SIAM J. CONTROL OPTIM.
Vol. 44, No. 3, pp. 777–800

ON THE ROBUSTNESS OF $\mathcal{KL}$-STABILITY FOR DIFFERENCE INCLUSIONS: SMOOTH DISCRETE-TIME LYAPUNOV FUNCTIONS*

CHRISTOPHER M. KELLETT† AND ANDREW R. TEEL‡

Abstract. We consider stability with respect to two measures of a difference inclusion, i.e., of a discrete-time dynamical system with the push-forward map being set-valued. We demonstrate that robust stability is equivalent to the existence of a smooth Lyapunov function and that, in fact, a continuous Lyapunov function implies robust stability. We also present a sufficient condition for robust stability that is independent of a Lyapunov function. Toward this end, we develop several new results on the behavior of solutions of difference inclusions. In addition, we provide a novel result for generating a smooth function from one that is merely upper semicontinuous.

Key words. difference inclusions, stability with respect to two measures, Lyapunov functions, robustness, smoothing upper-semicontinuous functions

AMS subject classifications. 39A12, 93C55, 93D09, 93D30

DOI. 10.1137/S0363012903435862

1. Introduction.

The close connection between robustness of stability properties for differential equations and the existence of Lyapunov functions has been implicit in the literature since the result of Kurzweil [13]. In particular, Kurzweil exploited the inherent robustness of asymptotic stability of the origin for differential equations defined by a continuous right-hand side in order to demonstrate the existence of a smooth Lyapunov function. Since Kurzweil, robustness of the assumed stability property has played a key role in deriving Lyapunov functions. Results on the existence of Lyapunov functions for asymptotically stable closed sets became available in the 1960s in the works of Hoppensteadt [7] and Wilson [23]. These results were extended by Lin, Sontag, and Wang [15] to consider asymptotic stability of closed sets for locally Lipschitz differential equations subject to disturbances. Recently, Clarke, Ledyaev, and Stern [4] demonstrated the existence of a smooth Lyapunov function for upper-semicontinuous differential inclusions with an asymptotically stable origin.

Rather than considering differential inclusions, we will consider the difference inclusion

$$x^+ \in F(x), \quad x \in \mathcal{G}, \tag{1.1}$$

where $\mathcal{G} \subseteq \mathbb{R}^n$ is open. Difference inclusions are a natural way to consider difference equations subject to disturbances or controlled difference equations. One may consider a set-valued map as

$$x^+ \in F(x) := f(x, \mathcal{V}),$$

where $\mathcal{V}$ is a set of disturbances or the admissible control set. We use $\phi \in \mathcal{S}(x)$ to denote a solution of the difference inclusion (1.1) from initial condition $x \in \mathcal{G}$, i.e., a

*Received by the editors October 10, 2003; accepted for publication (in revised form) March 6, 2005; published electronically September 15, 2005. This work was partially supported by AFOSR grants F49620-00-1-0106 and -03-1-0203 and by NSF grant ECS-9988813.

http://www.siam.org/journals/sicon/44-3/43586.html

†The Hamilton Institute, National University of Ireland, Maynooth, Ireland (chris.kellett@nuim.ie).

‡Department of Electrical and Computer Engineering, University of California, Santa Barbara, CA 93106 (teel@ece.ucsb.edu).

function satisfying $\phi(0, x) = x$ and

$$\phi(k+1, x) \in F(\phi(k, x)) \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

Whereas in the continuous-time case a solution was an absolutely continuous function, in the discrete-time case solutions are sequences of points. Solutions are defined for all $k \in \mathbb{Z}_{\geq 0}$ when $F(\cdot)$ maps $\mathcal{G}$ to subsets of $\mathcal{G}$, which is the discrete-time counterpart to forward completeness for continuous-time systems.

In the 1970s, Lakshmikantham and Salvadori [14] demonstrated a locally Lipschitz Lyapunov function for a differential equation under the assumption of stability with respect to two measures, a concept first introduced by Movchan [17]. Stability with respect to two measures can be seen to cover uniform global or local asymptotic stability of a point, prescribed motion, or closed set. In fact, Teel and Praly [22, Proposition 1] (following [15, Proposition 2.5]) demonstrated that $\mathcal{KL}$-stability with respect to two measures is equivalent to uniform stability and global boundedness coupled with uniform global attractivity (both properties being defined in an appropriate two-measure sense). In Proposition 2.2 we show that this property carries over to the discrete-time case.

A smooth Lyapunov function for output stability, a special case of stability with respect to two measures where one of the measures is the norm of the output function, was presented by Sontag and Wang [20, Theorem 2]. Teel and Praly [22] extended these results to consider the existence of a smooth Lyapunov function under the assumption of $\mathcal{KL}$-stability with respect to two measures for differential inclusions. It is this last result by Teel and Praly [22, Theorem 1] that we propose to develop in the discrete-time case, namely, the equivalence of robustness of $\mathcal{KL}$-stability with respect to two measures for a difference inclusion and the existence of a smooth Lyapunov function. This is the result of Theorem 2.7.

In Theorem 2.8 we present a result stating that when the set-valued map defining the difference inclusion (1.1) is compact and nonempty, a continuous Lyapunov function is sufficient to demonstrate robustness. This result has important implications in robustness analysis. The authors used this fact in [11, Theorem 14] to demonstrate robustness for a (discontinuous) difference equation. Frequently, in model predictive control, a continuous Lyapunov function is assumed (see Mayne et al. [16]) which guarantees robustness of stability. Recently, Grimm et al. [6] presented several examples where model predictive control is nonrobust. Intuitively, these results follow from the lack of a continuous Lyapunov function.

A question of great interest over many years is the so-called converse Lyapunov question, namely, what stability requirements guarantee the existence of a Lyapunov function? We see from Theorem 2.7 that, for $\mathcal{KL}$-stability with respect to two measures, this question is reduced to that of finding sufficient conditions for robustness. The result of Theorem 2.10 states that if the difference inclusion $x^+ \in F(x)$ is $\mathcal{KL}$-stable, the set-valued map $F(x)$ is nonempty and compact for each $x \in \mathcal{G}$, and $F(\cdot)$ is continuous, then the $\mathcal{KL}$-stability is robust. In [9] and [10], other sufficient conditions were presented for robustness of $\mathcal{KL}$-stability. For example, $\mathcal{KL}$-stability is robust when using a single measurement function that is a proper indicator function for a compact attractor. Each of these sufficient conditions then allows us to state a converse Lyapunov theorem.

