SIAM J. CONTROL OPTIM.
Vol. 42, No. 6, pp. 1934–1959

WEAK CONVERSE LYAPUNOV THEOREMS AND CONTROL-LYAPUNOV FUNCTIONS*

CHRISTOPHER M. KELLETT† AND ANDREW R. TEEL‡

Abstract. Given a *weakly uniformly globally asymptotically stable* closed (not necessarily compact) set $\mathcal{A}$ for a differential inclusion that is defined on $\mathbb{R}^n$, is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, and satisfies other basic conditions, we construct a weak Lyapunov function that is locally Lipschitz on $\mathbb{R}^n$. Using this result, we show that uniform global asymptotic controllability to a closed (not necessarily compact) set for a locally Lipschitz nonlinear control system implies the existence of a locally Lipschitz control-Lyapunov function, and from this control-Lyapunov function we construct a feedback that is robust to measurement noise.

Key words. converse Lyapunov theorem, weak set stability, differential inclusions, control-Lyapunov function, asymptotic controllability

AMS subject classifications. 34A60, 93B05, 93D15, 93D20, 93C57

DOI. 10.1137/S0363012901398186

1. Introduction.

Herein we consider the related questions of the existence of a weak Lyapunov function for differential inclusions, and the existence of a control-Lyapunov function for controlled differential equations, under the assumption of weak asymptotic stability (respectively, asymptotic controllability) of (to) a closed, not necessarily compact, set $\mathcal{A}$.

The converse question in Lyapunov theory has received a great deal of attention. In the case of the differential inclusion

$$\dot{x} \in F(x), \; x \in \mathbb{R}^n, \tag{1.1}$$

we ask the following question: Given the weak (or strong) asymptotic stability of an attractor $\mathcal{A}$, does there exist a positive definite, proper function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfying a specific decrease condition along at least one trajectory (respectively, all trajectories)?

Numerous results on the existence of smooth, strong Lyapunov functions for *strong asymptotic stability* have been established, that is, when *all* solutions of (1.1) satisfy stability and uniform attractivity properties with respect to the set $\mathcal{A}$. Among these results is the pioneering work of Kurzweil [17] and Wilson [31] with recent results by Lin, Sontag, and Wang [18], Clarke, Ledyaev, and Stern [7], and Teel and Praly [30]. A more complete summary of these results can be found in Teel and Praly [30].

All of the above references for strong asymptotic stability generate *smooth* Lyapunov functions. Lower semicontinuous Lyapunov functions for weak stability (not

*Received by the editors November 13, 2001; accepted for publication (in revised form) May 31, 2003; published electronically January 22, 2004. Portions of this article appeared in preliminary form in *Proceedings of Mathematical Theory of Networks and Systems*, Perpignan, France, June 2000 and *Proceedings of the* 39*th IEEE Conference on Decision and Control*, Sydney, Australia, December 2000. This research was supported in part by the NSF under grant ECS-9896140 and the AFOSR under grant F49620-00-1-0106.

http://www.siam.org/journals/sicon/42-6/39818.html

†Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106. Current address: Department of Electrical and Electronic Engineering, University of Melbourne, VIC 3052, Australia (ckellett@ee.mu.oz.au).

‡Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106 (teel@ece.ucsb.edu).

asymptotic) were given by Roxin [21, Theorem 9.5] for arbitrary closed sets $\mathcal{A}$ and by Deimling [10, Proposition 14.2] for $\mathcal{A} = \{0\}$. Clarke, Ledyaev, and Stern [7, Remark 1.5 and Theorem 6.1] noted that the existence of a continuously differentiable Lyapunov function for weak asymptotic stability of $\mathcal{A} = \{0\}$ implies the existence of a smooth Lyapunov function. However, they further demonstrated that the set-valued map $F(\cdot)$ must satisfy a nongeneric covering condition to admit a continuously differentiable weak Lyapunov function. Since we would like to consider inclusions which do not satisfy this covering condition, we require a decrease condition for nondifferentiable functions. We will make use of the Dini subderivate.

DEFINITION 1. *The Dini subderivate of a function $V : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ in the direction $v \in \mathbb{R}^n$ is defined as*

$$DV(x; v) := \liminf_{w \to v, \varepsilon \to 0^+} \frac{V(x + \varepsilon w) - V(x)}{\varepsilon} .$$

For a locally Lipschitz function $V : \mathcal{O} \to \mathbb{R}$ ($\mathcal{O}$ open) at $x \in \mathcal{O}$ in the direction $v \in \mathbb{R}^n$ this simplifies to (see [8, p. 136])

$$DV(x; v) = \liminf_{\varepsilon \to 0^+} \frac{V(x + \varepsilon v) - V(x)}{\varepsilon} .$$

Therefore, in order to consider a more general class of inclusions (i.e., the class that includes inclusions which do not satisfy the previously mentioned covering condition), we search for a locally Lipschitz function $V(\cdot)$ and specify the decrease condition as

$$\min_{w \in F(x)} DV(x; w) \leq -V(x) \ \forall x \in \mathbb{R}^n.$$

Under appropriate conditions on the map $F : \mathbb{R}^n \to$ subsets of $\mathbb{R}^n$, the existence of a locally Lipschitz weak Lyapunov function for (possibly) noncompact attractors is asserted in Theorem 2.1. This result first appeared in Kellett and Teel [14]. We note that, while their results were stated for controlled differential equations, Clarke et al. [5] and Rifford [20] used differential inclusions as an intermediary. Implicitly, Clarke et al. [5] constructed a weak Lyapunov function, locally Lipschitz on compact sets disjoint from the origin, given, essentially, weak asymptotic stability of the origin. Rifford [20] then combined these functions in a clever way to obtain a weak Lyapunov function that is locally Lipschitz on $\mathbb{R}^n$, given weak asymptotic stability to the origin.

Now consider the controlled differential equation

$$\dot{x} = f(x, u), \ x \in \mathbb{R}^n, \ u \in \mathcal{U} , \tag{1.2}$$

where $\mathcal{U}$ is the set of all possible controls, and, rather than a weak Lyapunov function, we are interested in a control-Lyapunov function. An early result related to the existence of control-Lyapunov functions came from Roxin [22]. Results on the existence of control-Lyapunov functions, given asymptotic controllability to the origin, came from Sontag [24] (cf. Sontag and Sussman [29]), Clarke et al. [6], Sontag [28], and Clarke et al. [5]. In [20], Rifford generated a locally Lipschitz control-Lyapunov function, given asymptotic controllability to the origin, answering a longstanding question on the existence of same. A continuous control-Lyapunov function was generated by Albertini and Sontag [1] under the assumption of asymptotic controllability to a closed (not necessarily compact) set.

It is well known that continuously differentiable control-Lyapunov functions fail to exist for generic systems; i.e., systems which fail to satisfy Brockett's covering condition (see Brockett [4] or Ryan [23] for a definition). Again, we would like to consider a broader class of systems, including those that do not satisfy Brockett's condition. Toward this end, we examine the differential inclusion obtained by allowing the controls to range over the admissable control set; i.e.,

$$F(x) := \{z \in \mathbb{R}^n : z = f(x,u), u \in \mathcal{U}\} \ .$$

Intuitively, then, weak asymptotic stability of $\mathcal{A}$ for the differential inclusion thus defined is equivalent to the asymptotic controllability of the differential equation to $\mathcal{A}$. That is, the trajectory of $\dot{x} \in F(x)$ which does not wander too far from and is attracted to the set $\mathcal{A}$ is generated by a particular control selection. The notion of controllability used herein will be made precise in section 3. Following the arguments presented for inclusions, the corresponding decrease condition of the control-Lyapunov function then becomes

$$\min_{w \in f(x,\mathcal{U})} DV(x;w) \leq -V(x) \ \forall x \in \mathbb{R}^n \ .$$

That is, there exists a control selection such that the direction of the vector field defining the system causes the Dini subderivate of $V(\cdot)$ to decrease. Note that this is an intuitive discussion and we have avoided concerning ourselves with specifics such as a necessary "small-control property," precise regularity conditions on $f(\cdot,\cdot)$, and other technical details. These specifics are addressed in the following sections.

Remark 1. The above decrease condition involving the Dini subderivate was also used in Clarke et al. [5], [6]. In both references, use is also made of an equivalent formulation in terms of the proximal subgradient; i.e.,

$$\min_{u \in \mathcal{U}} \langle f(x,u), \zeta \rangle \leq -V(x) \ \forall x \in \mathbb{R}^n \ \forall \zeta \in \partial_P V(x) \ . \qquad \square$$

The significance of this novel result stems from the important role that control-Lyapunov functions have played in the development of stabilizing state feedbacks over the years. As examples, we refer the reader to Artstein [2], Sontag [25], Freeman and Kokotović [12], and Krstić, Kanellakopoulos, and Kokotović [16] for the case of continuously differentiable control-Lyapunov functions and to Clarke et al. [6], Sontag [28], and Clarke et al. [5] for locally Lipschitz control-Lyapunov functions. Similar to the latter articles, in section 4 we present the design of a (discontinuous) stabilizing state feedback that is robust to small additive disturbances and measurement noise using our derived control-Lyapunov function.

Our approach is to convert the control system into a differential inclusion (which is the approach also taken in Clarke et al. [5] and Rifford [20]) and then use the result on the existence of a Lyapunov function for the differential inclusion to get the promised control-Lyapunov function. The novelty of the current control-Lyapunov function is that the result is derived for closed (possibly noncompact) attractors. The proof technique is also novel in that it follows directly from a comparison function formulation of the controllability or stability property. This result first appeared in Kellett and Teel [15].

Our paper is organized as follows. Section 2 contains the precise statement of our weak converse Lyapunov theorem for differential inclusions with the associated proof in section 5. Section 3 contains our control-Lyapunov function result. A stabilizing feedback construction for use with locally Lipschitz control-Lyapunov functions, such as the one presented in section 3, is given in section 4, with a robustness result for this feedback given in section 4.4. Section 6 contains necessary technical proofs.

2. A weak converse Lyapunov theorem. Having given some insight for the results which follow, we begin to make these ideas more precise. In what follows we let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^n$; i.e., $|x| = \sqrt{\langle x, x\rangle}$. For a closed set $\mathcal{A} \subset \mathbb{R}^n$ we write the distance from a point $x \in \mathbb{R}^n$ to the set $\mathcal{A}$ as $|x|_\mathcal{A} := \inf_{a\in\mathcal{A}} |x - a|$. We let $\overline{\mathcal{B}}_n(x, r)$ denote the closed ball in $\mathbb{R}^n$ of radius r centered at x; i.e., $\overline{\mathcal{B}}_n(x, r) := \{\xi \in \mathbb{R}^n : |\xi - x| \leq r\}$. We define $\overline{\mathcal{B}}_n := \overline{\mathcal{B}}_n(0, 1)$, where 0 denotes the origin in $\mathbb{R}^n$. Recall that a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it is continuous, zero at zero, strictly increasing, and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if, for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s\to 0^+} \beta(s, t) = 0$, and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s, t) = 0$.

A function $x : [0, T] \to \mathbb{R}^n$ ($T > 0$) is said to be a *solution* of the differential inclusion $\dot{x} \in F(x)$ if it is absolutely continuous and satisfies, for almost all $t \in [0, T]$, $\dot{x}(t) \in F(x(t))$. A function $x : [0, T) \to \mathbb{R}^n$ ($0 < T \leq +\infty$) is said to be a *maximal solution* of the differential inclusion if it does not have an extension which is a solution belonging to $\mathbb{R}^n$; i.e., either $T = \infty$ or there does not exist a solution $y : [0, T_+] \to \mathbb{R}^n$ with $T_+ > T$ such that $y(t) = x(t)$ for all $t \in [0, T)$. We use $\phi(\cdot, x)$ to denote a solution of $\dot{x} \in F(x)$ starting at x. We denote by $\mathcal{S}[0, T](x)$, or $\mathcal{S}[0, T)(x)$, the set of maximal solutions starting at x that are defined on the time interval $[0, T]$, or $[0, T)$.

The following basic conditions guarantee existence of solutions for differential inclusions.

Definition 2. *The set-valued map $F(\cdot)$ is said to satisfy the* basic conditions on $\mathbb{R}^n$ *if, for each $x \in \mathbb{R}^n$, $F(x)$ is nonempty, compact, and convex and if $F(\cdot)$ is upper semicontinuous on $\mathbb{R}^n$; i.e., for each $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $\xi \in \mathbb{R}^n$ satisfying $|x - \xi| < \delta$, we have $F(\xi) \subseteq F(x) + \varepsilon\mathcal{B}_n$.*

Previous results which obtained lower semicontinuous Lyapunov functions (see [21], [10]) assumed the set-valued $F(\cdot)$ was merely upper semicontinuous. Our stronger result (i.e., existence of a locally Lipschitz Lyapunov function) will require a correspondingly stronger regularity property.

Definition 3. *A set-valued map $F : \mathbb{R}^n \to$ subsets of $\mathbb{R}^n$ is* locally Lipschitz on $\mathcal{O} \subseteq \mathbb{R}^n$ *if for all $x \in \mathcal{O}$ there exists a neighborhood $\mathcal{U} \subseteq \mathcal{O}$ of x and $L > 0$ such that $x_1, x_2 \in \mathcal{U}$ implies $F(x_1) \subseteq F(x_2) + L|x_1 - x_2|\overline{\mathcal{B}}_n$. We say that this property is* uniform in distance to the closed set $\mathcal{A}$ *if for any $\ell > 0$ the above neighborhood can be defined as $\mathcal{U} := \{x \in \mathbb{R}^n : |x|_\mathcal{A} \leq \ell\}$.*

Definition 4. *Analogous to our terminology for set-valued maps, we say that a function $g : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, uniformly in distance to the closed set $\mathcal{A}$, if for any $\ell > 0$ and a closed set $\mathcal{U} := \{x \in \mathbb{R}^n : |x|_\mathcal{A} \leq \ell\}$ there exists $L > 0$ such that for every $x_1, x_2 \in \mathcal{U}$ we have $|g(x_1) - g(x_2)| \leq L|x_1 - x_2|$.*

Our stability condition is phrased in the language of comparison functions (as is our controllability condition in the next section). This presents a simple and concise way to summarize both uniform boundedness and attractivity.

Definition 5. *For the differential inclusion $\dot{x} \in F(x)$, the closed set $\mathcal{A} \subset \mathbb{R}^n$ is said to be* weakly uniformly globally asymptotically stable (weakly UGAS) *if there exists $\beta \in \mathcal{KL}$ such that, for each $x \in \mathbb{R}^n$, there exists a solution $\phi \in \mathcal{S}[0, \infty)(x)$ satisfying $|\phi(t, x)|_\mathcal{A} \leq \beta(|x|_\mathcal{A}, t)$ for all $t \geq 0$.*

We will make an assumption that follows [1, Definition 1.5]. This rules out finite time trajectory escape unobservable through distance to the set $\mathcal{A}$. Or, intuitively, it prevents the trajectory escaping to infinity in a direction parallel to the set $\mathcal{A}$.

Assumption 1. For each $r > 0$ there exists $M_r > 0$ such that $|x|_{\mathcal{A}} \leq r$ implies $\sup_{w \in F(x)} |w| \leq M_r$.

