$

Specifically by choosing the control u as in (43) and coming back to the original time t, we may rewrite the system (28) in the compact notation

$$\frac{d}{dt} x = f_{cl}(x) \tag{47}$$

and we therefore have

$$\frac{d}{dt} V_{123}(x) = L_{f_{cl}} V_{123}(x) = -\sqrt{\frac{\mu}{p_0^3}}\, U_{123}(x). \tag{48}$$

Note that $L_{f_{cl}} V_{123}(x)$ is negative definite in (x_2, x_3). Moreover, for $(x_2, x_3) \equiv 0$, we have $u \equiv 0$. This yields $0 = \bar{\varepsilon} \operatorname{sat}\left(\frac{\bar{\lambda}}{\bar{\varepsilon}} \sin(x_1)\right) \frac{2}{c_1 \lambda}$. So we get $\sin(x_1) \equiv 0$. This proves that, as the time goes to $+\infty$, the solution converges to $x_2 = x_3 = 0$ and $x_1 = 0$ or $x_1 = \pi$. Note that the latter is unstable. In other words the closed loop system (47) is $\mathcal{S}$ observable with the output function U_{123} and $\mathcal{S} = \{(0, 0, 0), (\pi, 0, 0)\}$.

Now, using (35) and (41), we observe that, in (43), u is not defined at $x_2 = -1$. However, it is reasonable to restrict our attention to solutions such that $x_2 > -1$. Assume that our initial conditions lie on a circular orbit. In this case, $e = 0$, which implies that $x_2 = \frac{p}{p_0} - 1$. Since p and p_0 are always positive, x_2 is, at least initially, greater than -1. At the end of Section 6, we shall argue that solutions, in fact, never leave the set $x_2 > -1$.

The control law we have proposed has nine free design parameters: $(\alpha, \varepsilon, \lambda, c_1, c_2, \bar{\varepsilon}, \bar{\lambda}, \kappa_b, \kappa_\ell)$. The parameters may be fixed to set the local performance $(c_1, \lambda, \bar{\lambda}, \kappa_\ell)$, while $(\alpha, \varepsilon, c_2, \bar{\varepsilon}, \kappa_b)$ are then available to satisfy the control constraints. We do not explicitly address the local behaviour in this work.

4.4 Control Bounds

We now proceed to show that u is bounded. We will do this by considering each term individually considering that $(c_1, \lambda, \bar{\lambda}, \kappa_\ell)$ are given.

The third term is the easiest, as, from (46) we clearly see that $|u_3| \leq \kappa_b$. So, using κ_b to bound u_3, we have four constants, $(\alpha, \varepsilon, c_2, \bar{\varepsilon})$, that may be used to bound the remaining two terms.

We now examine the second control term u_2 defined by (45). We can show that the expression $\frac{\lambda \varepsilon^2}{\varepsilon^2 + \lambda^2 x_2^2}(1 + x_2)^2$ obtains its maximum at $x_2 = \frac{\varepsilon^2}{\lambda^2}$ (it obtains its minimum at $x_2 = -1$) so that

$$\left|\frac{\lambda \varepsilon^2}{\varepsilon^2 + \lambda^2 x_2^2}(1 + x_2)^2\right| \leq \lambda\left(1 + \frac{\varepsilon^2}{\lambda^2}\right).$$

Furthermore, we see that

$$\left|\frac{1 + 3\alpha\phi_2^2(x_2)}{\sqrt{\alpha}}\right| \leq \frac{1 + 3\alpha\varepsilon^2 \frac{\pi^2}{4}}{\sqrt{\alpha}}. \tag{49}$$

Let $h(s) := \arctan\left(\sqrt{\alpha}(\phi_2 + s(x_3 - \phi_2))\right)$ so that $h(0) = \arctan(\sqrt{\alpha}\phi_2)$ and $h(1) = \arctan(\sqrt{\alpha} x_3)$; in other words, $\Delta\text{atg} = h(1) - h(0)$. We note that $h'(s) = \sqrt{\alpha}(x_3 - \phi_2)\frac{1}{1 + \alpha(\phi_2 + s(x_3 - \phi_2))^2}$ so that, by the fundamental theorem of calculus, $\Delta\text{atg} = \sqrt{\alpha}(x_3 - \phi_2)\int_0^1 \frac{1}{1 + \alpha(\phi_2 + s(x_3 - \phi_2))^2} ds$. Hence we have $|\Delta\text{atg}| \leq \min\{\pi, \sqrt{\alpha}(x_3 - \phi_2)\}$ and we may write

$$\begin{aligned}|x_3 \Delta\text{atg}| &= |x_3 \Delta\text{atg} + \phi_2\ \Delta\text{atg} - \phi_2\ \Delta\text{atg}| \\ &\leq |x_3 - \phi_2| \pi \left(1 + \sqrt{\alpha}\frac{\varepsilon}{2}\right).\end{aligned} \tag{50}$$