Previous converse Lyapunov theorems for discrete-time systems appeared in books by Agarwal [1, Theorem 5.12.5] and Stuart and Humphries [21, Theorem 1.7.6], where uniform global asymptotic stability of the origin or a compact attractor for a locally

Lipschitz single-valued mapping yields a locally Lipschitz Lyapunov function. Nešić, Teel, and Kokotović [18] demonstrated the equivalence of uniform global asymptotic stability of the origin for a difference equation (with no regularity) and the existence of a Lyapunov function (with no regularity).

Jiang and Wang [8, Theorem 1] showed that uniform global asymptotic stability to a closed set $\mathcal{A}$ for a difference equation with disturbances is equivalent to the existence of a smooth Lyapunov function under the assumption that the difference equation is continuous. The assumption of continuity on the difference equation (and compactness of the set of allowable disturbances) gives rise to a continuous set-valued map. This result can then be seen to be a special case of Theorem 2.7 and Theorem 2.10 with $\omega_1(\cdot) = \omega_2(\cdot) = |\cdot|_{\mathcal{A}}$, where $|x|_{\mathcal{A}} := \inf_{a\in\mathcal{A}} |x-a|$.

The authors [11] demonstrated that global asymptotic stability of a point for an upper-semicontinuous difference inclusion implied the existence of a smooth Lyapunov function. This result follows from the results presented here and in [10] (see also [9]).

We will require two sets of technical results, heretofore unknown in the literature. In section 5 we develop results pertaining to difference inclusions which parallel those found in the work of Filippov [5] for differential inclusions. Specifically, we prove results on closeness of solutions under perturbations (Lemmas 5.1 and 5.2) as well as on uniform convergence of sequences of solutions (Lemma 5.3). The second novel technical result involves smoothing nonsmooth functions on a given open domain. As in much previous work (e.g., [13], [15], [22], and [23]), we first construct a Lyapunov function satisfying the desired decrease condition, but with a rather weak regularity property, and then apply a smoothing result to obtain the smooth Lyapunov function without destroying the decrease property. In the past, these smoothing results applied to continuous functions. In section 3, we present a novel smoothing theorem which obtains a smooth function from one that is upper semicontinuous.

2. Smooth Lyapunov functions and robustness. We now turn to precise statements of our results. Recall that a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. A function is of class-$\mathcal{K}_\infty$ if, in addition to being class-$\mathcal{K}$, it is unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to belong to class-$\mathcal{KL}$ if, for each $t \geq 0$, $\beta(\cdot, t)$ is of class-$\mathcal{K}$ and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,t) = 0$.

Definition 2.1. *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous functions. Let $F(\cdot)$ be a set-valued map from $\mathcal{G}$ to subsets of $\mathcal{G}$. We say that the difference inclusion $x^+ \in F(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$ if there exists a function $\beta \in \mathcal{KL}$ such that for every initial condition $x \in \mathcal{G}$ all solutions $\phi \in \mathcal{S}(x)$ satisfy*

$$\omega_1(\phi(k,x)) \leq \beta(\omega_2(x), k) \quad \forall k \in \mathbb{Z}_{\geq 0}. \tag{2.1}$$

Note that appropriate choices for the measurement functions $\omega_1(\cdot)$ and $\omega_2(\cdot)$ as well as the domain $\mathcal{G}$ allow us to recover several classical stability notions. For instance, global asymptotic stability of the origin (for a given difference inclusion evolving in $\mathbb{R}^n$) corresponds to taking $\mathcal{G} = \mathbb{R}^n$ and the measurement functions $\omega_1(x) = \omega_2(x) = |x|$ for all $x \in \mathbb{R}^n$. Other stability notions, such as local asymptotic stability or partial state stability, can be covered by appropriately choosing the domain and measurement functions.

Lin et al. [15, Proposition 2.5] demonstrated that $\mathcal{KL}$-stability with respect to $(|\cdot|_{\mathcal{A}}, |\cdot|_{\mathcal{A}})$ (where $\mathcal{A}$ is a closed set) is equivalent to uniform stability and uniform attractivity of the set $\mathcal{A}$ (i.e., $\mathcal{KL}$-stability is equivalent to uniform global asymptotic stability of the set $\mathcal{A}$). Teel and Praly [22, Proposition1] extended this result to the consideration of the general two-measure case; that is, $\mathcal{KL}$-stability with respect to two measures is equivalent to uniform stability and global boundedness coupled with uniform global attractivity, where these properties are defined in an appropriate two-measure sense. This result also holds in the discrete time. The details are similar to the continuous-time result and may be found in [9].

PROPOSITION 2.2. *Let $\omega_i : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $i = 1, 2$, be continuous and let $F(\cdot)$ be a set-valued map from $\mathcal{G}$ to subsets of $\mathcal{G}$. The following are equivalent:*

1. *The difference inclusion $x^+ \in F(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.*
2. *The following hold:*
   (a) *(Uniform stability and global boundedness): There exists a function $\gamma \in \mathcal{K}_\infty$ such that, for each $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}(x)$ satisfy*
   $$\omega_1(\phi(k, x)) \leq \gamma(\omega_2(x)) \quad \forall k \in \mathbb{Z}_{\geq 0}.$$
   (b) *(Uniform global attractivity): For each $r, \varepsilon > 0$, there exists $K(r, \varepsilon) > 0$ such that, for each $x \in \mathcal{G}$, all solutions $\phi \in \mathcal{S}(x)$ satisfy*
   $$\omega_2(x) \leq r, \quad k \geq \mathbb{Z}_{\geq K} \quad \Longrightarrow \quad \omega_1(\phi(k, x)) \leq \varepsilon.$$

For a continuous function $\sigma : \mathcal{G} \to \mathbb{R}_{\geq 0}$ we define the σ-perturbation of $F(\cdot)$ as

$$F_\sigma(x) := \left\{ v \in \mathbb{R}^n : v \in \{\eta\} + \sigma(\eta)\overline{\mathcal{B}}, \eta \in F\left(x + \sigma(x)\overline{\mathcal{B}}\right) \right\}. \tag{2.2}$$

We denote the solution set of the difference inclusion $x^+ \in F_\sigma(x)$ starting from an initial condition $x \in \mathcal{G}$ by $\mathcal{S}_\sigma(x)$. We will use $\mathcal{B}$ (or $\overline{\mathcal{B}}$) to denote the open (or closed) unit ball in $\mathbb{R}^n$. For two sets $\mathcal{O}_1$ and $\mathcal{O}_2$, we denote the intersection of $\mathcal{O}_1$ and the complement of $\mathcal{O}_2$ by $\mathcal{O}_1 \backslash \mathcal{O}_2$.