We are now in a position to assert the existence of a locally Lipschitz weak Lyapunov function.

THEOREM 2.1. *Suppose $F(\cdot)$ satisfies the basic conditions on $\mathbb{R}^n$, is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, satisfies Assumption 1, and, for $\dot{x} \in F(x)$, the closed set $\mathcal{A}$ is weakly UGAS. Then there exists a (weak Lyapunov) function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ that is locally Lipschitz on $\mathbb{R}^n$ and $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}) \quad \text{and} \tag{2.1}$$

$$\min_{w \in F(x)} DV(x; w) \leq -V(x) \; . \tag{2.2}$$

Furthermore, if $F(\cdot)$ is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, uniformly in distance to the set $\mathcal{A}$, then there exists a locally Lipschitz weak Lyapunov function for $\dot{x} \in F(x)$ with respect to $\mathcal{A}$ where the local Lipschitz property is uniform in distance to the set $\mathcal{A}$.

The theorem is proved in section 5.

Remark 2. The use of the minimum is justified here, and throughout the paper, in place of an infimum by virtue of the fact that the set over which the infimum is taken is compact and the function is continuous; i.e., $DV(x; \cdot)$ is locally Lipschitz for all $x \in \mathbb{R}^n$ when $V(\cdot)$ is locally Lipschitz (see [8, Exercise 3.4.1a]). □

3. A control-Lyapunov function.

In this section we state our result that uniform asymptotic controllability to a set implies the existence of a locally Lipschitz control-Lyapunov function. In what follows, we take $\mathcal{U}$ to be a locally compact metric space with a unique zero element, "0," and, by abuse of notation, $|u| := d(u, 0)$. We define the closed unit ball in the metric space $\mathcal{U}$ as $\overline{\mathcal{B}}_{\mathcal{U}} := \{\xi \in \mathcal{U} : d(\xi, 0) \leq 1\}$.

DEFINITION 6. *Let $\mathcal{A} \subset \mathbb{R}^n$ be a closed, nonempty set, and let $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be nondecreasing. We say that* (1.2) *is* uniformly globally asymptotically controllable (UGAC) to $\mathcal{A}$ with $\mathcal{U} \cap \sigma$ controls *if there exists a function $\beta \in \mathcal{KL}$ such that for each $x \in \mathbb{R}^n$ there exist a measurable, essentially bounded function $u : [0, \infty) \to \mathcal{U}$ and a solution $\phi(\cdot, x, u)$ of $\dot{x} = f(x, u(t))$ satisfying*

$$\begin{aligned} |\phi(t, x, u)|_{\mathcal{A}} &\leq \beta(|x|_{\mathcal{A}}, t), \\ |u(t)| &\leq \sigma(|\phi(t, x, u)|_{\mathcal{A}}) \quad \text{for almost all } t \geq 0. \end{aligned} \tag{3.1}$$

Remark 3. We note that $\mathcal{U} \cap \sigma$ is an abuse of notation. It is shorthand for allowing controls from $\mathcal{U} \cap \sigma(|x|_{\mathcal{A}})\overline{\mathcal{B}}_{\mathcal{U}} = \{u \in \mathcal{U} : |u| \leq \sigma(|x|_{\mathcal{A}})\}$ for each $x \in \mathbb{R}^n$. □

Note that the usual definition of UGAC (such as in [24, Definition 2.2] or [27]) limits the control based on the size of the initial condition of the state, whereas for UGAC with $\mathcal{U} \cap \sigma$ controls we limit the control through the size of the trajectory. The following lemma is proved in [13, section 5.3.6].

LEMMA 3.1. *The system* (1.2) *is UGAC to $\mathcal{A}$ with $\mathcal{U} \cap \sigma$ controls if and only if there exist $\beta_c \in \mathcal{KL}$ and $\sigma_c : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ nondecreasing such that for each $x \in \mathbb{R}^n$ there exist a measurable, essentially bounded function $u : [0, \infty) \to \mathcal{U}$ and a maximal solution $\phi(t, x, u(t))$ of* (1.2) *such that $\|u\|_\infty \leq \sigma_c(|x|_{\mathcal{A}})$ and $|\phi(t, x, u(t))|_{\mathcal{A}} \leq \beta_c(|x|_{\mathcal{A}}, t)$.*

DEFINITION 7. *Let $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be nondecreasing. We say a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a* control-Lyapunov function with $\mathcal{U} \cap \sigma$ controls *for the* *system* (1.2) *if there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that* $\alpha_1(|x|_\mathcal{A}) \leq V(x) \leq \alpha_2(|x|_\mathcal{A})$, *and* $V(\cdot)$ *satisfies the weak infinitesimal decrease property*

$$\min_{w \in \overline{\text{co}} f(x, \mathcal{U} \cap \sigma(|x|_\mathcal{A})\overline{\mathcal{B}}_\mathcal{U})} DV(x; w) \leq -V(x) \ \forall x \in \mathbb{R}^n \ .$$

Our result will require the following technical assumption which parallels Assumption 1 and which is essentially [1, Definition 1.5]. Again, this rules out finite time escape of trajectories which is not observable through distance to the set $\mathcal{A}$.

Assumption 2. For all $r_1, r_2 \in \mathbb{R}_{>0}$, there exists $M_{r_1,r_2} > 0$ such that

$$\sup_{\{|x|_\mathcal{A} \leq r_1, |u| \leq \sigma(r_2)\}} |f(x,u)| \leq M_{r_1,r_2}.$$

With all the necessary definitions in hand, we state the following theorem.

THEOREM 3.2. *Suppose* (1.2) *satisfies Assumption* 2 *and is UGAC to the set* $\mathcal{A}$ *with* $\mathcal{U} \cap \sigma$ *controls. Furthermore, assume that the set-valued map*

$$F(x) := \overline{\text{co}} f\left(x, \mathcal{U} \cap \sigma(|x|_\mathcal{A})\overline{\mathcal{B}}_\mathcal{U}\right)$$

satisfies the basic conditions on $\mathbb{R}^n$ *and is locally Lipschitz on* $\mathbb{R}^n \backslash \mathcal{A}$. *Then there exists a locally Lipschitz control-Lyapunov function with* $\mathcal{U} \cap \sigma$ *controls for* (1.2).

Furthermore, if $F(\cdot)$ *is locally Lipschitz on* $\mathbb{R}^n \backslash \mathcal{A}$, *uniformly in distance to the set* $\mathcal{A}$, *then there exists a locally Lipschitz control-Lyapunov function with* $\mathcal{U} \cap \sigma$ *controls which is locally Lipschitz, uniformly in distance to the set* $\mathcal{A}$.

Remark 4. Two examples of regularity conditions on $f(\cdot, \cdot)$ which would give rise to a locally Lipschitz $F(\cdot)$ are as follows.

1. Let $c > 0$ be constant (possibly $+\infty$) and $\sigma(\cdot) \equiv c$. Furthermore, let $f(x, \cdot)$ be measurable for each $x \in \mathbb{R}^n$ and $f(\cdot, u)$ be locally Lipschitz uniformly in $u \in \mathcal{U} \cap c\overline{\mathcal{B}}_\mathcal{U}$. Then $F(\cdot)$ is locally Lipschitz.
2. Consider $\mathcal{U} = \mathbb{R}^m$, and let $\sigma(\cdot)$ be locally Lipschitz (and nondecreasing) and $f(\cdot, \cdot)$ be locally Lipschitz. Then $F(\cdot)$ is locally Lipschitz.

Note that we have not assumed that the set-valued map $f\left(x, \mathcal{U} \cap \sigma(|x|_\mathcal{A})\overline{\mathcal{B}}_\mathcal{U}\right)$ is convex. However, if $f\left(x, \mathcal{U} \cap \sigma(|x|_\mathcal{A})\overline{\mathcal{B}}_\mathcal{U}\right)$ is Lipschitz, then $\overline{\text{co}} f\left(x, \mathcal{U} \cap \sigma(|x|_\mathcal{A})\overline{\mathcal{B}}_\mathcal{U}\right)$ is also Lipschitz (see [3, section 1.1, Proposition 6]).

These examples extend easily to the case of generating a set-valued map $F(\cdot)$ which is locally Lipschitz, uniformly in distance to the set $\mathcal{A}$. This is done by requiring the corresponding Lipschitz property on f to be uniform in distance to the set $\mathcal{A}$. □

The proof follows by noting that the set $\mathcal{A}$ will be weakly UGAS for the set-valued map $F(\cdot)$, allowing the application of Theorem 2.1. See [15] and [13] for details.

4. Control construction. By making use of the control-Lyapunov function of Theorem 3.2, we can construct a (discontinuous) time-invariant feedback stabilizer that, when implemented with a sample-and-hold strategy, guarantees semiglobal practical asymptotic stability of the set $\mathcal{A}$ and robustness to small additive disturbances and measurement noise. By sample and hold we mean that the system state is "sampled," a control action is computed, and then it is implemented (or "held") for a fixed holding period. The procedure is then repeated.

Results of this type were presented by Clarke et al. [6] and Sontag [28] for the case where $\mathcal{A}$ is compact, and a construction is given by Clarke et al. [5] that applies to the case of noncompact sets $\mathcal{A}$. Our construction resembles the constructions used in both of these references. In comparison to the construction in [5], we use proximal aiming to a point that minimizes the control-Lyapunov function in a ball around the current point rather than proximal aiming to a sublevel set of the control-Lyapunov function. This permits a very concise statement of the control synthesis algorithm.

The following assumptions, under which we construct our feedback law, all follow directly from Theorem 3.2. However, these assumptions are somewhat weaker as they simplify the exposition. Specifically, as we use a sample-and-hold strategy to implement our feedback control, we are concerned with only the semiglobal practical qualities of the control-Lyapunov function.

4.1. Assumptions. For $V : \mathbb{R}^n \to \mathbb{R}$ and $\ell_1, \ell_2 \in \{-\infty\} \bigcup \mathbb{R}$ such that $\ell_1 < \ell_2$, we define $\mathcal{V}(\ell_1, \ell_2) := \{x \in \mathbb{R}^n : \ell_1 \leq V(x) \leq \ell_2\}$. We denote $\mathcal{V}(-\infty, \ell_2)$ by $\mathcal{V}(\ell_2)$. Suppose $\sigma(\cdot)$ is nondecreasing and we are given $\ell_1 < \ell_2$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$, $\varepsilon_4 > 0$, $c > 0$, $L_V > 0$, $L_f > 0$, $\widetilde{M} > 0$ such that

1. for $x_1, x_2 \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2) + \varepsilon_3 \overline{\mathcal{B}}_n$ and $u \in \mathcal{U} \cap \sigma(\max\{|x_1|_{\mathcal{A}}, |x_2|_{\mathcal{A}}\} + \varepsilon_4)\overline{\mathcal{B}}_{\mathcal{U}}$
   (a) $|V(x_1) - V(x_2)| \leq L_V |x_1 - x_2|$,
   (b) $|f(x_1, u) - f(x_2, u)| \leq L_f |x_1 - x_2|$,
   (c) $\min_{w \in \overline{\text{co}} f(x, \mathcal{U} \cap \sigma(|x|_{\mathcal{A}})\overline{\mathcal{B}}_{\mathcal{U}})} DV(x; w) \leq -2c$;
2. $f(\cdot, u)$ is continuous and bounded in norm by $\widetilde{M}$ on $\mathcal{V}(\ell_2 + \varepsilon_2) + \varepsilon_3 \overline{\mathcal{B}}_n$ for all $u \in \mathcal{U} \cap \sigma(|\cdot|_{\mathcal{A}} + \varepsilon_4)\overline{\mathcal{B}}_{\mathcal{U}}$.

Note again that, with appropriate values for the constants, these assumptions are all satisfied by the control-Lyapunov function of Theorem 3.2.

4.2. Control design. For the control system

$$\dot{x} = f(x, u) + d, \quad u \in \mathcal{U} \cap \sigma(|x|_{\mathcal{A}} + \varepsilon_4)\overline{\mathcal{B}}_{\mathcal{U}}, \tag{4.1}$$

we define a (discontinuous) control law as follows.

1. Let $r \in (0, \min\{\frac{\varepsilon_2}{L_V}, \varepsilon_3, \varepsilon_4, \frac{c}{L_f L_V}\}]$.
2. For each $x \in \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$,
   (a) let $s \in \overline{\mathcal{B}}_n(x, r)$ be such that $V(s) \leq V(\xi)$ for all $\xi \in \overline{\mathcal{B}}_n(x, r)$;
   (b) let $\alpha \in \mathcal{U} \cap \sigma(|x|_{\mathcal{A}} + r)\overline{\mathcal{B}}_{\mathcal{U}}$ be such that $\langle x - s, f(x, \alpha) \rangle \leq -\frac{c}{L_V}|x - s|$.
3. For any $x \notin \mathcal{V}(\ell_1, \ell_2 + \varepsilon_2)$ let $\alpha \in \mathcal{U} \cap \sigma(|x|_{\mathcal{A}})\overline{\mathcal{B}}_{\mathcal{U}}$ be arbitrary.
4. Take $u = \alpha(x)$.

4.3. Closed-loop results. We let $T_1 > 0$ be such that $\frac{cT_1}{16L_V} \leq r - \sqrt{r^2 - \frac{rcT_1}{4L_V}}$. Such a value exists since the derivative with respect to T_1 of the function on the right-hand side evaluated at $T_1 = 0$ is equal to $\frac{c}{8L_V}$. We define $M := \widetilde{M} + \frac{c}{2L_V}$, $a_1 := M^2 L_f$, $a_2 := M(M + rL_f)$, $a_3 := \frac{cr}{4L_V}$, and

$$T^* := \min\left\{T_1,\ \frac{\ell_2 - \ell_1}{L_V M},\ \frac{\varepsilon_3}{M},\ \frac{\sqrt{a_2^2 + 4a_1 a_3} - a_2}{2a_1}\right\}.$$

We note that $T^* > 0$ and $T^* \to 0$ as $r \to 0$. This is evident from the last term that defines T^*.

THEOREM 4.1. *Suppose $u = \alpha(x)$ is implemented by sampling and holding with holding period $T \in (0, T^*]$. Then for every $x_0 \in \mathcal{V}(\ell_2)$, for all $d(\cdot)$ such that $||d||_\infty \leq \frac{c}{2L_V}$, and for all $t \geq 0$, the resulting solutions satisfy*

$$V(x(t)) \leq \max\left\{V(x_0) - \frac{c^2 \max\{t - T, 0\}}{8L_V M}, \ell_1 + L_V M T\right\} + L_V r.$$

An outline of the proof of this theorem may be found in [15] with full details to be found in [13].

4.4. Robustness to measurement noise. In this section we will demonstrate that our control design is robust with respect to small measurement errors. That is, if we implement our control using a corrupted measurement $x + n$ rather than with the true state x, the trajectory of the controlled system will still approach the set $\mathcal{A}$. Similar results are established by Sontag [28] and Clarke et al. [5]. The important observation here is that, while the measurement noise may be persistent, nondifferentiable, and unknown, since we implement the control via a sample-and-hold procedure, it is only the noise values at the sampling instants that are important.