With (42), we see that $\left|\frac{x_3\ \Delta\text{atg}}{\psi(x_2,x_3)}\right| \leq 2\pi\left(1+\sqrt{\alpha}\frac{\varepsilon}{2}\right)$. We may then bound $|u_2|$ as follows:

$$\left|\frac{\frac{\lambda\varepsilon^2}{\varepsilon^2+\lambda^2x_2^2}(1+x_2)^2\left(\frac{1+3\alpha\phi_2^2(x_2)}{\sqrt{\alpha}}\right)x_3\ \Delta\text{atg}}{\psi(x_2,x_3)}\right|$$

$$\leq \lambda\left(1+\frac{\varepsilon^2}{\lambda^2}\right)\ \frac{1+3\alpha\varepsilon^2\frac{\pi}{4}}{\sqrt{\alpha}}\ 2\pi\left(1+\sqrt{\alpha}\frac{\varepsilon}{2}\right)$$

We observe that this can be made arbitrarily small by choosing $\sqrt{\alpha}$ large and the product $\varepsilon\sqrt{\alpha}$ small.

We now turn to the first term defining the control with only $(c_2,\bar{\varepsilon})$ not already constrained. We first show that we have

$$\left|\frac{y}{\arctan(y)}\right| \leq 1+\frac{2}{\pi}|y|. \tag{51}$$

To do so, we remark that the function $z\mapsto \frac{1}{z}-\frac{1}{\tan(z)}$ is well defined and increasing on $[0,\frac{\pi}{2})$. In particular its derivative is $-\frac{1}{z^2}+\frac{1+\tan(z)^2}{\tan(z)^2}$ and therefore positive since we have

$$\frac{\tan(z)^2}{1+\tan(z)^2}\ =\ \sin(z)^2\ \leq\ z^2. \tag{52}$$

We conclude that $\frac{1}{z}-\frac{1}{\tan(z)}\ <\ \frac{2}{\pi}$ for all $z\in[0,\frac{\pi}{2})$. We see that, by letting $z=\arctan(y)$, (51) follows for positive y's and by symmetry for negative y's. Therefore

$$\begin{aligned}|u_1| &\leq 2\bar{\varepsilon}\left|\frac{(2+x_2)x_2}{(c_1+c_2V_{23}(x_2,x_3))\varepsilon\arctan\left(\frac{\lambda}{\varepsilon}x_2\right)}\right|\\ &\leq 2\bar{\varepsilon}\left|\frac{(2+x_2)}{\lambda(c_1+c_2x_2^2)}\ \left(1+\frac{2\lambda}{\pi\varepsilon}|x_2|\right)\right|\\ &\leq 2\bar{\varepsilon}\ \max\left\{\frac{4}{\lambda c_1}\ ,\ \frac{\left(\frac{1}{2}+\frac{2\lambda}{\pi\varepsilon}\right)^2}{\lambda c_2}\right\}\ := 2\bar{\varepsilon}\kappa_1.\end{aligned}$$

Since κ_1 is fixed by an appropriate choice of c_2, the size of u_1 is fully controlled by $\bar{\varepsilon}$. Consequently,

$$\begin{aligned}&|u| \leq |u_1|+|u_2|+|u_3| \leq 2\bar{\varepsilon}\kappa_1\\ &+\lambda\left(1+\frac{\varepsilon^2}{\lambda^2}\right)\ \frac{1+3\alpha\varepsilon^2\frac{\pi}{4}}{\sqrt{\alpha}}\ 2\pi\left(1+\sqrt{\alpha}\frac{\varepsilon}{2}\right)+\kappa_b\ .\end{aligned}$$

Therefore, given the constant A_r in (24), we see that we can satisfy $\frac{\mu}{p_0^2}|u| = |a_r| \leq A_r$ by selecting appropriate values for $(\alpha,\varepsilon,c_2,\bar{\varepsilon},\kappa_b)$.