The following set will be used in what follows:

$$\mathcal{A} := \left\{ \xi \in \mathcal{G} : \sup_{k \in \mathbb{Z}_{\geq 0}, \phi \in \mathcal{S}(\xi)} \omega_1(\phi(k, \xi)) = 0 \right\}. \tag{2.3}$$

In most cases the set $\mathcal{A}$ will be nonempty, but we observe that this is not necessary for the following results to hold. When $\mathcal{A}$ is empty, we define $|x|_{\mathcal{A}} = \inf_{a \in \mathcal{A}} |x - a|$ to be infinite.

For stability with respect to (ω, ω) ($\omega : \mathcal{G} \to \mathbb{R}_{\geq 0}$ continuous) the closed set $\mathcal{A}$ is

$$\mathcal{A} := \{x \in \mathcal{G} : \omega(x) = 0\}.$$

This follows from the previous definition (2.3) by examining the $\mathcal{KL}$-estimate defining stability. Specifically, if

$$\omega(\phi(k, x)) \leq \beta(\omega(x), k) \quad \forall x \in \mathcal{G}, \ \phi \in \mathcal{S}(x), \ k \in \mathbb{Z}_{\geq 0},$$

then, for $\xi \in \mathcal{G}$, $\omega(\xi) = 0$ if and only if $\sup_{k \in \mathbb{Z}_{\geq 0}, \phi \in \mathcal{S}(\xi)} \omega(\phi(k, \xi)) = 0$.

Our robustness definition is defined relative to the above σ-perturbation.

DEFINITION 2.3. *Let $F(\cdot)$ be a set-valued map from $\mathcal{G}$ to subsets of $\mathcal{G}$. We say that the difference inclusion $x^+ \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$ if there exists a continuous function $\sigma : \mathcal{G} \to \mathbb{R}_{\geq 0}$ such that*

1. *for all $x \in \mathcal{G}$, $\{x\} + \sigma(x)\overline{\mathcal{B}} \subset \mathcal{G}$;*
2. *for all $x \in \mathcal{G} \backslash \mathcal{A}$, $\sigma(x) > 0$;*

3.

$$\mathcal{A} = \mathcal{A}_\sigma := \left\{ \xi \in \mathcal{G} : \sup_{k \in \mathbb{Z}_{\geq 0},\ \phi \in \mathcal{S}_\sigma(\xi)} \omega_1(\phi(k,\xi)) = 0 \right\}; \quad \text{and} \tag{2.4}$$

4. *the difference inclusion* $x^+ \in F_\sigma(x)$ *is* $\mathcal{KL}$*-stable with respect to* (ω_1, ω_2) *on* $\mathcal{G}$.

In what follows, we denote the exponential function by e.

DEFINITION 2.4. *A function* $V : \mathcal{G} \to \mathbb{R}_{\geq 0}$ *is said to be a Lyapunov function with respect to* (ω_1, ω_2) *on* $\mathcal{G}$ *for the difference inclusion* $x^+ \in F(x)$ *if there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *such that for all* $x \in \mathcal{G}$,

$$\alpha_1(\omega_1(x)) \leq V(x) \leq \alpha_2(\omega_2(x)), \tag{2.5}$$

$$\sup_{f \in F(x)} V(f) \leq V(x)e^{-1}, \quad \text{and} \tag{2.6}$$

$$V(x) = 0 \iff x \in \mathcal{A}. \tag{2.7}$$

We claim that the above decrease condition (2.6) can be stated as

$$\sup_{f \in F(x)} V(f) \leq V(x) - \alpha(V(x)) \quad \forall x \in \mathcal{G}, \tag{2.8}$$

where $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous and positive definite. However, we prefer (2.6) because of the symmetry with the continuous time decrease condition

$$\sup_{\omega \in F(x)} \langle \nabla V(x), \omega \rangle \leq -V(x),$$

which yields an exponential decrease of the Lyapunov function along trajectories. The following claim is proved in section 8.

CLAIM 1. *Suppose we are given functions* $V : \mathcal{G} \to \mathbb{R}_{\geq 0}$, $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, *and* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *satisfying* (2.5), (2.7), *and* (2.8). *Then there exist* $W : \mathcal{G} \to \mathbb{R}_{\geq 0}$ *and functions* $\hat{\alpha}_1$, $\hat{\alpha}_2 \in \mathcal{K}_\infty$ *such that for all* $x \in \mathcal{G}$

$$\hat{\alpha}_1(\omega_1(x)) \leq W(x) \leq \hat{\alpha}_2(\omega_2(x)), \tag{2.9}$$

$$\sup_{f \in F(x)} W(f) \leq e^{-1}W(x), \quad \text{and} \tag{2.10}$$

$$W(x) = 0 \iff x \in \mathcal{A}. \tag{2.11}$$

Prior to stating our first result we require two definitions related to set-valued maps.

DEFINITION 2.5. *The set-valued map* $F(\cdot)$ *is said to be upper semicontinuous on (the open set)* $\mathcal{O}$ *if for each* $x \in \mathcal{O}$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that, for all* $\xi \in \mathcal{O}$ *satisfying* $|x - \xi| < \delta$, *we have* $F(\xi) \subseteq F(x) + \varepsilon\mathcal{B}$.