Consider the system

$$\dot{x} = f(x, \alpha(x_i + n_i)),$$

where n_i represents samples of a bounded noise function $n : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. We construct a fake noise function $n_L(\cdot)$ that is globally Lipschitz and matches $n(\cdot)$ at sampling instances. If N is a bound for $|n(\cdot)|$ and T is the sampling period, then $n_L(\cdot)$ can be constructed so that it is bounded by N and its Lipschitz constant is $2N/T$. Note that such an approximation always exists. For instance, taking a linear interpolation between each noise sample yields such a function. Also note that, in what follows, precise knowledge of this signal is unnecessary. That is, we have not assumed that we know this function, only that it is bounded and we know its Lipschitz constant. We perform a coordinate change, $z = x + n_L$, in order to write

$$\dot{z} = f(z - n_L, \alpha(z_i)) + \dot{n}_L \ .$$

We rewrite the system as

$$\dot{z} = f(z - n_L, \alpha(z_i)) + f(z, \alpha(z_i)) - f(z, \alpha(z_i)) + \dot{n}_L = f(z, \alpha(z_i)) + d \ ,$$

where we have defined $d := f(z - n_L, \alpha(z_i)) - f(z, \alpha(z_i)) + \dot{n}_L$. Utilizing the Lipschitz constant for $f(\cdot, u)$ and the bound on $|\dot{n}_L|$ we have that $|d| \leq N(L_f + 2/T)$. Therefore the result of Theorem 4.1 applies if we insist that $N \leq \frac{Tc}{2L_V(2+L_fT)}$. That is, selecting the noise bound N appropriately yields $|d| \leq c/2L_V$, which allows us to appeal to the result of the theorem.

Remark 5. It is well known that fast sampling is advantageous for stability properties. Specifically, fast sampling is needed for large states to guarantee stability and is desirable for small states to decrease the size of the "practical stability" region. However, as shown by the bound required on the noise, as the sampling time T becomes small, the allowable noise also becomes small. Therefore, one wants to sample fast but not too fast. This observation was also made by Sontag [28, Theorem 1]. However, the observation is worth repeating as the result is made transparent via the above coordinate change. □

5. Proof of weak converse result. We turn now to the proof of Theorem 2.1. We prove the result without the assumption that $F(\cdot)$ is locally Lipschitz, uniform in distance to the set $\mathcal{A}$. It is easy to see that the Lyapunov function then inherits this property.

5.1. Technical preliminaries. In what follows we will appeal to the following lemma to demonstrate the local Lipschitz property of our Lyapunov function by demonstrating an appropriate bound on the Dini subderivate (see [9, Corollary 3.7]).

LEMMA 5.1. *Let the function $V : \mathcal{O} \to (-\infty, \infty]$ be lower semicontinuous. Let $\mathcal{U} \subset \mathcal{O}$ be open and convex. The function $V(\cdot)$ is Lipschitz with Lipschitz constant M on $\mathcal{U}$ if and only if $DV(x; v) \leq M|v|$, for all $x \in \mathcal{U}$, and all $v \in \mathbb{R}^n$.*

The following lemma is [26, Corollary 10].

LEMMA 5.2. *For each $\omega_0 \in \mathcal{K}_\infty$ there exist $\omega_1, \omega_2 \in \mathcal{K}_\infty$ such that $\omega_0(rs) \leq \omega_1(r)\omega_2(s)$ for all $r, s \geq 0$.*

Next, we state some lemmas that are proved in section 6. We start with a slight refinement of Sontag's lemma on $\mathcal{KL}$-estimates wherein we specify the required regularity property of one of the $\mathcal{K}_\infty$ functions [26, Proposition 7].

LEMMA 5.3. *For each $\beta \in \mathcal{KL}$ and $\lambda > 0$, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(\cdot)$ is Lipschitz on its domain, continuously differentiable (C^1) on $(0, \infty)$, $\alpha_1(s) \leq s\alpha_1'(s)$ for all $s > 0$, and $\alpha_1(\beta(s, t)) \leq \alpha_2(s)e^{-\lambda t}$ for all $(s, t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.*

The next lemma comes from [19, Lemmas 11 and 12].

LEMMA 5.4. *For each continuous, positive definite function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ there exists $\rho \in \mathcal{K}_\infty$ such that $\rho(\cdot)$ is locally Lipschitz on its domain, continuously differentiable on $(0, \infty)$, and $\rho(s) \leq \alpha(s)\rho'(s)$ for all $s > 0$.*

LEMMA 5.5. *For each $\omega_2 \in \mathcal{K}_\infty$ there exist a locally Lipschitz, strictly increasing, unbounded function $\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ and a continuous, nonincreasing function $\vartheta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ such that, with $s_0 := \omega_2^{-1}(0.5)$,*

$$\kappa(0) = 1, \tag{5.1}$$

$$\kappa(t)\kappa(T) \geq \kappa(t + T) \qquad \forall t, T \geq 0\ , \tag{5.2}$$

$$\frac{\kappa(t)}{\kappa(T)} \leq e^{2(t-T)} \qquad \forall t \geq T \geq 0\ , \tag{5.3}$$

$$\kappa(t) \leq \min\left\{e^t, \frac{1}{\omega_2(s_0 e^{-t})}\right\}\ \forall t \geq 0, \tag{5.4}$$

$$\max_{s\in[0,T]} \frac{\kappa(s)}{\kappa(s + t)} \leq 1 - \vartheta(T)t \quad \forall t \in [0, 1]\ . \tag{5.5}$$

The next fact follows [18, Lemma 4.3].

LEMMA 5.6. *Let $\mathcal{A} \subset \mathbb{R}^n$ be a closed set. Suppose $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, continuous on $\mathbb{R}^n$, $V(x) = 0$ for all $x \in \mathcal{A}$, and $V(x) > 0$ for all $x \in \mathbb{R}^n \backslash \mathcal{A}$. Then there exists a function $\rho \in \mathcal{K}_\infty$ that is C^1 on $(0, \infty)$, satisfies $\rho(s) \leq s\rho'(s)$ for all $s > 0$, and is such that $V_L := \rho \circ V$ is locally Lipschitz on $\mathbb{R}^n$ and $DV_L(x; v) = 0$ for all $x \in \mathcal{A}$ and all $v \in \mathbb{R}^n$.*

The next lemma is used twice in the proof of our main result.

LEMMA 5.7. *Suppose we are given the following:*

1. *A function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ defined as*

$$V(x) := \inf_{\phi \in \mathcal{S}[0,\infty)(x)} \sup_{t \geq 0} g(\phi(t, x))\kappa(t),$$

where $g(\cdot)$ is locally Lipschitz on $\mathbb{R}^n$ and $\kappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ is locally Lipschitz, strictly increasing, unbounded, $\kappa(t)\kappa(T) \geq \kappa(t + T)$ for all $t, T \geq 0$, and $\kappa(0) = 1$.

2. *$\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|x|_\mathcal{A}) \leq g(x)$, $V(x) \leq \alpha_2(|x|_\mathcal{A})$.*

3. *A sequence $\{\ell_j\}_{j=1}^\infty$, where $\ell_j \to 0$ as $j \to \infty$ such that $|x|_\mathcal{A} = \ell_j$, implies $V(x) = g(x)$. Then V is continuous on $\mathbb{R}^n$, locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, and there exists a function $T : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is continuous on $\mathbb{R}_{>0}$ and, for each $x \in \mathbb{R}^n$,*

there exists $\phi^* \in \mathcal{S}[0,\infty)(x)$ *such that*

$$V(x) \;=\; \sup_{t\geq 0} g(\phi^*(t,x))\kappa(t) \;=\; \max_{t\in[0,T(|x|_{\mathcal{A}})]} g(\phi^*(t,x))\kappa(t)\ .$$

5.2. Construction and Lipschitz property of $V(x)$. The following proposition is the key to constructing our Lyapunov function and is proved in section 5.5.

Proposition 5.8. *Under the assumptions of Theorem* 2.1*, there exists a family of locally Lipschitz functions* $W_i : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$*,* $i \in \mathbb{Z}_{>0}$*, that have the following properties.*

1. *There exist* $\tilde{\alpha}_1, \alpha \in \mathcal{K}_\infty$ *such that, for each* $x \in \mathbb{R}^n$*,* $i \in \mathbb{Z}_{>0}$*,*

$$\tilde{\alpha}_1(|x|_{\mathcal{A}}) \leq W_i(x) \leq \alpha(|x|_{\mathcal{A}}). \tag{5.6}$$

2. *There exists* $\tilde{\alpha}_2 \in \mathcal{K}_\infty$ *such that, for every* $x \in \mathbb{R}^n$*, there exists* $\hat{\phi} \in \mathcal{S}[0,\infty)(x)$ *such that, for all* $i \in \mathbb{Z}_{>0}$*,*

$$W_i(\hat{\phi}(t,x)) \leq \tilde{\alpha}_2(|x|_{\mathcal{A}})e^{-2t}\ \forall t \geq 0. \tag{5.7}$$

3. *For every* $i \in \mathbb{Z}_{>0}$ *and* $|x|_{\mathcal{A}} \geq \frac{1}{i}$*, there exists* $\phi_i \in \mathcal{S}[0,\infty)(x)$ *such that the set* $\left\{t : |\phi_i(t,x)|_{\mathcal{A}} = \frac{1}{i}\right\}$ *is nonempty and, with* $T := \inf\left\{t : |\phi_i(t,x)|_{\mathcal{A}} = \frac{1}{i}\right\}$*,*

$$W_i(\phi_i(t,x)) \leq W_i(x)e^{-2t} \qquad \forall t \in [0,T]\ . \tag{5.8}$$

Remark 6. Note that the functions $W_i(\cdot)$ are locally Lipschitz weak Lyapunov functions on sets defined by $|x|_{\mathcal{A}} \geq \frac{1}{i}$. This follows from items 1 and 3 above. □

Let $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ come from Proposition 5.8, and let $\omega_1, \omega_2 \in \mathcal{K}_\infty$ come from Lemma 5.2 satisfying

$$\tilde{\alpha}_2 \circ \tilde{\alpha}_1^{-1}(rs) \leq \omega_1(r)\omega_2(s) \quad \forall r,s \geq 0\ . \tag{5.9}$$

Define $s_0 := \omega_2^{-1}(0.5)$, and let $\kappa(\cdot)$ and $\vartheta(\cdot)$ come from Lemma 5.5 satisfying (5.1)–(5.5). Let $\widetilde{\omega} \in \mathcal{K}_\infty$ satisfy

$$\widetilde{\omega}(s) \leq s \qquad\qquad \forall s \geq 0 \text{ and} \tag{5.10}$$

$$\widetilde{\omega}(s) \leq s_0^2 \frac{\left(\omega_1^{-1}(s)\right)^2}{s} \quad \forall s \in (0,1]\ . \tag{5.11}$$

Let $\alpha \in \mathcal{K}_\infty$ also come from Proposition 5.8, and define

$$\alpha_f := \tilde{\alpha}_2^{-1} \circ \widetilde{\omega} \circ \tilde{\alpha}_1 \circ \alpha^{-1} \circ \tilde{\alpha}_1\ . \tag{5.12}$$

The function $\alpha_f(\cdot)$ belongs to class-$\mathcal{K}_\infty$. Also

$$\alpha_f(s) \leq s \qquad \forall s \geq 0 \tag{5.13}$$

since, from (5.6), $\alpha^{-1} \circ \tilde{\alpha}_1(s) \leq s$, and from (5.6), (5.7) with $t = 0$, and (5.10), $\tilde{\alpha}_2^{-1} \circ \widetilde{\omega} \circ \tilde{\alpha}_1(s) \leq s$. Now choose a function $q : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{>0}$ satisfying

$$q(0) = 1\ , \quad q(j+1) \geq \frac{1}{\alpha_f(\frac{1}{q(j)+1})}\ ,\ j \in \mathbb{Z}_{\geq 0}\ . \tag{5.14}$$

It follows from (5.13) that $\frac{1}{q(j)+1} \geq \frac{1}{q(j+1)}$; i.e., $q(j)+1 \leq q(j+1)$. Therefore, the sequence of integers $q(j)$ is strictly increasing with j. Let the locally Lipschitz functions $\lambda_i : \mathbb{R}^n \to [0,1]$ $(i \in \mathbb{Z}_{>0})$ be such that

$$\frac{1}{q(j+1)} \leq |x|_{\mathcal{A}} \leq \frac{1}{q(j)+1} \Longrightarrow \lambda_{q(j+1)}(x) = 1 \ , \tag{5.15}$$

and for each $x \in \mathbb{R}^n$ there exists a finite index set $\mathcal{I}_x$ such that $\sum_i \lambda_i(x) = \sum_{i \in \mathcal{I}_x} \lambda_i(x)$ $= 1$. Let the functions $W_i(\cdot)$ come from Proposition 5.8, and define

$$f(x) = \sum_i \lambda_i(x) W_i(x) \ . \tag{5.16}$$

Note that, on the strips defined by (5.15), this corresponds to $f(x) = W_{q(j+1)}(x)$ and a convex combination of the $W_i(\cdot)$ functions between these strips. It follows from the properties of $W_i(\cdot)$, given in Proposition 5.8, and $\lambda_i(\cdot)$ that $f(\cdot)$ is locally Lipschitz,

$$\tilde{\alpha}_1(|x|_{\mathcal{A}}) \leq f(x) \leq \alpha(|x|_{\mathcal{A}}) \quad \forall x \in \mathbb{R}^n, \tag{5.17}$$

and, for each $x \in \mathbb{R}^n$, there exists $\hat{\phi} \in \mathcal{S}[0,\infty)(x)$ such that

$$f(\hat{\phi}(t,x)) \leq \tilde{\alpha}_2(|x|_{\mathcal{A}}) e^{-2t} \quad \forall t \geq 0 \ . \tag{5.18}$$

Next define

$$V_1(x) := \inf_{\phi \in \mathcal{S}[0,\infty)(x)} \sup_{t \geq 0} f(\phi(t,x))\kappa(t) \tag{5.19}$$

and note that, using (5.1) and (5.17), we have

$$V_1(x) \geq f(x)\kappa(0) \geq \tilde{\alpha}_1(|x|_{\mathcal{A}}) \tag{5.20}$$

and, using (5.4) and (5.18), we have

$$V_1(x) \leq \sup_{t \geq 0} f(\hat{\phi}(t,x))\kappa(t) \leq \tilde{\alpha}_2(|x|_{\mathcal{A}}) \ . \tag{5.21}$$

The following claim is proved at the end of this section.

Claim 1. There exists a decreasing sequence $\{\ell_j\}_{j=1}^{\infty}$ such that $\ell_j \to 0$ as $j \to \infty$ and $|x|_{\mathcal{A}} = \ell_j$ implies $V_1(x) = f(x)$.