5. FOURTH ORDER SYSTEM

We have designed a feedback law for the reduced third order dynamics leading to the system (47) when $p=p_0$. We now consider the fourth order system (27) with a_r chosen as above. We then write (27) as

$$\left\{\begin{aligned}\frac{d}{dt}\ x\ &=\ f_{cl}(x)+\left(\sqrt{\frac{p}{p_0}}-1\right)g(x,p)\\ \frac{d}{dt}\ p\ &=\ 2p_0\sqrt{\frac{p}{\mu}}\frac{1}{1+x_2}a_\theta\end{aligned}\right. \tag{53}$$

with

$$g(x,p)=\begin{bmatrix}\frac{\mu}{p_0^3}(1+x_2)^2\\ 0\\ \left(\sqrt{\frac{p}{p_0}}+1\right)\frac{\mu}{p_0^3}(1+x_2)^3\end{bmatrix}. \tag{54}$$

Note that when $p=p_0$, the x-subsystem reduces to the third-order closed-loop system (47).

We define a Lyapunov function for the fourth order system as

$$V_{1234}(x,p)=V_{123}(x)\ +\ \rho\left(V_4(p)\right) \tag{55}$$

where

$$V_4(p):=2\sqrt{p}-\sqrt{p_0}\log p-[2\sqrt{p_0}-\sqrt{p_0}\log p_0]$$

and V_{123} is nothing but (36). The function $\rho:\mathbb{R}_{\geq 0}\rightarrow\mathbb{R}_{\geq 0}$, as γ previously, is continuously differentiable with a strictly positive derivative and will be used to provide an extra degree of design freedom. Therefore, with (48), we obtain

$$\begin{aligned}&\frac{d}{dt}\ V_{1234}(x,p)=-\sqrt{\frac{\mu}{p_0^3}}\,U_{123}(x)+\left(\sqrt{\frac{p}{p_0}}-1\right)\\ &\left(L_gV_{123}(x)+\frac{2}{\sqrt{p}}\sqrt{\frac{p_0^3}{\mu}}\frac{1}{1+x_2}\rho'(V_4(p))a_\theta\right).\end{aligned}$$

We then define the control a_θ as

$$a_\theta := -\frac{\frac{\sqrt{p}}{2}\sqrt{\frac{\mu}{p_0^3}}(1+x_2)L_gV_{123}(x)}{\rho'\left(V_4(p)\right)}\ -\ k_p\left(\sqrt{\frac{p}{p_0}}-1\right) \tag{56}$$

where $k_p(\cdot)$ is any continuous function which has the same sign as its argument.

The time-derivative of the closed-loop Lyapunov function is then

$$\begin{aligned}&-U_{1234}(x,p):=\frac{d}{dt}\ V_{1234}(x,p)=-\sqrt{\frac{\mu}{p_0^3}}\,U_{123}(x)\\ &-\frac{\sqrt{\frac{p_0^3}{\mu}}}{1+x_2}\rho'(V_4(p))\left(\sqrt{\frac{p}{p_0}}-1\right)k_p\left(\sqrt{\frac{p}{p_0}}-1\right).\end{aligned} \tag{57}$$

With the previous analysis for the third order system, we can conclude that $p=p_0, x_2=x_3, \sin(x_1)=0$ is stable and attractive for the fourth order system.

Concerning the bound for a_θ, we have no problem with its second term since the function $k_p(\cdot)$ is completely free except for its sign. We do, however, have a problem with its first term. Its magnitude with respect to p can be bounded by an appropriate choice of the function ρ but there is nothing we can do concerning the presence of x. If follows that we cannot guarantee a global bound but only a local bound on the domain of practical solutions.

Having specified the controls a_θ and a_r, the closed loop system (26) or equivalently (22), with $a_\theta = 0$ and without the (h_x, h_y) components, can be written in a compact form as

$$\frac{d}{dt}\,\mathcal{X} = \varphi_{cl}(\mathcal{X})\ . \tag{58}$$

We also rewrite V_{1234} from (57) as

$$\frac{d}{dt}\,V_{1234}(\mathcal{X}) \;=\; -U_{1234}(\mathcal{X}). \tag{59}$$

We observe that the closed loop system (58) is $\mathcal{S}$ observable with

$$\mathcal{S} = \{(p = p_0\ ,\ e_x = e_y = 0\ ,\ L = L_0 \bmod \pi\}$$

and output function U_{1234}.