We point out that the concept of upper semicontinuity for a set-valued map is not the same as that for extended real-valued functions. In fact, for $f : \mathbb{R}^n \to \mathbb{R}^n$, the set-valued map $x \mapsto \{f(x)\}$ is upper semicontinuous if and only if $x \mapsto f(x)$ is continuous.

DEFINITION 2.6. *We say that the set-valued map* $F(\cdot)$ *satisfies the basic conditions on* $\mathcal{G}$ *if* $F(\cdot)$ *is upper semicontinuous on* $\mathcal{G}$ *and, for each* $x \in \mathcal{G}$, $F(x)$ *is nonempty and compact.*

In continuous time the "basic conditions" also require convexity of $F(x)$ for each $x \in \mathcal{G}$. This is necessary to guarantee solutions of the differential inclusion $\dot{x} \in F(x)$ (see [5]). Obviously, solutions to the difference inclusion $x^+ \in F(x)$ will exist so long as $F(x)$ is nonempty.

With all the necessary definitions in hand, we may state under what conditions robust stability is equivalent to the existence of a Lyapunov function. The following is the discrete-time analogue of [22, Theorem 1] and is proved in section 6.

THEOREM 2.7. *Let $F(\cdot)$ mapping $\mathcal{G}$ to subsets of $\mathcal{G}$ satisfy the basic conditions on $\mathcal{G}$. Then, for the difference inclusion $x^+ \in F(x)$, there exists a smooth Lyapunov function with respect to (ω_1, ω_2) on $\mathcal{G}$ if and only if the inclusion is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.*

Remark 1. We note that (2.4) and (2.7) were not required in the corresponding definitions of robustness and a Lyapunov function in [22]. The addition of (2.4) to the definition of robustness significantly simplifies the proof. In order to maintain the equivalence of robustness and the existence of a Lyapunov function, one would then expect that an extra property is required of $V(\cdot)$. This property is (2.7). This is not unreasonable as, in the case of a single measure, we see that the upper and lower bounds (2.5) actually imply (2.7). □

It is possible to weaken the conditions of Theorem 2.7 and still maintain the necessity. This means that, in order to demonstrate robustness, it is only necessary to exhibit a continuous Lyapunov function (rather than a smooth one). Furthermore, note that we can drop the regularity requirement on the set-valued map. This allows application of the theorem, for example, to the consideration of discontinuous difference equations.

THEOREM 2.8. *Let $F(\cdot)$ mapping $\mathcal{G}$ to subsets of $\mathcal{G}$ be compact and nonempty, and suppose we have a continuous Lyapunov function. Then $x^+ \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.*

Since Lyapunov functions can sometimes be difficult to find, we would like a sufficient condition for robustness that is independent of having a Lyapunov function. Intuitively, if the set-valued map $F(\cdot)$ of (1.1) is sufficiently regular, robustness should follow since small perturbations will lead to small deviations. In fact, continuity of $F(\cdot)$ outside of the set $\mathcal{A}$ is sufficient.

DEFINITION 2.9. *We say the set-valued map $F(\cdot)$ is continuous on (the open set) $\mathcal{O}$ if, in addition to being upper semicontinuous on $\mathcal{O}$, for each $x \in \mathcal{O}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that, for $z \in \mathcal{O}$ satisfying $|z - x| < \delta$, we have $F(x) \subseteq F(z) + \varepsilon\mathcal{B}$.*

The following theorem is the discrete-time counterpart of [22, Theorem 2] and is proved in section 7.

THEOREM 2.10. *Let $F(\cdot)$ be a set-valued map from $\mathcal{G}$ to subsets of $\mathcal{G}$ satisfying the basic conditions on $\mathcal{G}$ and continuous on an open set containing $\mathcal{G}\backslash\mathcal{A}$. Under these conditions, if $x^+ \in F(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$, then the inclusion is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$.*

3. Smoothing functions. Frequently one wishes to prove that certain assumptions such as asymptotic stability of a set or asymptotic controllability to a set imply the existence of a function satisfying certain boundedness and decrease properties, as well as a given regularity property. Typically, one constructs a function which satisfies all the given properties (i.e., boundedness and decrease properties) excepting the desired regularity property. One may then take the additional step of "smoothing" the constructed function without destroying the boundedness or decrease properties. Such techniques were first used by Kurzweil [13] and Wilson [23]. Throughout this section, we take $\mathcal{O}$ to be an open set.

We will smooth nonsmooth functions via an integration which involves a change of variables. We will require the following assumption on the function we wish to smooth.

ASSUMPTION 1. *The function* $V : \mathcal{O} \to \mathbb{R}_{\geq 0}$ *is such that*

1. $V(\cdot)$ *is upper semicontinuous and locally bounded on* $\mathcal{O}$*,*
2. $V(x) > 0$ *implies that there exists* $\delta > 0$ *such that* $|z-x| < \delta$ *implies* $V(z) > 0$.

Define

$$\mathcal{A} := \{x \in \mathcal{O} : V(x) = 0\}. \tag{3.1}$$

We observe that, under the above assumption on $V(\cdot)$, the set $\mathcal{O}\backslash\mathcal{A}$ is open. Note that we need not assume that $\mathcal{A}$ is nonempty.

We will also require an assumption on the "smoothing perturbation."

ASSUMPTION 2. *The smooth function* $\sigma : \mathcal{O}\backslash\mathcal{A} \to \mathbb{R}_{>0}$ *satisfies the following:*

1. *for each* $x^* \in \mathcal{A}$ *and* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that*

$$x \in \mathcal{O}\backslash\mathcal{A}\ , \quad |x - x^*| \leq \delta \qquad \Longrightarrow \qquad \sigma(x) \leq \varepsilon, \tag{3.2}$$

2.

$$x \in \mathcal{O}\backslash\mathcal{A} \qquad \Longrightarrow \qquad \{x\} + \sigma(x)\overline{\mathcal{B}} \subset \mathcal{O}. \tag{3.3}$$

Item 1 implies that the function $\sigma(\cdot)$ can be continuously extended to the set $\mathcal{A}$ by defining it to be identically zero on $\mathcal{A}$. For the case where $\mathcal{A}$ is empty, item 1 is trivially satisfied.