Since (5.1) and (5.2) hold, we can apply Lemma 5.7 with $V(x) := V_1(x)$, $g(x) :=$ $f(x)$, $\alpha_1(s) := \tilde{\alpha}_1(s)$, $\alpha_2(s) := \tilde{\alpha}_2(s)$, and the sequence $\{\ell_j\}_{j=1}^{\infty}$ constructed in Claim 1. So $V_1(\cdot)$ is continuous on $\mathbb{R}^n$, locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, and there exists a function $T : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ that is continuous on $\mathbb{R}_{>0}$ and for each $x \in \mathbb{R}^n$ there exists $\phi^* \in$ $\mathcal{S}[0,\infty)(x)$ such that

$$V_1(x) = \max_{t \in [0, T(|x|_{\mathcal{A}})]} f(\phi^*(t,x))\kappa(t) \ . \tag{5.22}$$

We define $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $\alpha_3(0) = 0$ and

$$\alpha_3(s) := \min_{r \in [\tilde{\alpha}_2^{-1}(s), \tilde{\alpha}_1^{-1}(s)]} \vartheta(T(r)+1)s \ . \tag{5.23}$$

Since $\vartheta(\cdot)$ is continuous and positive, $T(\cdot)$ is continuous, and $\tilde{\alpha}_1^{-1}(\cdot)$ and $\tilde{\alpha}_2^{-1}(\cdot)$ are continuous, $\alpha_3(\cdot)$ is continuous. It is also positive definite. We apply Lemma 5.4 with $\alpha_3(\cdot)$ to get $\rho_1 \in \mathcal{K}_\infty$, locally Lipschitz on its domain, and C^1 on $(0,\infty)$ such that $\rho_1(s) \le \alpha_3(s)\rho_1'(s)$ for all $s > 0$. We define $V_2 := \rho_1 \circ V_1$, and we note that $V_2(\cdot)$ is continuous on $\mathbb{R}^n$, locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, and from (5.20) and (5.21) we have

$$\rho_1 \circ \tilde{\alpha}_1(|x|_{\mathcal{A}}) \le V_2(x) \le \rho_1 \circ \tilde{\alpha}_2(|x|_{\mathcal{A}}) \ . \tag{5.24}$$

It follows that we can apply Lemma 5.6 with $V_2(\cdot)$ to get $\rho_2 \in \mathcal{K}_\infty$ that is C^1 on $(0,\infty)$ such that $\rho(s) \le s\rho'(s)$ for all $s > 0$ and $V := \rho_2 \circ V_2$ is locally Lipschitz on $\mathbb{R}^n$ and satisfies $DV(x;v) = 0$ for all $x \in \mathcal{A}$ and all $v \in \mathbb{R}^n$. Moreover, it follows from (5.24) that (2.1) holds with the class-$\mathcal{K}_\infty$ functions $\alpha_1 := \rho_2 \circ \rho_1 \circ \tilde{\alpha}_1$ and $\alpha_2 := \rho_2 \circ \rho_1 \circ \tilde{\alpha}_2$.

5.3. Infinitesimal decrease. The infinitesimal decrease condition holds for each $x \in \mathcal{A}$ since $DV(x;v) = 0 = -V(x)$, for any $x \in \mathcal{A}$, and $v \in \mathbb{R}^n$. Let $x \in \mathbb{R}^n \backslash \mathcal{A}$, and let $\phi^* \in \mathcal{S}[0,\infty)(x)$ satisfy (5.22). Define $T := T(|x|_{\mathcal{A}})$, and let $t^* \in (0,1]$ be such that $T(|\phi^*(t,x)|_{\mathcal{A}}) \le T+1$ for all $t \in [0,t^*]$. With (5.22), for each $t \in [0,t^*]$ let $\psi^* \in \mathcal{S}[0,\infty)(\phi^*(t,x))$ satisfy

$$V_1(\phi^*(t,x)) = \max_{s \in [0,T(|\phi^*(t,x)|_{\mathcal{A}})]} f(\psi^*(s,\phi^*(t,x)))\kappa(s) \ .$$

Then, for each $t \in [0,t^*]$, we have

$$\begin{aligned} V_1(\phi^*(t,x)) &= \max_{s \in [0,T(|\phi^*(t,x)|_{\mathcal{A}})]} f(\psi^*(s,\phi^*(t,x)))\kappa(s) \le \max_{s \in [0,T+1]} f(\phi^*(s+t,x))\kappa(s) \\ &= \max_{s \in [0,T+1]} f(\phi^*(s+t,x))\kappa(s+t)\frac{\kappa(s)}{\kappa(s+t)} \\ &\le \sup_{\tau \ge 0} f(\phi^*(\tau,x))\kappa(\tau)\left[1-\vartheta(T+1)t\right] = V_1(x)\left[1-\vartheta(T+1)t\right]. \end{aligned} \tag{5.25}$$

For almost all s, let $g(s)$ be the unique closest point in $F(x)$ to $\overbrace{\phi^*(s,x)}^{\cdot}$. Such a unique closest point exists since $F(x)$ is convex (see [11, section 5, Lemma 2]). Since $F(\cdot)$ is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, $x \in \mathbb{R}^n \backslash \mathcal{A}$, $\phi^*(\cdot,x)$ is (absolutely) continuous, $\overbrace{\phi^*(s,x)}^{\cdot} \in F(\phi^*(s,x))$ for almost all s, and $F(\cdot)$ is locally bounded, there exist $\bar{s}, M, L > 0$ such that, for almost all $s \in [0,\bar{s}]$,

$$|g(s) - \overbrace{\phi^*(s,x)}^{\cdot}| \le L|x - \phi^*(s,x)| \le LMs \ . \tag{5.26}$$

We can also write

$$\frac{\phi^*(t,x)-x}{t} = t^{-1}\int_0^t \overbrace{\phi^*(s,x)}^{\cdot} ds = t^{-1}\int_0^t g(s)ds + t^{-1}\int_0^t [\overbrace{\phi^*(s,x)}^{\cdot} - g(s)]ds \ . \tag{5.27}$$

Then note that the first term on the right-hand side belongs to $F(x)$ for all t since $F(x)$ is convex (see [11, section 5, Lemma 12]) and the second term on the right-hand side converges to zero as t converges to zero (using (5.26)).

Call the first term on the last line of (5.27) $v(t)$ and the sum $w(t)$. Since $F(x)$ is compact, there exists an accumulation point $v \in F(x)$ for $v(t)$, i.e., a sequence of times t_j converging to zero so that $v(t_j) \to v$. So, we can write $\phi^*(t_j, x) = x + t_j w(t_j)$, where $w(t_j) \to v$ as $t_j \to 0$. Then, using (5.23) and (5.25),

$$\begin{aligned} DV_1(x; v) &\leq \liminf_{j\to\infty} [V_1(x + t_j w(t_j)) - V_1(x)]/t_j = \liminf_{j\to\infty} [V_1(\phi^*(t_j, x)) - V_1(x)]/t_j \\ &\leq -V_1(x)\vartheta(T(|x|_{\mathcal{A}}) + 1) \leq -\alpha_3(V_1(x)). \end{aligned}$$

Since $V_2 = \rho_1 \circ V_1$ and $\rho_1(s) \leq \alpha_3(s)\rho_1'(s)$ for all $s > 0$, we have, for all $x \in \mathbb{R}^n \backslash \mathcal{A}$,

$$\min_{w \in F(x)} DV_2(x; w) \leq -\rho_1'(V_1(x))\alpha_3(V_1(x)) \leq -\rho_1(V_1(x)) = -V_2(x).$$

Similarly, since $V = \rho_2 \circ V_2$ and $\rho_2(s) \leq s\rho_2'(s)$ for all $s > 0$, we have, for all $x \in \mathbb{R}^n \backslash \mathcal{A}$,

$$\min_{w \in F(x)} DV(x; w) \leq -\rho_2'(V_2(x))V_2(x) \leq -\rho_2(V_2(x)) = -V(x).$$

5.4. Proof of Claim 1. For each $j \in \mathbb{Z}_{\geq 0}$ define

$$\ell_j := \tilde{\alpha}_1^{-1} \circ \widetilde{\omega}^{-1} \circ \tilde{\alpha}_2 \left(\frac{1}{q(j+1)} \right) . \tag{5.28}$$

Since $q(j)$ is strictly increasing and unbounded, it follows that ℓ_j decreases monotonically to zero. We now restrict our attention to integers j sufficiently large so that $\alpha(\ell_j) \leq 1$. We consider $x \in \mathbb{R}^n \backslash \mathcal{A}$ such that $|x|_{\mathcal{A}} = \ell_j$, and we will show that $V_1(x) = f(x)$. Clearly, $V_1(x) \geq f(x)$. It remains to show that $V_1(x) \leq f(x)$. First, it follows from (5.17) that $0 < f(x) \leq 1$. Second, it follows from (5.17), (5.18) with $t = 0$, and (5.28) that

$$|z|_{\mathcal{A}} = \frac{1}{q(j+1)} \quad \Longrightarrow \quad f(z) \leq \widetilde{\omega}(f(x)) . \tag{5.29}$$

Third, it follows from (5.11) and $0 < f(x) \leq 1$ that

$$\widetilde{\omega}(f(x)) \leq s_0^2 \frac{\omega_1^{-1}(f(x))^2}{f(x)} . \tag{5.30}$$

Fourth, since $\tilde{\alpha}_1^{-1} \circ \widetilde{\omega}^{-1} \circ \tilde{\alpha}_2(s) \geq s$ for all $s \geq 0$, it follows from (5.28) that $|x|_{\mathcal{A}} \geq \frac{1}{q(j+1)}$. From property 3 of Proposition 5.8, there exists a trajectory $\phi_{q(j+1)} \in \mathcal{S}[0, \infty)(x)$ such that the set $\{t : |\phi_{q(j+1)}(t, x)|_{\mathcal{A}} = \frac{1}{q(j+1)}\}$ is nonempty and for all $t \in [0, T]$, where $T := \inf\{t : |\phi_{q(j+1)}(t, x)|_{\mathcal{A}} = \frac{1}{q(j+1)}\}$, we have

$$W_{q(j+1)}(\phi_{q(j+1)}(t, x)) \leq W_{q(j+1)}(x)e^{-2t}. \tag{5.31}$$

It follows from the definition of T that, for all $t \in [0, T]$,

$$|\phi_{q(j+1)}(t, x))|_{\mathcal{A}} \geq \frac{1}{q(j+1)} . \tag{5.32}$$

With (5.6), $|x|_{\mathcal{A}} = \ell_j$, (5.28), (5.12), and (5.14), one can show that (5.31) implies that, for all $t \in [0, T]$, $|\phi_{q(j+1)}(t, x)|_{\mathcal{A}} \leq \frac{1}{q(j)+1}$. Combining this with (5.32), (5.31), (5.15), and (5.16), we have

$$f(\phi_{q(j+1)}(t, x)) \leq f(x)e^{-2t} \quad \forall t \in [0, T] \tag{5.33}$$

and thus, using (5.4), we have for all $t \in [0, T]$,

$$f(\phi_{q(j+1)}(t,x))\kappa(t) \le f(x)e^{-2t}\kappa(t) \le f(x)\ . \tag{5.34}$$

Now define $z := \phi_{q(j+1)}(T, x)$ and note that

$$|z|_{\mathcal{A}} = \frac{1}{q(j+1)}\ . \tag{5.35}$$

Let $\tilde{\phi} \in \mathcal{S}[0,\infty)(z)$ satisfy (see (5.18))

$$f(\tilde{\phi}(t,z)) \le \tilde{\alpha}_2(|z|_{\mathcal{A}})e^{-2t} \qquad \forall t \ge 0\ . \tag{5.36}$$

Define $\tilde{\psi} \in \mathcal{S}[0,\infty)(x)$ as

$$\tilde{\psi}(t,x) := \begin{cases} \phi_{q(j+1)}(t,x), & t \in [0,T], \\ \tilde{\phi}(t-T,z), & t \ge T. \end{cases}$$

Now, using (5.36), (5.3), (5.17), (5.33), (5.9), (5.4), (5.35), (5.29), and (5.30) one may show that, for $t \ge T$, $f(\tilde{\phi}(t-T,z))\kappa(t) \le f(x)$. Combining this with (5.34), we have $f(\tilde{\psi}(t,x))\kappa(t) \le f(x)$ for all $t \ge 0$, which implies $V_1(x) \le f(x)$. □

5.5. Proof of Proposition 5.8. Given $\dot{x} \in F(x)$, where $F(\cdot)$ satisfies the basic conditions and is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$, let M_1 come from Assumption 1. We define a modified inclusion

$$\dot{x} \in F_i(x) := F(x) + \gamma_i(|x|_{\mathcal{A}})\,[M_1+1]\,\overline{\mathcal{B}}_n, \tag{5.37}$$

where $\gamma_i : [0,\infty) \to [0,1]$ is locally Lipschitz and $\gamma(s) = 0$ for $s \ge \frac{1}{i}$ and $\gamma(s) = 1$ for $s \le \frac{1}{i+1}$. It follows from the properties of $F(\cdot)$ and $\gamma_i(\cdot)$ that $F_i(\cdot)$ satisfies the basic conditions on $\mathbb{R}^n$ and is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$.

Note that $F(x) \subseteq F_i(x)$ for all $x \in \mathbb{R}^n$ by taking the origin in $\overline{\mathcal{B}}_n$. Furthermore, note that for all

$$|x|_{\mathcal{A}} \le \frac{1}{i+1} \quad \Longrightarrow \quad \overline{\mathcal{B}}_n \subseteq F_i(x). \tag{5.38}$$

To see this, let $x_1 \in \overline{\mathcal{B}}_n$ and $x_2 \in F(x)$, where $|x|_{\mathcal{A}} \le \frac{1}{i+1}$. Therefore, $-x_2 + x_1 \in \left[\max_{z \in F(x)} |z| + 1\right] \overline{\mathcal{B}}_n \subseteq [M_1+1]\,\overline{\mathcal{B}}_n$, which implies that $x_1 = x_2 + (-x_2 + x_1) \in F(x) + [M_1+1]\,\overline{\mathcal{B}}_n = F_i(x)$. We will denote the set of maximal solutions of (5.37) starting at the point x as $\mathcal{S}_i(x)$.

Since $F(x) \subseteq F_i(x)$, the assumption of weak-UGAS of $\mathcal{A}$ for $\dot{x} \in F(x)$ gives $\hat{\phi} \in \mathcal{S}_i[0,\infty)(x)$ such that

$$|\hat{\phi}(t,x)|_{\mathcal{A}} \le \beta(|x|_{\mathcal{A}}, t)\ \forall t \ge 0\ . \tag{5.39}$$

Let $\lambda = 4$, and then let $\widehat{\alpha}_1 \in \mathcal{K}_\infty$ and $\widehat{\alpha} \in \mathcal{K}_\infty$ come from Lemma 5.3 so that $\widehat{\alpha}_1(\cdot)$ is Lipschitz on its domain, C^1 on $(0,\infty)$, $\widehat{\alpha}_1(s) \le s\widehat{\alpha}_1'(s)$ for all $s \in (0,\infty)$, and

$$\widehat{\alpha}_1(\beta(s,t)) \le \widehat{\alpha}(s)e^{-4t} \quad \forall s \ge 0,\ t \ge 0\ . \tag{5.40}$$

We make the following definitions:

- $\widetilde{W}_i(x) := \inf_{\phi\in\mathcal{S}_i[0,\infty)(x)} \sup_{t\geq 0} \widehat{\alpha}_1(|\phi(t,x)|_{\mathcal{A}})e^{2t}$;
- $\beta_2(s,t) := \widehat{\alpha}\circ\widehat{\alpha}_1^{-1}\left(\widehat{\alpha}(s)e^{-4t}\right)$ (note that $\beta_2\in\mathcal{KL}$);
- applying Lemma 5.3 with $\beta_2\in\mathcal{KL}$ and $\lambda=2$, we get $\alpha_m\in\mathcal{K}_\infty$ and $\tilde{\alpha}_2\in\mathcal{K}_\infty$ such that $\alpha_m(\cdot)$ is Lipschitz on its domain, C^1 on $(0,\infty)$, $\alpha_m(s)\leq s\alpha_m'(s)$ for all $s\in(0,\infty)$, and

$$\alpha_m(\beta_2(s,t)) \leq \tilde{\alpha}_2(s)e^{-2t} \quad \forall s\geq 0,\ t\geq 0; \tag{5.41}$$

- $W_i := \alpha_m\circ\widetilde{W}_i$, $\alpha := \alpha_m\circ\widehat{\alpha}$, $\tilde{\alpha}_1 := \alpha_m\circ\widehat{\alpha}_1$.