6. FULL SYSTEM

Coming back to the full system with the controls a_θ and a_r as specified above and the notation of (58), we may rewrite (22) in the compact form

$$\left\{\begin{array}{lcl} \dfrac{d}{dt}\,\mathcal{X} &=& \varphi_{cl}(\mathcal{X}) + \sigma(\mathcal{X}, h_x, h_y)a_h \\ \dfrac{d}{dt}\,h_x &=& \dfrac{k}{2}\,X\,\cos(L)\,a_h \\ \dfrac{d}{dt}\,h_y &=& \dfrac{k}{2}\,X\,\sin(L)\,a_h \end{array}\right. \tag{60}$$

with

$$\sigma(\mathcal{X}, h_x, h_y) = \begin{bmatrix} 0 \\ -ke_yY \\ ke_xY \\ kY \end{bmatrix}\ . \tag{61}$$

Recall that $Y = h_x \sin(L) - h_y \cos(L)$. Let

$$\nu(\mathcal{X}, h_x, h_y) = \begin{bmatrix} \sigma(\mathcal{X}, h_x, h_y) \\ \dfrac{k}{2}\,X\,\cos(L) \\ \dfrac{k}{2}\,X\,\sin(L) \end{bmatrix}\ . \tag{62}$$

With (59), we observe that (60) is $\mathcal{C}^1$-dissipative from the Lyapunov function

$$W(x) := V_{1234}(\mathcal{X}) + \frac{1}{2}h_x^2 + \frac{1}{2}h_y^2. \tag{63}$$

Furthermore, it is not difficult to see that a solution which satisfies $L_\nu W = 0$ must have $h_x = h_y = 0$. It follows that (60) is $\mathcal{S}$ observable with the output function $(U_{1234}\ \ L_\nu W)$ and $\mathcal{S} = \{p = p_0,\ e_x = e_y = h_x = h_y = 0,\ L = L_0 \bmod \pi\}$. Therefore (see Theorem 1), by picking the control a_h as

$$a_h = -k_h(L_\nu W) \tag{64}$$

with k_h any continuous function with the same sign as its argument, the set $\mathcal{S}$ is made attractive with $(p = p_0\ ,\ e_x = e_y = h_x = h_y = 0\ ,\ L = L_0 + \pi)$ unstable and $(p = p_0\ ,\ e_x = e_y = h_x = h_y = 0\ ,\ L = L_0)$ stable.

Finally, we return to the issue of whether or not it is possible for the solutions to leave the set $x_2 > -1$. Let $[0, T)$ be the right maximal interval of definition of a solution in this set. From our Lyapunov analysis we know that all solutions, and in particular x_3, are bounded on $[0, T)$. If T is finite, we have $\lim_{t\to T} x_2(t) = -1$. However, this is not possible as, for some $c > 0$ we know that $\overbrace{|1 + x_2|}^{\cdot} \leq c|1 + x_2|$. Consequently, $T = +\infty$ and the previous Lyapunov analysis implies that $x_2 \to 0$.

7. CONCLUSIONS

We have derived a state feedback stabilizer for orbital transfer under low thrust using several recent nonlinear control techniques. We note that we have not addressed the issue of performance. In particular, we made certain choices for various functions (such as in equations (37), (41), and (64)) where these choices were not motivated by performance issues. We also have not considered the problem of periodic occultations. Ion engines are usually dependent on solar power. Solar energy is clearly not available over an entire orbit due to the satellite periodically passing through the Earth's shadow.

REFERENCES

R. H. Battin. *An Introduction to the Mathematics and Methods of Astrodynamics*. American Institute of Aeronautics and Astronautics, 1987.

D. E. Chang, D. F. Chichka, and J. E. Marsden. Lyapunov-based transfer between elliptic keplerian orbits. *Discrete and Continuous Dynamical Systems - Series B*, 2(1):57–67, February 2002.

M. Krstić, I. Kanellakopoulos, and P. Kokotović. *Nonlinear and Adaptive Control Design*. John Wiley and Sons, Inc., 1995.

L. Praly. An introduction to forwarding. In *Control of Complex Systems*. Springer, 2001.

L. Praly, B. d'Andréa Novel, and J.-M. Coron. Lyapunov design of stabilizing controllers for cascaded systems. *IEEE Transactions on Automatic Control*, 36(10), October 1991.

L. Praly, R. Ortega, and G. Kaliora. Stabilization of nonlinear systems via forwarding mod $\{L_gV\}$. *IEEE Transactions on Automatic Control*, 46(9):1461–1466, September 2001.

R. Sepulchre, M. Janković, and P. Kokotović. *Constructive Nonlinear Control*. Springer, 1997.