We define $V_s : \mathcal{O} \to \mathbb{R}_{\geq 0}$ as

$$\begin{aligned} V_s(x) &:= 0, && x \in \mathcal{A}, \\ V_s(x) &:= \int V(x + \sigma(x)\xi)\psi(\xi)\, d\xi, && x \in \mathcal{O}\backslash\mathcal{A}, \end{aligned} \tag{3.4}$$

where $\psi : \mathbb{R}^n \to [0, 1]$ is smooth, vanishes on $\mathbb{R}^n\backslash\overline{\mathcal{B}}$, and satisfies $\int \psi(\xi)d\xi = 1$.

The following theorem is a generalization of [11, Theorem 20], where the smoothing was carried out on $\mathbb{R}^n\backslash\{0\}$.

THEOREM 3.1. *Under Assumptions* 1 *and* 2*, the function* $V_s : \mathcal{O} \to \mathbb{R}_{\geq 0}$ *defined by* (3.4) *is well defined, continuous on* $\mathcal{O}$*, and smooth on* $\mathcal{O}\backslash\mathcal{A}$*.*

Proof. The properties of $\sigma(\cdot)$ and $\psi(\cdot)$ and the upper semicontinuity of $V(\cdot)$ guarantee that the (Lebesgue) integral in (3.4) is well defined.

Continuity at $\mathcal{A}$. Since $V_s(x) \equiv 0$ for $x \in \mathcal{A}$, the function is clearly continuous in the interior of $\mathcal{A}$. It remains to check continuity at the boundary of $\mathcal{A}$. Let x^* belong to the boundary of $\mathcal{A}$ so that $V_s(x^*) = 0$. Let $\varepsilon > 0$ be given. Since $V(\cdot)$ is upper semicontinuous, there exists $\delta_2 > 0$ such that $V(z) \leq \varepsilon$ for all $z \in \mathcal{O}$ satisfying $|z - x^*| \leq \delta_2$. Since (3.2) holds, there exists $\delta > 0$ such that $\{x\} + \sigma(x)\overline{\mathcal{B}} \subseteq \{x^*\} + \delta_2\overline{\mathcal{B}}$ for all $|x - x^*| \leq \delta$. Consequently, with the fact that $\int \psi(\xi)d\xi = 1$, if $|x - x^*| \leq \delta$ and $x \in \mathcal{O}\backslash\mathcal{A}$, then

$$V_s(x) = \int V(x + \sigma(x)\xi)\psi(\xi)d\xi \leq \sup_{z \in \{x^*\} + \delta_2\overline{\mathcal{B}}} V(z) \leq \varepsilon;$$

i.e., $V_s(x)$ is continuous for x in the boundary of $\mathcal{A}$.

Finally, if we can establish that V_s is smooth on $\mathcal{O}\backslash\mathcal{A}$, then it will be continuous on $\mathcal{O}$.

Smoothness on $\mathcal{O}\backslash\mathcal{A}$. For each $x \in \mathcal{O}\backslash\mathcal{A}$, we perform a change of variables under the integration with $z = x + \sigma(x)\xi$ so that

$$V_s(x) = \sigma(x)^{-n} \int V(z)\psi(\sigma(x)^{-1}(z-x))\,dz.$$

For notational purposes, we define $h(x,z) := \psi(\sigma(x)^{-1}(z-x))$ so that, for each $x \in \mathcal{O}\backslash\mathcal{A}$,

$$V_s(x) = \sigma(x)^{-n} \int V(z)h(x,z)\,dz.$$

For $x, z \in \mathcal{O}\backslash\mathcal{A}$ such that $|z-x| > \sigma(x)$ we note that $h(x,z)$ and all of its higher order partial derivatives with respect to x vanish. From this, (3.4), and the fact that $\psi(\cdot)$ and $\sigma(\cdot)$ are smooth (the latter on $\mathcal{O}\backslash\mathcal{A}$) it follows that each of these partial derivatives is continuous in x uniformly in z.

Because of the properties of $\sigma(\cdot)$, to establish smoothness of $V_s(\cdot)$ on $\mathcal{O}\backslash\mathcal{A}$ it is enough to establish smoothness of

$$W_s(x) := \int V(z)h(x,z)\,dz.$$

We note that, using the mean value theorem, for each $x \in \mathcal{O}\backslash\mathcal{A}$, $\varepsilon > 0$, and $v \in \mathbb{R}^n$, there exists $\lambda \in [0,1]$ such that

$$\begin{aligned}\frac{W_s(x+\varepsilon v) - W_s(x)}{\varepsilon} &= \int V(z)\frac{h(x+\varepsilon v, z) - h(x,z)}{\varepsilon}\,dz \\ &= \int V(z)\langle \nabla h(x+\varepsilon\lambda v, z), v\rangle\,dz \\ &= r(x,\varepsilon,v) + \int V(z)\langle \nabla h(x,z), v\rangle\,dz,\end{aligned}$$

where $r(x,\varepsilon,v) := \int V(z)\langle \nabla h(x+\varepsilon\lambda v, z) - \nabla h(x,z), v\rangle dz$.

Using that $V(\cdot)$ is locally bounded on $\mathcal{O}$, (3.3), the fact that $\nabla h(x,z) = 0$ when $|z-x| > \sigma(x)$, and the continuity of $\nabla h(\cdot, z)$, which is uniform in z, for each $\rho > 0$ and $M > 0$ there exists $\varepsilon^* > 0$ such that if $\varepsilon \in (0, \varepsilon^*]$ and $|v| \leq M$, then $|r(x,\varepsilon,v)| \leq \rho$. It follows that $W_s(\cdot)$ is (Fréchet) differentiable (hence continuous) and

$$\langle \nabla W_s(x), v\rangle = \int V(z)\langle \nabla h(x,z), v\rangle\,dz.$$

Repeating this argument for higher order derivatives, we conclude that $W_s(\cdot)$ is smooth on $\mathcal{O}\backslash\mathcal{A}$. ∎

The following lemma can be applied to the function $V_s(\cdot)$ obtained from Theorem 3.1 in order to obtain a function that is smooth on the entire domain $\mathcal{O}$. The lemma appeared as [22, Lemma 17], which derives from [15, Lemma 4.3] and [13, Theorem 6].