5.5.1. Proof of (5.6). Given the bound for $\hat{\phi}\in\mathcal{S}_i[0,\infty)(x)$ in (5.39) and the $\mathcal{KL}$-estimate given by (5.40), we have

$$W_i(x) \leq \alpha_m\left(\sup_{t\geq 0}\widehat{\alpha}_1(\beta(|x|_{\mathcal{A}},t))e^{2t}\right) \leq \alpha_m\left(\sup_{t\geq 0}\widehat{\alpha}(|x|_{\mathcal{A}})e^{-2t}\right) \leq \alpha(|x|_{\mathcal{A}}), \tag{5.42}$$

while the lower bound follows from

$$W_i(x) \geq \alpha_m\left(\inf_{\phi\in\mathcal{S}_i[0,\infty)(x)}\widehat{\alpha}_1(|\phi(t,x)|_{\mathcal{A}})e^{2t}\bigg|_{t=0}\right) = \alpha_m(\widehat{\alpha}_1(|x|_{\mathcal{A}})) = \tilde{\alpha}_1(|x|_{\mathcal{A}}). \tag{5.43}$$

5.5.2. Proof of (5.7). From the upper bound on $W_i(x)$, the $\mathcal{KL}$-bound in (5.39), and the $\mathcal{KL}$-estimates in (5.40) and (5.41) we can write, for all $x\in\mathbb{R}^n$ and all $t\geq 0$,

$$\begin{aligned} W_i(\hat{\phi}(t,x)) &\leq \alpha(|\hat{\phi}(t,x)|_{\mathcal{A}}) = \alpha_m\circ\widehat{\alpha}(|\hat{\phi}(t,x)|_{\mathcal{A}}) \leq \alpha_m\circ\widehat{\alpha}(\beta(|x|_{\mathcal{A}},t)) \\ &\leq \alpha_m\circ\widehat{\alpha}\circ\widehat{\alpha}_1^{-1}\left(\widehat{\alpha}(|x|_{\mathcal{A}})e^{-4t}\right) = \alpha_m\left(\beta_2(|x|_{\mathcal{A}},t)\right) \leq \tilde{\alpha}_2(|x|_{\mathcal{A}})e^{-2t}\ . \end{aligned}$$

5.5.3. Proof of (5.8). We make the following claim and defer its proof to the end of this section.

Claim 2. For $|x|_{\mathcal{A}}\leq\frac{1}{i+1}$, we have $\widetilde{W}_i(x) = \widehat{\alpha}_1(|x|_{\mathcal{A}})$.

Construct any decreasing sequence $\{\ell_j\}_{j=1}^\infty$ such that $\ell_j\to 0$ as $j\to\infty$ and $\ell_1 = \frac{1}{i+1}$. Apply Lemma 5.7 with $V(x) := \widetilde{W}_i(x)$, $\alpha_1(s) := \widehat{\alpha}_1(s)$, $\alpha_2(s) := \widehat{\alpha}(s)$, $\kappa(t) = e^{2t}$, and the sequence $\{\ell_j\}_{j=1}^\infty$ constructed above. Therefore, for each $x\in\mathbb{R}^n$, there exists $\tilde{\phi}_i\in\mathcal{S}_i[0,\infty)(x)$ such that $\widetilde{W}_i(x) = \sup_{t\geq 0}\widehat{\alpha}_1(|\tilde{\phi}_i(t,x)|_{\mathcal{A}})e^{2t}$. Then

$$\begin{aligned} \widetilde{W}_i(x)e^{-2t} &= \sup_{\tau\geq 0}\widehat{\alpha}_1(|\tilde{\phi}_i(\tau,x)|_{\mathcal{A}})e^{2\tau}e^{-2t} \geq \sup_{\tau\geq t}\widehat{\alpha}_1(|\tilde{\phi}_i(\tau,x)|_{\mathcal{A}})e^{2(\tau-t)} \\ &= \sup_{s\geq 0}\widehat{\alpha}_1(|\tilde{\phi}_i(s+t,x)|_{\mathcal{A}})e^{2s} \geq \inf_{\phi\in\mathcal{S}_i[0,\infty)(\tilde{\phi}_i(t,x))}\sup_{s\geq 0}\widehat{\alpha}_1(|\phi(s,\tilde{\phi}_i(t,x))|_{\mathcal{A}})e^{2s} \\ &= \widetilde{W}_i(\tilde{\phi}_i(t,x)). \end{aligned} \tag{5.44}$$

If we define $U(t) := \alpha_m(\widetilde{W}_i(x)e^{-2t})$, then $U(\cdot)$ is C^1 on $(0,\infty)$ since $\alpha_m(\cdot)$ is C^1 on $(0,\infty)$, and we can write $\dot{U}(t) = (-2\widetilde{W}_i(x)e^{-2t})\alpha_m'(\widetilde{W}_i(x)e^{-2t}) \leq -2U(t)$, where the inequality follows from the property $\alpha_m(s)\leq s\alpha_m'(s)$ for all $s\in(0,\infty)$. By a standard comparison lemma we obtain $U(t)\leq U(0)e^{-2t}$; i.e., $\alpha_m(\widetilde{W}_i(x)e^{-2t}) \leq \alpha_m(\widetilde{W}_i(x))e^{-2t}$. Combining this result with (5.44) we get

$$\begin{aligned} W_i(\tilde{\phi}_i(t,x)) &= \alpha_m(\widetilde{W}_i(\tilde{\phi}_i(t,x))) \leq \alpha_m\left(\widetilde{W}_i(x)e^{-2t}\right) \leq \alpha_m\left(\widetilde{W}_i(x)\right)e^{-2t} \\ &= W_i(x)e^{-2t}. \end{aligned} \tag{5.45}$$

Fix $i \in \mathbb{Z}_{>0}$, and let $|x|_{\mathcal{A}} \geq \frac{1}{i}$. It follows from (5.45) with (5.6) that the set $\{t : |\tilde{\phi}_i(t,x)|_{\mathcal{A}} = \frac{1}{i}\}$ is nonempty. Moreover, since $F_i(x) = F(x)$ on the set $|x|_{\mathcal{A}} \geq \frac{1}{i}$, it follows, with $T := \inf\{t : |\tilde{\phi}_i(t,x)|_{\mathcal{A}} = \frac{1}{i}\}$, that $\tilde{\phi} \in \mathcal{S}[0,T](x)$. Let $\psi \in \mathcal{S}[0,\infty)(\tilde{\phi}_i(T,x))$ be arbitrary. Define

$$\phi_i(t,x) := \begin{cases} \tilde{\phi}_i(t,x), & t \in [0,T], \\ \psi(t-T, \tilde{\phi}_i(T,x)), & t \geq T\ . \end{cases}$$

Therefore, $\phi_i \in \mathcal{S}[0,\infty)(x)$. Furthermore, for all $t \in [0,T]$, $W_i(\phi_i(t,x)) \leq W_i(x)e^{-2t}$.

5.5.4. Local Lipschitz property of $W_i(x)$. From Lemma 5.7 we know that $\widetilde{W}_i(\cdot)$ is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$. Consequently, since $\alpha_m(\cdot)$ is locally Lipschitz, $W_i(\cdot)$ is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$. This, coupled with Claim 2 and the local Lipschitz property of $\widehat{\alpha}_1(\cdot)$ and $\alpha_m(\cdot)$, implies that $W_i(\cdot)$ is locally Lipschitz on $\mathbb{R}^n$.

5.5.5. Proof of Claim 2. Via the same argument as in (5.43), we observe that $\widetilde{W}_i(x) \geq \widehat{\alpha}_1(|x|_{\mathcal{A}})$. So, we just need to show that $\widetilde{W}_i(x) \leq \widehat{\alpha}_1(|x|_{\mathcal{A}})$ for all $|x|_{\mathcal{A}} \leq \frac{1}{i+1}$.

Let $|x|_{\mathcal{A}} \leq \frac{1}{i+1}$, $a \in \mathcal{A}$, be a closest point to x, and define

$$\psi(t,x) := \begin{cases} x + \frac{a-x}{|a-x|}t & \forall t \in [0, |a-x|], \\ a & \forall t \geq |a-x|\ . \end{cases} \tag{5.46}$$

We see that $\psi(\cdot,x)$ is absolutely continuous and

$$\left|\frac{d}{dt}\psi(t,x)\right| = \begin{cases} \left|\frac{a-x}{|a-x|}\right| = 1 \in \overline{\mathcal{B}}_n & \forall t \in (0, |a-x|), \\ 0 \in \overline{\mathcal{B}}_n & \forall t > |a-x|, \end{cases}$$

which, with (5.38), implies that (5.46) is a solution to (5.37); i.e., $\psi \in \mathcal{S}_i[0,\infty)(x)$. For $t \in [0, |a-x|]$ we can write

$$\begin{aligned} |x - \psi(t,x)| &= \left|\tfrac{a-x}{|a-x|}t\right| = t \ \text{ and} \\ |\psi(t,x) - a| &= \left|(x-a)\left(1 - \tfrac{1}{|a-x|}t\right)\right| = |x-a| - t \end{aligned} \tag{5.47}$$

so that $|a-x| = |x-\psi(t,x)| + |\psi(t,x)-a|$. Furthermore, $|x-a| = |x|_{\mathcal{A}}$, and $|x|_{\mathcal{A}} \leq \frac{1}{i+1}$. These facts, together with (5.47), imply that $|\psi(t,x)|_{\mathcal{A}} \leq \frac{1}{i+1}$ for all $t \geq 0$.

Now, for some $t \in (0, |a-x|]$, assume there exists $b \in \mathcal{A}$ such that $|b - \psi(t,x)| < |a - \psi(t,x)|$; i.e., b is closer than a to $\psi(t,x)$. The triangle inequality yields

$$|x-b| \leq |x-\psi(t,x)| + |b-\psi(t,x)| < |x-\psi(t,x)| + |a-\psi(t,x)| = |x-a|\ ,$$

which contradicts that $a \in \mathcal{A}$ is a closest point to x. Therefore, $a \in \mathcal{A}$ is also a closest point to $\psi(t,x)$ for $t \in [0, |a-x|]$. More explicitly, $|\psi(t,x)|_{\mathcal{A}} = |\psi(t,x) - a|$. Combining this with (5.47) yields, for almost all $t \in [0, |a-x|]$,

$$\frac{d}{dt}|\psi(t,x)|_{\mathcal{A}} = \frac{d}{dt}\left(|x-a| - t\right) = -1\ . \tag{5.48}$$

Using (5.47) and (5.48), and since $\widehat{\alpha}_1(\cdot)$ is C^1 on $(0,\infty)$ and $\widehat{\alpha}_1(s) \le s\widehat{\alpha}_1'(s)$, we can write, for almost all $t \in [0, |a-x|]$,

$$\frac{d}{dt}\left(\widehat{\alpha}_1(|\psi(t,x)|_{\mathcal{A}})e^{2t}\right) = \left(2\widehat{\alpha}_1(|\psi(t,x)|_{\mathcal{A}}) + \widehat{\alpha}_1'(|\psi(t,x)|_{\mathcal{A}})(\overbrace{|\psi(t,x)|_{\mathcal{A}}}^{\cdot})\right)e^{2t}$$
$$\le \widehat{\alpha}_1'(|\psi(t,x)|_{\mathcal{A}})\,(2(|x|_{\mathcal{A}} - t) - 1)\,e^{2t} \quad \le \quad 0,$$

where the final step follows from $|x|_{\mathcal{A}} \le \frac{1}{i+1} \le \frac{1}{2}$. Furthermore, for $t > |a-x|$ we have

$$\frac{d}{dt}\left(\widehat{\alpha}_1(|\psi(t,x)|_{\mathcal{A}})e^{2t}\right) = \frac{d}{dt}\left(\widehat{\alpha}_1(0)e^{2t}\right) = 0\ .$$

This gives $\frac{d}{dt}\left(\widehat{\alpha}_1(|\psi(t,x)|_{\mathcal{A}})e^{2t}\right) \le 0$ for almost all $t \ge 0$. It follows by integrating that $\sup_{t\ge 0}\widehat{\alpha}_1(|\psi(t,x)|_{\mathcal{A}})e^{2t} = \widehat{\alpha}_1(|x|_{\mathcal{A}})$. So $\widetilde{W}_i(x) \le \widehat{\alpha}_1(|x|_{\mathcal{A}})$ for all $|x|_{\mathcal{A}} \le \frac{1}{i+1}$. □

6. Proofs of Lemmas 5.3–5.7. Collected in this section are the proofs of lemmas utilized in proving Theorem 2.1.

6.1. Proof of Lemma 5.3. The proof relies on the proof of [30, Lemma 3]. First we pick $\rho \in \mathcal{K}_\infty$ and a function $\theta : \mathbb{R}_{\ge 0} \to \mathbb{R}_{>0}$ continuous and strictly decreasing with $\lim_{t\to\infty}\theta(t) = 0$ such that, for all $t \ge 0$, $\beta(\rho(t), t) \le \theta(t)$. (The proof that such functions exist can be found at the beginning of [30, Lemma 3].) Next, let $\theta^{-1}(\cdot)$ be the inverse of $\theta(\cdot)$, which is defined and continuous on $(0, \theta(0)]$. It is also strictly decreasing with $\lim_{s\to 0}\theta^{-1}(s) = +\infty$. It follows that the function $e^{-2\lambda\theta^{-1}(s)}$ is well defined, continuous, positive and strictly increasing on $(0, \theta(0)]$. Then we define a continuous, positive definite, nondecreasing function $\pi(\cdot)$ as

$$\pi(s) := \begin{cases} 0, & s = 0, \\ \frac{1}{\theta(0)}e^{-2\lambda\theta^{-1}(s)}, & s \in (0, \theta(0)], \\ \frac{1}{\theta(0)}, & s \ge \theta(0). \end{cases}$$

We then define $\alpha_1(s) := \int_0^s \pi(\tau)d\tau$. It follows that $\alpha_1(\cdot)$ is Lipschitz on its domain (with global Lipschitz constant $\frac{1}{\theta(0)}$), continuously differentiable on $(0,\infty)$, and of class-$\mathcal{K}_\infty$. From the definition of $\alpha_1(\cdot)$, $\alpha_1(s) \le s\pi(s) = s\alpha_1'(s)$, for all $s > 0$, and $\alpha_1(\cdot)$ satisfies, for any $s \in (0, \theta(0)]$,

$$\alpha_1(s) \le s\pi(s) \le \theta(0)\pi(s) \le e^{-2\lambda\theta^{-1}(s)}\ . \tag{6.1}$$

One may now follow [30, Lemma 3].