LEMMA 3.2. *Let $\mathcal{A} \subset \mathcal{O}$ be a closed set, and assume that $V_s : \mathcal{O} \to \mathbb{R}_{\geq 0}$ is continuous, the restriction of V_s to $\mathcal{O}\backslash\mathcal{A}$ is smooth, $V_s(x) = 0$ for all $x \in \mathcal{A}$, and $V_s(x) > 0$ for all $x \in \mathcal{O}\backslash\mathcal{A}$. Then there exists a strictly convex function $\rho \in \mathcal{K}_\infty$, smooth on $(0,\infty)$, such that $V := \rho \circ V_s$ is smooth on $\mathcal{O}$.*

Frequently, to construct the function $\sigma(\cdot)$ used in the integral smoothing of Theorem 3.1, we will first specify a constraint which $\sigma(\cdot)$ must satisfy and then take a smaller smooth function.

LEMMA 3.3. *Given a function $\sigma_2 : \mathcal{O} \to \mathbb{R}_{>0}$ bounded away from zero on compact subsets of $\mathcal{O}$ there exists a smooth function $\sigma_1 : \mathcal{O} \to \mathbb{R}_{>0}$, also bounded away from zero on compact subsets of $\mathcal{O}$, such that, for all $x \in \mathcal{O}$, $\sigma_1(x) \leq \sigma_2(x)$.*

Proof. We let $\{\mathcal{U}_i\}_{i=1}^{\infty}$ be a locally finite open cover of $\mathcal{O}$ with $\overline{\mathcal{U}}_i$ a compact subset of $\mathcal{O}$ and let $\{\kappa_i\}_{i=1}^{\infty}$ be a smooth partition of unity on $\mathcal{O}$ subordinate to $\{\mathcal{U}_i\}$. Define $\varepsilon_i := \inf_{\xi \in \mathcal{U}_i} \sigma_2(\xi)$,

$$\sigma_1(x) := \sum_{i=1}^{\infty} \kappa_i(x)\varepsilon_i,$$

and, for each $x \in \mathcal{O}$, $\mathcal{I}_x := \{j : x \in \mathcal{U}_j\}$. The set $\mathcal{I}_x$ is finite for each $x \in \mathcal{O}$. We also note that

$$\max_{j \in \mathcal{I}_x} \varepsilon_j = \max_{j \in \mathcal{I}_x} \inf_{\xi \in \mathcal{U}_j} \sigma_2(\xi) \leq \sigma_2(x).$$

Since $\overline{\mathcal{U}}_i$ is a compact subset of $\mathcal{O}$ for each i and $\sigma_2(\cdot)$ is bounded away from zero on compact subsets of $\mathcal{O}$, we have $\varepsilon_i > 0$ for each i. Thus $\sigma_1(x) > 0$ for all $x \in \mathcal{O}$. Also,

$$\sigma_1(x) \leq \max_{j \in \mathcal{I}_x} \varepsilon_j \leq \sigma_2(x).$$

Finally, σ_1 is smooth on $\mathcal{O}$, inheriting this property from the κ_i. ∎

4. Set-valued maps. Prior to stating our novel results for difference inclusions, we require certain facts from set-valued analysis. Our primary sources for set-valued analysis include the books by Aubin and Cellina [2], Aubin and Frankowska [3], Filippov [5], and Kisielewicz [12].

Given a set-valued map $F(\cdot)$ from an open set $\mathcal{O} \subset \mathbb{R}^n$ to subsets of $\mathbb{R}^n$, we define the mapping of a compact set M by

$$F(M) := \bigcup_{\xi \in M} F(\xi).$$

We also define the composition of two set-valued maps $F(\cdot)$ and $G(\cdot)$ to be

$$F(G(x)) := \bigcup_{w \in G(x)} F(w),$$

and we denote the n-times composition of $F(\cdot)$ with itself by $F^n(\cdot)$ (e.g., $F(F(x)) = F^2(x)$).

The following is well known. See, for example, [12, Proposition 2.3].

CLAIM 2. *Let $F(\cdot)$ be an upper-semicontinuous set-valued map from $\mathcal{O}$ to subsets of $\mathbb{R}^n$, let $M \subset \mathcal{O}$ be compact, and let $F(x)$ be compact for all $x \in \mathcal{O}$. Then the set $F(M)$ is compact.*

For $\delta \geq 0$, we define the δ-perturbation of the set-valued map $F(\cdot)$ by

$$F_\delta(x) := F\left(\{x\} + \delta\overline{\mathcal{B}}\right) + \delta\overline{\mathcal{B}}$$

and the δ-inflation of a set M by

$$M_\delta := M + \delta\overline{\mathcal{B}}.$$

The following claim, which is not difficult to prove, extends the concept of upper semicontinuity to the consideration of compact sets rather than merely points.

Claim 3. *Let $F(\cdot)$ be an upper-semicontinuous set-valued map from $\mathcal{O}$ to subsets of $\mathbb{R}^n$ and let $M \subset \mathcal{O}$ be compact. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$F_\delta(M_\delta) \subseteq F(M) + \varepsilon\overline{\mathcal{B}} \ .$$

Claim 4. *Let $F(\cdot)$ be an upper-semicontinuous set-valued map from $\mathcal{O}$ to subsets of $\mathbb{R}^n$. Let $k \in \mathbb{Z}_{>0}$, $i \in \{1, 2, \dots, k\}$, and S_i compact subsets of $\mathcal{O}$. Then there exist $\rho \in \mathcal{K}_\infty$ and $c > 0$ such that for every $\delta \in (0, c]$*

$$F_\delta(S_{i_\delta}) \subseteq F(S_i) + \rho(\delta)\overline{\mathcal{B}}.$$

Proof. For a particular S_i, let $\varepsilon > 0$. Then, from the result of Claim 3, there exists $\delta_i > 0$ such that $F_{\delta_i}(S_{i_\delta}) \subseteq F(S_i) + \varepsilon\overline{\mathcal{B}}$. For fixed $\varepsilon > 0$, let $\bar{\delta}_i(\varepsilon)$ be the supremum of all applicable $\delta_i(\varepsilon)$. Therefore,

$$F_{\bar{\delta}_i(\varepsilon)}(S_{i_{\bar{\delta}_i(\varepsilon)}}) \subseteq F(S_i) + \varepsilon\overline{\mathcal{B}}.$$

We note that $\bar{\delta}_i(\varepsilon)$ is positive and nondecreasing, but not necessarily continuous. Choose $\alpha_i \in \mathcal{K}$ such that $\alpha_i(r) \leq k\bar{\delta}_i(r)$ for all $r \in \mathbb{R}_{\geq 0}$ with $k \in (0, 1)$. Let $c_i := \lim_{r\to\infty} \alpha_i(r)$ and $\rho_i(r) := \alpha_i^{-1}(r)$ for all $r \in [0, c_i)$. Then ρ_i is continuous, zero at zero, strictly increasing, and is defined on $[0, c_i)$. Given $\delta_i < c_i$, let $\varepsilon = \rho_i(\delta_i)$. Then $\delta_i < \bar{\delta}_i(\varepsilon)$ and

$$F_{\delta_i}(S_{i_\delta}) \subseteq F(S_i) + \rho_i(\delta_i)\overline{\mathcal{B}}.$$

Let $c^* := \min_{i\in\{1,\dots,k\}}\{c_i\}$ and, for each $r \in [0, c^*)$, let $\hat{\rho}(r) := \max_{i\in\{1,\dots,k\}} \rho_i(r)$. Then, for each $i \in \{1, 2, \dots, k\}$, $F_\delta(S_{i_\delta}) \subseteq F(S_i) + \hat{\rho}(\delta)\overline{\mathcal{B}}$ for all $\delta \in (0, c^*)$. Finally, let $\rho \in \mathcal{K}_\infty$ be such that $\rho(r) \geq \hat{\rho}(r)$ for all $r \in [0, \frac{1}{2}c^*]$. Therefore, with $c := \frac{1}{2}c^*$,

$$F_\delta(S_{i_\delta}) \subseteq F(S_i) + \rho(\delta)\overline{\mathcal{B}} \quad \forall \delta \in (0, c]. \qquad \square$$

Claim 5. *Suppose $F(\cdot)$ is an upper-semicontinuous set-valued map from $\mathcal{O}$ to subsets of $\mathbb{R}^n$ and that, for each $x \in \mathcal{O}$, $F(x)$ is nonempty and compact. Let M be a compact set in $\mathcal{O}$ and $K \in \mathbb{Z}_{>0}$. Assume $F^k(M) \subset \mathcal{O}$ for all $k \in \{1, \dots, K\}$. Then there exist $\tilde{\rho} \in \mathcal{K}_\infty$ and $\tilde{c} > 0$ such that, for every $\delta \in (0, \tilde{c}]$ and $k \in \{1, \dots, K\}$,*

$$F_\delta^k(M_\delta) \subseteq F^k(M) + \tilde{\rho}(\delta)\overline{\mathcal{B}}.$$

Proof. Define the compact sets $S_0 := M$, $S_1 := F(M), \dots, S_k := F^k(M)$. Let $\rho \in \mathcal{K}_\infty$ and $c > 0$ come from Claim 4. Without loss of generality, assume $\rho(s) \geq s$ for all $s \in \mathbb{R}_{\geq 0}$. Let $\tilde{c} > 0$ be such that $\rho^{k-1}(\tilde{c}) < c$ and define $\tilde{\rho}(r) := \rho^k(r)$ for all $r \in [0, c)$ (where $\rho^k(\cdot)$ is the k-times composition of $\rho(\cdot)$ with itself). From Claim 4 we may write

$$F_\delta(M_\delta) = F_\delta(S_0 + \delta\overline{\mathcal{B}}) \subseteq F(M) + \rho(\delta)\overline{\mathcal{B}}.$$

Since $\delta < \tilde{c}$, we have that $\rho(\delta) < c$.

Assume the result holds for $k-1$; i.e., $F_\delta^{k-1}(M_\delta) \subseteq F^{k-1}(M) + \rho^{k-1}(\delta)\overline{\mathcal{B}}$. Noting that $\delta \leq \rho^{k-1}(\delta) < c$ we may write

$$\begin{aligned} F_\delta^k(M_\delta) &= F_\delta\left(F_\delta^{k-1}(M_\delta)\right) \ \subseteq \ F_\delta\left(F^{k-1}(M) + \rho^{k-1}(\delta)\overline{\mathcal{B}}\right) \\ &\subseteq F_{\rho^{k-1}(\delta)}\left(F^{k-1}(M) + \rho^{k-1}(\delta)\overline{\mathcal{B}}\right) \ \subseteq \ F^k(M) + \rho^k(\delta)\overline{\mathcal{B}} \\ &= F^k(M) + \tilde{\rho}(\delta)\overline{\mathcal{B}}, \end{aligned}$$

where the final subset is obtained by appealing to Claim 4. ∎

We will need to apply the lemmas in the following section to the difference inclusion defined by $x^+ \in F_\sigma(x)$ with $F_\sigma(\cdot)$ as in (2.2). To do this, we need to know that $F_\sigma(\cdot)$ satisfies the basic conditions.

CLAIM 6. *If $F(\cdot)$ is a set-valued map from $\mathcal{G}$ to subsets of $\mathcal{G}$ satisfying the basic conditions on $\mathcal{G}$, and $\sigma : \mathcal{G} \to \mathbb{R}_{\geq 0}$ satisfies item 1 of Definition 2.3, then $F_\sigma(\cdot)$ satisfies the basic conditions on $\mathcal{G}$.*

Proof. That $F_\sigma(x)$ is nonempty follows from $F(x)$ nonempty. Similarly, $F_\sigma(x)$ being compact follows from $F(x)$ compact, the compactness of the closed unit ball, $F(\cdot)$ upper semicontinuous, and the fact that upper-semicontinuous maps send compacts to compacts (see Claim 2).