6.2. Proof of Lemma 5.4. We follow the proofs of [19, Lemmas 11 and 12]. Without loss of generality, we assume that $\alpha(\cdot)$ is C^1 at the origin, $\alpha(s) \le s$, and $\alpha'(0) = 0$. Now define $\rho(0) = 0$ and

$$\rho(s) := \exp\left(2\int_1^s \frac{1}{\alpha(\tau)}d\tau\right) \qquad \forall s > 0\ . \tag{6.2}$$

Since $\alpha(s) \le s$, for $s \ge 1$ we have $\rho(s) \ge \exp\left(2\int_1^s \frac{d\tau}{\tau}\right) = s^2$ and for $s \le 1$ we have $\rho(s) \le \exp(-2\int_s^1 \frac{d\tau}{\tau}) = s^2$. It follows with the fact that $\rho(\cdot)$ is C^1 on $(0,\infty)$ that $\rho \in \mathcal{K}_\infty$. By differentiating (6.2), we have, for all $s > 0$, $\rho'(s) = \frac{2\rho(s)}{\alpha(s)}$ so that $\rho(s) = \frac{1}{2}\alpha(s)\rho'(s) \le \alpha(s)\rho'(s)$. The Lipschitz property for $\rho(\cdot)$ on its domain, demonstrated by showing the boundedness of $\rho'(s)$ for small s, follows from the proof of [19, Lemma 12].

6.3. Proof of Lemma 5.5. Pick $\kappa(t) = 1 + \int_0^t q(s)ds$, where $q : \mathbb{R}_{\geq 0} \to (0, 1]$ is piecewise continuous, nonintegrable (so that $\kappa(\cdot)$ is unbounded), nonincreasing, and such that

$$1 + \int_0^t q(s)ds \leq \frac{1}{\omega_2(s_0 e^{-t})}. \tag{6.3}$$

We can always pick a nonintegrable, nonincreasing, piecewise continuous function $q(\cdot)$ taking values in $(0, 1]$ satisfying (6.3) as follows: We take $q(\cdot)$ to be piecewise constant with $q(s) = q_j$ for $s \in [j - 1, j)$, where $\{q_j\}_{j=1}^{\infty}$ is a sequence of nonincreasing strictly positive numbers chosen inductively as follows: Define $q_0 = 1$. Let $\kappa_i = \sum_{j=0}^{i} q_j$. Note that $1 = \kappa_0 \leq 1/\omega_2(s_0 e^{-0}) = 2$. Define

$$m_i := \frac{1}{\omega_2(s_0 e^{-i})} - \kappa_i \tag{6.4}$$

and pick $q_{i+1} = \min\{m_i, q_i\}$. It follows that q_i is nonincreasing. It also follows that

$$\kappa_{i+1} = \kappa_i + q_{i+1} \leq \frac{1}{\omega_2(s_0 e^{-i})} < \frac{1}{\omega_2(s_0 e^{-(i+1)})} ,$$

from which it follows that $m_i > 0$ and so $q_i > 0$. Let $t \geq 0$, and let i be such that $t \in [i, i + 1)$. Then

$$\kappa_i \leq 1 + \int_0^t q(s)ds \leq \kappa_{i+1} \leq \frac{1}{\omega_2(s_0 e^{-i})} \leq \frac{1}{\omega_2(s_0 e^{-t})} . \tag{6.5}$$

Now, suppose that $q(\cdot)$ is integrable. Since the function $\frac{1}{\omega_2(s_0 e^{-t})}$ is unbounded, (6.4) and (6.5) imply that $\lim_{i\to\infty} m_i = \infty$, which implies that there exists $j > 0$ such that $q_k = q_j > 0$ for all $k \geq j$. This contradicts $q(\cdot)$ being integrable.

That $\kappa(\cdot)$ is strictly increasing and (5.1) holds (i.e., $\kappa(0) = 0$) is obvious from the definition. We observe that

$$\frac{\kappa(t)}{\kappa(T)} = \frac{\kappa(T) + \int_T^t q(s)ds}{\kappa(T)} \leq 1 + \frac{t - T}{\kappa(T)} \leq 1 + (t - T) \leq e^{t-T}; \tag{6.6}$$

that is, (5.3) holds. Furthermore, by considering $T = 0$ in (6.6) and the constraint (6.3) we see that (5.4) holds.

Since $q(s) > 0$ for all $s \in \mathbb{R}_{\geq 0}$ and $q(\cdot)$ is nonincreasing, we have

$$\begin{aligned}\kappa(t)\kappa(T) &\geq 1 + \int_0^t q(s)ds + \int_0^T q(s)ds \geq 1 + \int_0^t q(s)ds + \int_t^{t+T} q(s)ds \\ &= 1 + \int_0^{t+T} q(s)ds = \kappa(t + T);\end{aligned}$$

that is, (5.2) holds.

To see that we can simultaneously satisfy (5.5), let $q_s : \mathbb{R}_{\geq 0} \to (0, 1]$ be continuous, nonincreasing, and such that $q_s(t) \leq q(t)$. Then, using the mean value theorem for locally Lipschitz functions, we have that for all $t \in [0, 1]$,

$$\max_{s\in[0,T]} \frac{\kappa(s)}{\kappa(s + t)} = \max_{s\in[0,T]} 1 - \frac{\kappa(s + t) - \kappa(s)}{\kappa(s + t)} \leq 1 - \frac{q_s(T + 1)}{\kappa(T + 1)} t =: 1 - \vartheta(T)t .$$

Clearly, $\vartheta(\cdot)$ is continuous, nonincreasing, and takes values in $\mathbb{R}_{>0}$. ∎

6.4. Proof of Lemma 5.6. We require some preliminary definitions. Let $\Omega \subset \mathbb{R}^n \backslash \mathcal{A}$ denote the set (of measure zero) where the gradient of $V(\cdot)$ is not defined in $\mathbb{R}^n \backslash \mathcal{A}$. For $(c,d) \in \mathbb{R}_{\geq 0} \times (0,1]$ define

$$\eta(c,d) := \sup_{\{|x|\leq c, V(x)\in[d,1], x\notin\Omega\}} |\nabla V(x)| \ .$$

For each $(c,d) \in \mathbb{R}_{\geq 0} \times (0,1]$, $\eta(c,d)$ is finite since for these (c,d) the set

$$\{x \in \mathbb{R}^n : |x| \leq c,\ V(x) \in [d,1]\}$$

is a compact subset of $\mathbb{R}^n \backslash \mathcal{A}$, the latter being a set on which $V(\cdot)$ is locally Lipschitz. Also $\eta(\cdot,d)$ is nondecreasing and $\eta(c,\cdot)$ is nonincreasing so that $\sigma : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ defined as

$$\sigma(s) := \begin{cases} \eta\left(\dfrac{1}{s}, s\right) & \forall s \in (0,1), \\ \eta(1,1) & \forall s \geq 1 \end{cases}$$

is nonincreasing. We claim that

$$(6.7) \qquad x \in \left\{\xi \in (\mathbb{R}^n \backslash \mathcal{A}) \backslash \Omega : \max\{1,|\xi|\} \leq \frac{1}{V(\xi)}\right\} \quad \Longrightarrow \quad |\nabla V(x)| \leq \sigma(V(x))$$

since, in the indicated set, $V(x) \in (0,1]$, $|x| \leq \frac{1}{V(x)}$ so that $\sigma(V(x)) \geq \eta(|x|, V(x)) \geq |\nabla V(x)|$.

Let $\varphi : \mathbb{R}_{\geq 0} \to [0,1]$ be continuous, zero at zero, positive definite, nondecreasing, and let $\pi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be continuous, positive definite, nondecreasing and such that

$$(6.8) \qquad \pi(s)\sigma(s) \leq \varphi(s) \leq 1 \qquad \forall s > 0 \ .$$

Define $\rho(r) := \int_0^r \pi(s)ds$. Since $\pi(\cdot)$ is continuous, positive definite, and nondecreasing, we have that $\rho \in \mathcal{K}_\infty$ and $\rho(r) \leq r\pi(r) = r\rho'(r)$ for all $r > 0$.

We now demonstrate the local Lipschitz property for $V_L := \rho \circ V$ on $\mathbb{R}^n$. Since both $\rho(\cdot)$ and $V(\cdot)$ are continuous on $\mathbb{R}^n$, $V_L := \rho \circ V$ is continuous on $\mathbb{R}^n$. Since $\rho(\cdot)$ is locally Lipschitz and $V(\cdot)$ is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$ and on the interior of $\mathcal{A}$, $V_L(\cdot)$ is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$ and on the interior of $\mathcal{A}$. To show that $V_L(\cdot)$ is locally Lipschitz on $\mathbb{R}^n$ it is sufficient to establish that, for each $x \in \text{bdry}\mathcal{A}$ (the boundary of $\mathcal{A}$), there exists $M > 0$ and an open, convex neighborhood $\mathcal{U}$ of x such that $DV_L(\xi; v) \leq M|v|$ for all $\xi \in \mathcal{U}, v \in \mathbb{R}^n$, and then use Lemma 5.1. Using $\text{int}(\mathcal{A})$ for the interior of $\mathcal{A}$,

$$(6.9) \qquad \xi \in \text{int}(\mathcal{A}) \quad \Longrightarrow \quad DV_L(\xi; v) = 0\ \forall v \in \mathbb{R}^n \ .$$

Also, it follows from (6.7), (6.8), and $\rho'(s) = \pi(s)$ that

$$\xi \in \left\{\zeta \in (\mathbb{R}^n \backslash \mathcal{A}) \backslash \Omega\ ,\ \max\{1,|\zeta|\} \leq \frac{1}{V(\zeta)}\right\} \quad \Longrightarrow \quad |\nabla V_L(\xi)| \leq \varphi(V(\xi)) \leq 1.$$

That $DV_L(\xi; v) \leq \limsup_{y\to\xi} \{\langle \nabla V_L(y), v\rangle : y \notin \Omega\}$ is a combination of [8, Corollary 2.8.2] and [8, Exercise 3.4.1]. Consequently, we have

$$(6.10) \quad \xi \in \left\{\zeta \in \mathbb{R}^n \backslash \mathcal{A} : \max\{1, 2|\zeta|\} \leq \frac{1}{V(\zeta)}\right\} \quad \Longrightarrow \quad DV_L(\xi; v) \leq |v|\ \forall v \in \mathbb{R}^n.$$

With (6.9) and (6.10), we will be done if we can show that

$$x \in \mathrm{bdry}\mathcal{A} \quad \Longrightarrow \quad DV_L(x; v) = 0 \ \forall v \in \mathbb{R}^n \ . \tag{6.11}$$

Since $V(x) = 0$ and $V(\xi) \geq 0$ for all $\xi \in \mathbb{R}^n$, we just need to show $DV_L(x; v) \leq 0$. Let $x \in \mathrm{bdry}\mathcal{A}$. Since $\varphi \circ V(\cdot)$ is continuous and $\varphi(V(x)) = 0$, for each $\varepsilon > 0$ there exists a convex neighborhood $\mathcal{U}$ of x such that

$$\xi \in (\mathbb{R}^n \backslash \mathcal{A}) \cap \mathcal{U} \quad \Longrightarrow \quad \begin{cases} \max\{1, |\xi|\} \leq \frac{1}{V(\xi)} \ , \\ \varphi(V(\xi)) \leq \varepsilon. \end{cases}$$

It follows from (6.7), (6.8), and $\rho'(s) = \pi(s)$ that

$$\xi \in ((\mathbb{R}^n \backslash \mathcal{A}) \backslash \Omega) \cap \mathcal{U} \quad \Longrightarrow \quad |\nabla V_L(\xi)| \leq \varepsilon \ . \tag{6.12}$$

It follows from continuity of $V_L(\cdot)$ that for each $\xi \in (\mathbb{R}^n \backslash \mathcal{A}) \bigcap \mathcal{U}$ there exists $s^*(\xi) \in (0, 1)$ such that

$$\begin{aligned} V_L(x + s(\xi - x)) &\geq \frac{V_L(\xi)}{2} \qquad \forall s \in [s^*(\xi), 1] \ , \\ V_L(x + s^*(\xi)(\xi - x)) &= \frac{V_L(\xi))}{2} \ . \end{aligned}$$

It follows from the convexity of $\mathcal{U}$ that $\xi \in (\mathbb{R}^n \backslash \mathcal{A}) \cap \mathcal{U}$, and $s \in [s^*(\xi), 1]$ imply $x + s(\xi - x) \in (\mathbb{R}^n \backslash \mathcal{A}) \cap \mathcal{U}$. So, with (6.12) and the mean value theorem for locally Lipschitz functions (see [8, Theorem 2.2.4]) together with [8, Theorem 2.8.1], for all $\xi \in (\mathbb{R}^n \backslash \mathcal{A}) \bigcap \mathcal{U}$,

$$\begin{aligned} 0 \leq V_L(\xi) - V_L(x) &= 2 \left(V_L(\xi) - V_L(x + s^*(\xi)(\xi - x)) \right) \\ &\leq 2\varepsilon(1 - s^*(\xi))|\xi - x| \leq 2\varepsilon|\xi - x|. \end{aligned}$$

Since, for every $\xi \in \mathcal{A} \bigcap \mathcal{U}$, we have $V_L(\xi) - V_L(x) = 0$, it follows that, for all $\xi \in \mathcal{U}$, we have $0 \leq V_L(\xi) - V_L(x) \leq 2\varepsilon|\xi - x|$. Since $\varepsilon > 0$ can be taken arbitrarily small, (6.11) holds. ∎

6.5. Proof of Lemma 5.7. For $\dot{x} \in F(x)$ we denote by $\mathcal{R}_{\leq T}(\mathcal{C})$ the set of points reachable from a compact set $\mathcal{C} \subset \mathbb{R}^n$ in time $T > 0$; i.e., $\mathcal{R}_{\leq T}(\mathcal{C}) := \{\xi \in \mathbb{R}^n : \xi = \phi(t, x), t \in [0, T], x \in \mathcal{C}, \phi \in \mathcal{S}(x)\}$.

DEFINITION 8. *The differential inclusion is said to be forward complete if, for every $x \in \mathbb{R}^n$, all $\phi \in \mathcal{S}(x)$ are defined on $[0, \infty)$.*

We require several preliminary lemmas prior to proving Lemma 5.7. The first is also [11, section 7, Theorem 3].

LEMMA 6.1. *Let $F(\cdot)$ satisfy the basic conditions on $\mathbb{R}^n$, and suppose $\dot{x} \in F(x)$ is forward complete. For each compact set $\mathcal{C} \subset \mathbb{R}^n$ and $T \in \mathbb{R}_{>0}$, $\mathcal{R}_{\leq T}(\mathcal{C})$ is a compact subset of $\mathbb{R}^n$ and $\mathcal{S}[0, T](\mathcal{C})$ is a compact set in the metric of uniform convergence.*

An easy corollary of the above appeared as [30, Lemma 5].

LEMMA 6.2. *Let $F(\cdot)$ satisfy the basic conditions on $\mathbb{R}^n$, and let $\dot{x} \in F(x)$ be forward complete. Then each sequence $\{\phi_n\}_{n=1}^{\infty}$ of solutions satisfying $\phi_n \in \mathcal{S}[0, \infty)(x_n)$, where $x_n \to x$, has a subsequence converging to a function $\phi \in \mathcal{S}[0, \infty)(x)$, and the convergence is uniform on each compact time interval.*

The following appeared as [30, Lemma 10] (which derives from [10, Lemma 8.3] and [8, Theorem 4.3.11]) and describes how solutions to locally Lipschitz inclusions depend on initial conditions in a locally Lipschitz manner.

LEMMA 6.3. *Let $F(\cdot)$ satisfy the basic conditions on $\mathbb{R}^n$ and be locally Lipschitz on the open set $\mathcal{O} \subseteq \mathbb{R}^n$. For each $T > 0$ and each compact set $\mathcal{C} \subset \mathcal{O}$, there exist L and $\delta > 0$ such that, for each $\xi \in \mathcal{C}$, each $\phi \in \mathcal{S}(\xi)$, and each $\bar{v}$ satisfying $|\bar{v}| \leq \delta$, there exists $\psi \in \mathcal{S}(\xi + \bar{v})$ with the property $|\phi(t,\xi) - \psi(t, \xi + \bar{v})| \leq L|\bar{v}|$ for all $t \in [0, T_\phi]$, where $T_\phi \in [0,T]$ is such that $\phi(t,\xi) \in \mathcal{C}$ for all $t \in [0, T_\phi]$.*

6.5.1. Preliminaries. We define, for all $(s,\mu) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$,

$$\gamma(s,\mu) := \begin{cases} 1 & \text{if} \quad s \leq 2 + \mu, \\ 3 + \mu - s & \text{if} \quad s \in [2+\mu, 3+\mu], \\ 0 & \text{if} \quad s \geq 3 + \mu. \end{cases}$$

It follows from this definition that $\gamma(\cdot,\mu)$ is Lipschitz of rank one (i.e., Lipschitz with a Lipschitz constant of one). We define the set-valued map

$$F_\mu(x) := \gamma\left(|x|_{\mathcal{A}}, \mu\right) F(x) \ .$$

We note that, for each μ, $F_\mu(\cdot)$ satisfies the basic conditions and is locally Lipschitz on $\mathbb{R}^n \backslash \mathcal{A}$. Also, using Assumption 1 and the definition of $F_\mu(x)$, we have $\max_{w \in F_\mu(x)} |w| \leq M_{3+\mu}$ for all $x \in \mathbb{R}^n$. Therefore, for each $\mu \geq 0$, the inclusion $\dot{x} \in F_\mu(x)$ is forward complete. We define $\mathcal{S}_\mu(x)$ to be the set of maximal solutions for $\dot{x} \in F_\mu(x)$. We let $\mathcal{Q}_\mu(x)$ be the subset of $\mathcal{S}_\mu(x)$ whose elements $\phi(\cdot, x)$ satisfy

$$|\phi(t,x)|_{\mathcal{A}} \leq \alpha_1^{-1}\left(\alpha_1(1+\mu)/\kappa(t)\right) \leq 1 + \mu \quad \forall t \geq 0 \ . \tag{6.13}$$

Notice that $\mathcal{Q}_\mu(x)$ may be empty. It follows from the definitions of $\gamma(\cdot,\mu)$ and $F_\mu(\cdot)$ and (6.13) that when $\mathcal{Q}_\mu(x)$ is nonempty we have $\mathcal{Q}_\mu(x) \subseteq \mathcal{S}[0,\infty)(x)$.

6.5.2. An optimal trajectory. We require the following claim.

Claim 3. There exists $\phi^* \in \mathcal{S}[0,\infty)(x)$ such that $V(x) = \sup_{t \geq 0} g(\phi^*(t,x))\kappa(t)$ and, for every $\mu \geq \alpha_1^{-1} \circ \alpha_2(|x|_{\mathcal{A}})$, $\phi^* \in \mathcal{Q}_\mu(x)$.

Proof. Let $\mu \geq \alpha_1^{-1} \circ \alpha_2(|x|_{\mathcal{A}})$, which, since $\alpha_1 \in \mathcal{K}_\infty$, implies $\alpha_1(1+\mu) - \alpha_2(|x|_{\mathcal{A}}) > 0$, and let $\varepsilon > 0$ be such that $\varepsilon \leq \alpha_1(1+\mu) - \alpha_2(|x|_{\mathcal{A}})$. Let $\{\phi_k\}_{k=1}^\infty$ be a sequence within $\mathcal{S}[0,\infty)(x)$, and let $\{\varepsilon_k\}_{k=1}^\infty$ be a sequence of positive real numbers satisfying $\varepsilon_k \leq \varepsilon$ and monotonically decreasing to zero such that

$$\sup_{t \geq 0} g(\phi_k(t,x))\kappa(t) \leq V(x) + \varepsilon_k \ . \tag{6.14}$$

Since $V(x) + \varepsilon_k \leq \alpha_2(|x|_{\mathcal{A}}) + \varepsilon \leq \alpha_1(1+\mu)$ and $g(x) \geq \alpha_1(|x|_{\mathcal{A}})$, it follows that $|\phi_k(t,x)|_{\mathcal{A}} \leq \alpha_1^{-1}(\alpha_1(1+\mu)/\kappa(t)) \leq 1+\mu$ for all $t \geq 0$; i.e., $\phi_k \in \mathcal{Q}_\mu(x) \subseteq \mathcal{S}_\mu(x)$. Since $\dot{x} \in F_\mu(x)$ is forward complete and $F_\mu(\cdot)$ satisfies the basic conditions, it follows from Lemma 6.2 that the sequence ϕ_k has a converging subsequence converging to an element $\phi^* \in \mathcal{S}_\mu(x)$ and that the convergence is uniform on each compact time interval. (We now use ϕ_k to refer to this converging subsequence and ε_k to refer to the corresponding values in (6.14).)

We claim that $\phi^* \in \mathcal{Q}_\mu(x)$. If not, then there exist $\widetilde{t} \geq 0$ and $\widetilde{\rho} > 0$ such that

$$|\phi^*(\widetilde{t},x)|_{\mathcal{A}} \geq \alpha_1^{-1}\left(\alpha_1(1+\mu)/\kappa(t)\right) + \widetilde{\rho} \ . \tag{6.15}$$

From the uniform convergence of ϕ_k to ϕ^* on the interval $[0,\widetilde{t}]$ and since $|\cdot|_{\mathcal{A}}$ is Lipschitz of rank one, there exists $\widetilde{k}$ such that

$$|\phi_{\widetilde{k}}(\widetilde{t},x)|_{\mathcal{A}} \geq |\phi^*(\widetilde{t},x)|_{\mathcal{A}} - \frac{\widetilde{\rho}}{2}\ . \tag{6.16}$$

Combining (6.15) and (6.16), we get that $\phi_{\widetilde{k}} \notin \mathcal{Q}_\mu(x)$, which is a contradiction.

Next we claim that

$$V(x) = V^*(x) := \sup_{t\geq 0} g(\phi^*(t,x))\kappa(t)\ . \tag{6.17}$$

Since $\mathcal{Q}_\mu(x) \subseteq \mathcal{S}[0,\infty)(x)$, we have that $V(x) \leq V^*(x)$. So if (6.17) does not hold, then there exists $\rho > 0$ and $\bar{t} \geq 0$ such that

$$\sup_{t\in[0,\bar{t}]} g(\phi^*(t,x))\kappa(t) \geq V(x) + \rho. \tag{6.18}$$

Now, from the uniform convergence of ϕ_k to ϕ^* on the interval $[0,\bar{t}]$, the compactness of $\mathcal{R}^\mu_{\leq\bar{t}}(\{x\})$ (the set of points reachable from x in time $\bar{t}$ for the modified inclusion $\dot{x} \in F_\mu(x)$) from Lemma 6.1, and the continuity of $g(\cdot)$, there exists k such that $\varepsilon_k < \rho/2$ and

$$\sup_{t\in[0,\bar{t}]} g(\phi_k(t,x))\kappa(t) \geq \sup_{t\in[0,\bar{t}]} g(\phi^*(t,x))\kappa(t) - \frac{\rho}{2}\ . \tag{6.19}$$

Combining (6.19) with (6.18) and $\varepsilon_k < \rho/2$ we get

$$\sup_{t\in[0,\bar{t}]} g(\phi_k(t,x))\kappa(t) \geq V(x) + \frac{\rho}{2} > V(x) + \varepsilon_k,$$

which contradicts (6.14). This establishes (6.17). ▯

6.5.3. The local Lipschitz property. We make the following claim.

Claim 4. $V(\cdot)$ is lower semicontinuous on $\mathbb{R}^n$; i.e., for each $x \in \mathbb{R}^n$ and any sequence $x_k \to x$, $\liminf_{k\to\infty} V(x_k) \geq V(x)$.

Proof. Fix $x \in \mathbb{R}^n$, and let $\mu \geq \alpha_1^{-1} \circ \alpha_2\left(|x|_{\mathcal{A}} + 1\right)$. Let x_k be a sequence converging to x and, without loss of generality, assume that $|x_k - x| \leq 1$ for all k so that $\mu \geq \alpha_1^{-1} \circ \alpha_2\left(|x_k|_{\mathcal{A}}\right)$ for all k. Define $\underline{V}(x) := \liminf_{k\to\infty} V(x_k)$. By extracting a suitable subsequence from x_k (and using x_k to denote the subsequence), we can construct a sequence ε_k monotonically decreasing to zero so that

$$V(x_k) \leq \underline{V}(x) + \varepsilon_k\ . \tag{6.20}$$

Let $\phi_k^* \in \mathcal{S}[0,\infty)(x_k)$ come from Claim 3 so that $\phi_k^* \in \mathcal{Q}_\mu(x_k)$. With Lemma 6.2 this sequence has a converging subsequence converging to an element $\psi \in \mathcal{S}_\mu(x)$, and the convergence is uniform on each compact time interval. (We now use ϕ_k and x_k to refer to this convergent subsequence and ε_k to refer to the corresponding values in (6.20).) We can use the same argument as in the proof of Claim 3 to establish that $\psi \in \mathcal{Q}_\mu(x)$. We claim

$$\underline{V}(x) \geq \sup_{t\geq 0} g(\psi(t,x))\kappa(t)\ . \tag{6.21}$$

If this is not the case, then there exist $\rho > 0$ and $\bar{t} \geq 0$ such that

$$\sup_{t\in[0,\bar{t}]} g(\psi(t,x))\kappa(t) \geq \underline{V}(x) + \rho \ . \tag{6.22}$$

Now, from the uniform convergence of $\phi_k^*(\cdot, x_k)$ to $\psi(\cdot, x)$ on the interval $[0, \bar{t}]$, the compactness of $\mathcal{R}^{\mu}_{\leq\bar{t}}(\{x\} + \overline{\mathcal{B}}_n)$ from Lemma 6.1, and the continuity of $g(\cdot)$, there exists k such that $\varepsilon_k < \rho/2$ and

$$\sup_{t\in[0,\bar{t}]} g(\phi_k^*(t,x_k))\kappa(t) \geq \sup_{t\in[0,\bar{t}]} g(\psi(t,x))\kappa(t) - \frac{\rho}{2} \ . \tag{6.23}$$

Combining (6.23) with (6.22) and $\varepsilon_k < \rho/2$ we get

$$V(x_k) \geq \sup_{t\in[0,\bar{t}]} g(\phi_k^*(t,x_k))\kappa(t) \geq \underline{V}(x) + \frac{\rho}{2} > \underline{V}(x) + \varepsilon_k,$$

which contradicts (6.20); i.e., (6.21) holds. Using (6.21) and $\psi \in \mathcal{Q}_\mu(x) \subseteq \mathcal{S}[0,\infty)(x)$, it follows that $\underline{V}(x) \geq V(x)$. □

We now proceed to set ourselves up to apply Lemma 5.1 in order to prove that $V(\cdot)$ is locally Lipschitz. Without loss of generality, we can assume that the sequence ℓ_j is strictly decreasing. For notational convenience, we set $\ell_0 = +\infty$. For $s > 0$, define $\underline{j}(s) := \inf\{j : s \geq \ell_j\}$ and then let $\Upsilon : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a continuous, nondecreasing function satisfying $\Upsilon(s) \leq \ell_{\underline{j}(s)+3}$. To see that such a function exists, we note that $\underline{j}(s)$ is nonincreasing and $\underline{j}(s) \to \infty$ as $s \to 0$. Consequently, $\ell_{\underline{j}(s)+3}$ is nondecreasing and $\ell_{\underline{j}(s)+3} \to 0$ as $s \to 0$.

We note that $s \geq \ell_{\underline{j}(s)} \geq \ell_{\underline{j}(s)+3} \geq \Upsilon(s)$. With the fact that $s \leq \alpha_1^{-1} \circ \alpha_2(s)$ for all $s \geq 0$ and that $\kappa(\cdot)$ has a continuous inverse on $[1, \infty)$, this allows us to define $T : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $T(0) = 0$ and

$$T(s) := \kappa^{-1}\left(\frac{\alpha_1(1 + \alpha_1^{-1} \circ \alpha_2(s+2))}{\alpha_1(\Upsilon(s))}\right) \quad \forall s > 0 \ .$$

It follows that $T(\cdot)$ is continuous on $\mathbb{R}_{>0}$.

Let $\zeta \in \mathbb{R}^n \backslash \mathcal{A}$, $j := \underline{j}(|\zeta|_{\mathcal{A}})$, $\mu := \alpha_1^{-1} \circ \alpha_2(|\zeta|_{\mathcal{A}} + 1)$ and

$$\mathcal{U} := \{\zeta\} + \frac{1}{2}\min\{1, (\ell_{j-2} - \ell_{j-1}), (\ell_j - \ell_{j+1})\}\,\mathcal{B}_n.$$

Notice that $\mathcal{U}$ is open and convex and that $\ell_j \leq |\zeta|_{\mathcal{A}} < \ell_{j-1}$. Since $\mu > \ell_j > \ell_{j+2}$, we can define $\overline{T} := \kappa^{-1}(\frac{\alpha_1(1+\mu)}{\alpha_1(\ell_{j+2})})$. It follows from the definition of $\mathcal{U}$ and since $|\cdot|_{\mathcal{A}}$ is globally Lipschitz of rank one that, for all $\xi \in \overline{\mathcal{U}}$,

$$|\zeta|_{\mathcal{A}} - \frac{1}{2}\min\{1, (\ell_j - \ell_{j+1})\} \leq |\xi|_{\mathcal{A}} \leq |\zeta|_{\mathcal{A}} + \frac{1}{2}\min\{1, \ell_{j-2} - \ell_{j-1}\} \ . \tag{6.24}$$

Therefore for all $\xi \in \overline{\mathcal{U}}$ we have that

$$\mu \geq \alpha_1^{-1} \circ \alpha_2(|\xi|_{\mathcal{A}}) \ , \tag{6.25}$$

$$|\xi|_{\mathcal{A}} + 2 \geq |\zeta|_{\mathcal{A}} + 1 \ , \text{ and} \tag{6.26}$$

$$|\xi|_{\mathcal{A}} \geq \ell_{j+1} \ . \tag{6.27}$$

Finally, using the upper bound on $|\xi|_{\mathcal{A}}$ and $|\zeta|_{\mathcal{A}} < \ell_{j-1}$ one may show that $|\xi|_{\mathcal{A}} \le \ell_{j-2}$. We note that from the definition of $\underline{j}$ we have that $\underline{j}(s) \ge i+1$ if $s < \ell_i$. Consequently, $\underline{j}(|\xi|_{\mathcal{A}}) \ge j-1$, which implies

$$\Upsilon(|\xi|_{\mathcal{A}}) \le \ell_{\underline{j}(|\xi|_{\mathcal{A}})+3} \le \ell_{j+2} \; . \tag{6.28}$$

Inequalities (6.28) and (6.26) imply that $\overline{T} \le \min_{\xi \in \overline{\mathcal{U}}} T(|\xi|_{\mathcal{A}})$.

Define $\mathcal{C} := \mathcal{R}^{\mu}_{\le \overline{T}}(\overline{\mathcal{U}}) \bigcap \{z : |z|_{\mathcal{A}} \ge \ell_{j+2}\}$. It follows that $\mathcal{C}$ is a compact subset of $\mathbb{R}^n \backslash \mathcal{A}$, the latter being an open set where $F_\mu(\cdot)$ is locally Lipschitz. We apply Lemma 6.3 to generate $L > 0$ and $\delta > 0$.

Let $\xi \in \mathcal{U}$. Using (6.25), let $\phi^* \in \mathcal{Q}_\mu(\xi)$ come from Claim 3 so that

$$V(\xi) = \sup_{t \ge 0} g(\phi^*(t,\xi))\kappa(t) \text{ and} \tag{6.29}$$

$$|\phi^*(t,\xi)|_{\mathcal{A}} \le \alpha_1^{-1}\left(\alpha_1(1+\mu)/\kappa(t)\right) \;\; \forall t \ge 0 \; . \tag{6.30}$$

It follows from combining (6.30) and (6.27) that the set $\{t : |\phi^*(t,\xi)|_{\mathcal{A}} = \ell_{j+2}\}$ is nonempty and $T_\xi := \inf\{t : |\phi^*(t,\xi)|_{\mathcal{A}} = \ell_{j+2}\}$ is well defined. Since $\phi^*(\cdot,\xi)$ is continuous, we see that

$$\ell_{j+2} = |\phi^*(T_\xi,\xi)|_{\mathcal{A}} \le \alpha_1^{-1}\left(\alpha_1(1+\mu)/\kappa(T_\xi)\right), \tag{6.31}$$

which leads to $\kappa(T_\xi) \le \alpha_1(1+\mu)/\alpha_1(\ell_{j+2}) = \kappa(\overline{T})$, which, since $\kappa(\cdot)$ is strictly increasing, implies that $T_\xi \le \overline{T}$.

Again since $\phi^*(\cdot,\xi)$ is continuous, it follows that $|\phi^*(t,\xi)|_{\mathcal{A}} \ge \ell_{j+2}$ for all $t \in [0,T_\xi]$; i.e., $\phi^*(t,\xi) \in \mathcal{C}$ for all $t \in [0,T_\xi]$. According to the result of Lemma 6.3, for all $\bar{v} \in \mathbb{R}^n$ such that $|\bar{v}| \le \delta$, there exists $\psi \in \mathcal{S}_\mu(\xi+\bar{v})$ such that

$$|\phi^*(t,\xi) - \psi(t,\xi+\bar{v})| \le L|\bar{v}| \;\; \forall t \in [0,T_\xi] \; . \tag{6.32}$$

We define $\delta_1 := \min\left\{\delta, \frac{1}{L}, \frac{1}{L}\left(\ell_{j+1} - \ell_{j+2}\right), \frac{1}{2}\left(\ell_j - \ell_{j+1}\right)\right\}$. Henceforth we assume $|\bar{v}| \le \delta_1$. It follows with (6.24) that

$$|\xi+\bar{v}|_{\mathcal{A}} \ge |\xi|_{\mathcal{A}} - \frac{1}{2}\left(\ell_j - \ell_{j+1}\right) \ge |\zeta|_{\mathcal{A}} - \left(\ell_j - \ell_{j+1}\right) \ge \ell_{j+1} \tag{6.33}$$

and with (6.32) that

$$|\phi^*(t,\xi) - \psi(t,\xi+\bar{v})| \le L|\bar{v}| \le \min\left\{1, \left(\ell_{j+1} - \ell_{j+2}\right)\right\} \quad \forall t \in [0,T_\xi]. \tag{6.34}$$

The relations (6.34), (6.30), and (6.31) imply that

$$|\psi(t,\xi+\bar{v})|_{\mathcal{A}} \le |\phi^*(t,\xi)|_{\mathcal{A}} + 1 \le 2+\mu \;\; \forall t \in [0,T_\xi] \text{ and} \tag{6.35}$$

$$|\psi(T_\xi,\xi+\bar{v})|_{\mathcal{A}} \le |\phi^*(T_\xi,\xi)|_{\mathcal{A}} + \ell_{j+1} - \ell_{j+2} = \ell_{j+1} \; . \tag{6.36}$$

It follows from (6.35) and the definition of $F_\mu(x)$ that

$$\psi \in \mathcal{S}[0,T_\xi](\xi+\bar{v}) \; . \tag{6.37}$$

The inequalities (6.33) and (6.36) imply that the set $\{t : |\psi(t,\xi+\bar{v})|_{\mathcal{A}} = \ell_{j+1}\}$ is nonempty and we can define $T_{\psi,\bar{v}} := \inf\{t : |\psi(t,\xi+\bar{v})|_{\mathcal{A}} = \ell_{j+1}\}$. We see that $T_{\psi,\bar{v}} \le T_\xi \le \overline{T}$, and, since $\psi(\cdot,\xi+\bar{v})$ is continuous, $|\psi(T_{\psi,\bar{v}},\xi+\bar{v})|_{\mathcal{A}} = \ell_{j+1}$. For clarity, define

$z := \psi(T_{\psi,\bar v}, \xi + \bar v)$. From the third assumption of the lemma $g(z) = V(z)$; so, with (6.37) and $\kappa(\tau)\kappa(T_{\psi,\bar v}) \geq \kappa(\tau + T_{\psi,\bar v})$ for all $\tau \geq 0$, $T_{\psi,\bar v} \geq 0$, we have

$$\begin{aligned}
\max_{t\in[0,T_{\psi,\bar v}]} g(\psi(t,\xi+\bar v))\kappa(t) &= \max\left\{\max_{t\in[0,T_{\psi,\bar v}]} g(\psi(t,\xi+\bar v))\kappa(t), V(z)\kappa(T_{\psi,\bar v})\right\} \\
&= \max\left\{\max_{t\in[0,T_{\psi,\bar v}]} g(\psi(t,\xi+\bar v))\kappa(t), \inf_{\phi\in\mathcal{S}[0,\infty)(z)} \sup_{\tau\geq 0} g(\phi(\tau,z))\kappa(\tau)\kappa(T_{\psi,\bar v})\right\} \\
&\geq \inf_{\phi\in\mathcal{S}[0,\infty)(\xi+\bar v)} \sup_{t\geq 0} g(\phi(t,\xi+\bar v))\kappa(t) \;=\; V(\xi+\bar v). \qquad (6.38)
\end{aligned}$$

One consequence of this calculation, which comes by taking $\bar v = 0$ and $\psi = \phi^*$ so that (6.32) is satisfied, is that for each $\xi \in \mathcal{U}$,

$$V(\xi) \geq \max_{t\in[0,T(|\xi|_{\mathcal{A}})]} g(\phi^*(t,\xi))\kappa(t) \geq \max_{t\in[0,T_{\phi^*,0}]} g(\phi^*(t,\xi))\kappa(t) \geq V(\xi)\ ;$$

i.e., $\max_{t\in[0,T(|\xi|_{\mathcal{A}})]} g(\phi^*(t,\xi))\kappa(t) = V(\xi)$. Let $L_{\mathcal{C}} > 0$ be such that $|g(x_1) - g(x_2)| \leq L_{\mathcal{C}}|x_1 - x_2|$ for all $x_1, x_2 \in \mathcal{C} + \overline{\mathcal{B}}_n$. From (6.29), (6.32), and (6.38) it follows that

$$\begin{aligned}
V(\xi) &\geq \max_{t\in[0,T_{\psi,\bar v}]} g(\phi^*(t,\xi))\kappa(t) \geq \max_{t\in[0,T_{\psi,\bar v}]} g(\psi(t,\xi+\bar v))\kappa(t) - LL_{\mathcal{C}}|\bar v|\kappa(\overline{T}) \\
&\geq V(\xi+\bar v) - LL_{\mathcal{C}}|\bar v|\kappa(\overline{T})\ .
\end{aligned}$$

Therefore, with the definition $M := LL_{\mathcal{C}}\kappa(\overline{T})$ for all $\xi \in \mathcal{U}$ and all $v \in \mathbb{R}^n$, we have

$$DV(\xi; v) = \liminf_{w\to v,\varepsilon\to 0^+} \frac{V(\xi+\varepsilon w) - V(\xi)}{\varepsilon} \leq \liminf_{w\to v,\varepsilon\to 0^+} LL_{\mathcal{C}}|w|\kappa(\overline{T}) = M|v|\ .$$

Having already shown that $V(\cdot)$ is lower semicontinuous on $\mathbb{R}^n$ (Claim 4), it follows from Lemma 5.1 that $V(\cdot)$ is locally Lipschitz on the open, convex set $\mathcal{U}$ with Lipschitz constant M. Since $\zeta \in \mathbb{R}^n\backslash\mathcal{A}$ was arbitrary, $V(\cdot)$ is locally Lipschitz on $\mathbb{R}^n\backslash\mathcal{A}$. With this Lipschitz property and the relation $0 \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$, it follows that $V(\cdot)$ is continuous on $\mathbb{R}^n$. □

REFERENCES

[1] F. Albertini and E. D. Sontag, *Continuous control-Lyapunov functions for asymptotically controllable time-varying systems*, Internat. J. Control, 72 (1999), pp. 1630–1641.

[2] Z. Artstein, *Stabilization with relaxed controls*, Nonlinear Anal., 7 (1983), pp. 1163–1173.

[3] J. P. Aubin and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Springer-Verlag, Berlin, 1984.

[4] R. W. Brockett, *Asymptotic stability and feedback stabilization*, in Differential Geometric Control Theory, Birkhäuser Boston, Cambridge, MA, 1982, pp. 181–191.

[5] F. H. Clarke, Yu. S. Ledyaev, L. Rifford, and R. J. Stern, *Feedback stabilization and Lyapunov functions*, SIAM J. Control Optim., 39 (2000), pp. 25–48.

[6] F. H. Clarke, Y. S. Ledyaev, E. D. Sontag, and A. I. Subbotin, *Asymptotic controllability implies feedback stabilization*, IEEE Trans. Automat. Control, 42 (1997), pp. 1394–1407.

[7] F. H. Clarke, Y. S. Ledyaev, and R. J. Stern, *Asymptotic stability and smooth Lyapunov functions*, J. Differential Equations, 149 (1998), pp. 69–114.

[8] F. H. Clarke, Y. S. Ledyaev, R. J. Stern, and P. R. Wolenski, *Nonsmooth Analysis and Control Theory*, Springer-Verlag, New York, 1998.

[9] F. H. Clarke, R. J. Stern, and P. R. Wolenski, *Subgradient criteria for monotonicity, the Lipschitz condition, and convexity*, Canad. J. Math., 45 (1993), pp. 1167–1183.

[10] K. Deimling, *Multivalued Differential Equations*, Walter de Gruyter, Berlin, 1992.

[11] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.

[12] R. Freeman and P. Kokotović, *Robust Nonlinear Control Design: State-Space and Lyapunov Techniques*, Birkhäuser Boston, Cambridge, MA, 1996.
[13] C. M. Kellett, *Advances in Converse and Control Lyapunov Functions*, Ph.D. thesis, University of California, Santa Barbara, CA, 2002.
[14] C. M. Kellett and A. R. Teel, *A converse Lyapunov theorem for weak uniform asymptotic stability of sets*, in Proceedings of Mathematical Theory of Networks and Systems, Perpignan, France, 2000.
[15] C. M. Kellett and A. R. Teel, *Uniform asymptotic controllability to a set implies locally Lipschitz control-Lyapunov function*, in Proceedings of the 39th IEEE Conference on Decision and Control, Sydney, Australia, 2000.
[16] M. Krstić, I. Kanellakopoulos, and P. Kokotović, *Nonlinear and Adaptive Control Design*, John Wiley and Sons, New York, 1995.
[17] J. Kurzweil, *On the inversion of Ljapunov's second theorem on stability of motion*, Amer. Math. Soc. Transl. Ser. 2, 24 (1956), pp. 19–77.
[18] Y. Lin, E. D. Sontag, and Y. Wang, *A smooth converse Lyapunov theorem for robust stability*, SIAM J. Control Optim., 34 (1996), pp. 124–160.
[19] L. Praly and Y. Wang, *Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability*, Math. Control Signals Systems, 9 (1996), pp. 1–33.
[20] L. Rifford, *Existence of Lipschitz and semiconcave control-Lyapunov functions*, SIAM J. Control Optim., 39 (2000), pp. 1043–1064.
[21] E. Roxin, *Stability in general control systems*, J. Differential Equations, 1 (1965), pp. 115–150.
[22] E. Roxin, *On asymptotic stability in control systems*, Rend. Circ. Mat. Palermo (2), 15 (1966), pp. 193–208.
[23] E. P. Ryan, *On Brockett's condition for smooth stabilizability and its necessity in a context of nonsmooth feedback*, SIAM J. Control Optim., 32 (1994), pp. 1597–1604.
[24] E. D. Sontag, *A Lyapunov-like characterization of asymptotic controllability*, SIAM J. Control Optim., 21 (1983), pp. 462–471.
[25] E. D. Sontag, *A "universal" construction of Artstein's theorem on nonlinear stabilization*, Systems Control Lett., 13 (1989), pp. 117–123.
[26] E. D. Sontag, *Comments on integral variants of ISS*, Systems Control Lett., 34 (1998), pp. 93–100.
[27] E. D. Sontag, *Stability and stabilization: Discontinuities and the effect of disturbances*, in Nonlinear Analysis, Differential Equations, and Control, NATO Sci. Ser. C. Math. Phys. Sci. 528, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999, pp. 551–598.
[28] E. D. Sontag, *Clocks and insensitivity to small measurement errors*, ESAIM Control Optim. Calc. Var., 4 (1999), pp. 537–557.
[29] E. D. Sontag and H. J. Sussman, *Nonsmooth control-Lyapunov functions*, in Proceedings of the 34th Conference on Decision and Control, New Orleans, LA, 1995, pp. 2799–2805.
[30] A. R. Teel and L. Praly, *A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions*, ESAIM Control Optim. Calc. Var., 5 (2000), pp. 313–367.
[31] F. W. Wilson, *Smoothing derivatives of functions and applications*, Trans. Amer. Math. Soc., 139 (1969), pp. 413–428.