Appealing to Claim 3 with $M := \{x\} + \sigma(x)\overline{\mathcal{B}}$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $F(M_\delta) = F\left(\{x\} + (\sigma(x) + \delta)\overline{\mathcal{B}}\right) \subseteq F\left(\{x\} + \sigma(x)\overline{\mathcal{B}}\right) + \varepsilon\overline{\mathcal{B}}$. Let $\varepsilon_\sigma = \frac{\delta}{2} > 0$. Then, since $\sigma(\cdot)$ is continuous, there exists $\delta_\sigma \in (0, \frac{\delta}{2}]$ such that if $|x - z| < \delta_\sigma$, then $|\sigma(x) - \sigma(z)| < \varepsilon_\sigma = \frac{\delta}{2}$. Therefore, $\{z\} + \sigma(z)\overline{\mathcal{B}} \subseteq \{x\} + (\sigma(x) + \delta)\overline{\mathcal{B}}$ and

$$F\left(\{z\} + \sigma(z)\overline{\mathcal{B}}\right) \subseteq F\left(\{x\} + \sigma(x)\overline{\mathcal{B}}\right) + \varepsilon\overline{\mathcal{B}};$$

i.e., $F_\sigma(\cdot)$ is upper semicontinuous on $\mathcal{G}$. ∎

5. Difference inclusions. In this section we present three new results for difference inclusions which will be necessary for the proofs of the results presented in section 2.

The first result makes use of a perturbed difference inclusion. Let $F(\cdot)$ map an open set $\mathcal{O} \subset \mathbb{R}^n$ to subsets of $\mathbb{R}^n$, let $\delta \geq 0$, and consider

$$x^+ \in F_\delta(x) := F(x + \delta\overline{\mathcal{B}}) + \delta\overline{\mathcal{B}}, \quad x \in \mathcal{O}. \tag{5.1}$$

We denote the solution set of (5.1) from the point $x \in \mathbb{R}^n$ by $\mathcal{S}_\delta(x)$. This result is similar to a result on closeness of solutions for differential inclusions (see, for example, [5, section 8, Corollary 2]).

LEMMA 5.1. *Let $\mathcal{O}$ be open, $\omega : \mathcal{O} \to \mathbb{R}_{\geq 0}$ be continuous, and $F(\cdot)$ map $\mathcal{O}$ to subsets of $\mathbb{R}^n$ satisfy the basic conditions on $\mathcal{O}$. Let the triple (K, ε, M) be such that $K \in \mathbb{Z}_{>0}$, $\varepsilon > 0$, $M \subset \mathcal{O}$ compact, and $F^k(M) \subset \mathcal{O}$ for all $k \in \{1, \dots, K\}$. Under these conditions, there exists $\delta > 0$ such that for every $x \in M_\delta$*

1. *every solution $\psi \in \mathcal{S}_\delta(x)$ satisfies $\psi(k, x) \in \mathcal{O}$ for $k \in \{0, \dots, K\}$, and*
2. *for every $\psi \in \mathcal{S}_\delta(x)$ there exists $\bar{x} \in M$ and $\phi \in \mathcal{S}(x)$ such that for all $k \in \{0, \dots, K\}$ we have*

$$|\omega(\psi(k, x)) - \omega(\phi(k, \bar{x}))| \leq \varepsilon. \tag{5.2}$$

Proof. The first item follows from Claim 5 and the fact that $F^k(M)$ is compact for each $k \in \mathbb{Z}_{>0}$.

If the second item is not true, then no matter how small we pick δ, there is an initial condition in M_δ and a solution to $x^+ \in F_\delta(x)$ starting at this initial condition such that, no matter which initial condition in M and solution of $x^+ \in F(x)$ we pick, the condition (5.2) is violated for some $k \in \{0, \dots, K\}$. In particular, there exist sequences $x_i \in M_{1/i}$ and $\psi_i \in \mathcal{S}_{1/i}(x_i)$ such that, no matter which initial condition in M and solution of $x^+ \in F(x)$ we pick, the condition (5.2) is violated for some $k \in \{0, \dots, K\}$. The sequence x_i has a subsequence, which we will not relabel, converging to a point $f_0^* \in M$. Associated with this subsequence is a sequence of points $\psi_i(1,x_i) \in F_{1/i}(x_i)$. This sequence has a converging subsequence and, from the upper semicontinuity of F and compactness of $F(f_0^*)$, its accumulation point, denoted f_1^*, belongs to $F(f_0^*)$. Continuing in this way we get a sequence of initial conditions $x_i \in M_{1/i}$ and a sequence of solutions $\psi_i \in \mathcal{S}_{1/i}(x_i)$ such that $x_i \to f_0^* \in M$ and $\psi_i(k,x_i) \to f_k^* \in F(f_{k-1}^*)$. Now with the solution $\phi \in \mathcal{S}(f_0^*)$ given by $\phi(k,f_0^*) = f_k^* \in F(\phi(k-1,f_0^*))$ for all $k \in \{1,\ldots,K\}$, and using the continuity of ω, condition (5.2) holds for all i sufficiently large. This is a contradiction and thus proves the lemma. ∎

We will require the following lemma on closeness of solutions for difference inclusions defined by continuous set-valued maps.

LEMMA 5.2. *Suppose $F(\cdot)$ is a set-valued map from $\mathcal{O}$ to subsets of $\mathbb{R}^n$ continuous on an open set $\mathcal{O}_1 \subseteq \mathcal{O}$ and that, for each $x \in \mathcal{O}$, $F(x)$ is compact and nonempty. Furthermore, suppose $\omega : \mathcal{O} \to \mathbb{R}_{\geq 0}$ is continuous. For each triple (K,ε,x_0) such that $K \in \mathbb{Z}_{>0}$, $\varepsilon > 0$, and $x_0 \in \mathcal{O}$, and for each solution $\phi \in \mathcal{S}(x_0)$ such that $\phi(k,x_0) \in \mathcal{O}_1$ for all $k \in \{0,\ldots,K\}$ there exists a $\delta > 0$ such that, for every $x \in \{x_0\} + \delta\overline{\mathcal{B}}$, there exists a solution $\psi \in \mathcal{S}(x)$ such that, for all $k \in \{0,\ldots,K+1\}$,*

$$|\omega(\phi(k,x_0)) - \omega(\psi(k,x))| \leq \varepsilon.$$

Proof. Define the compact set

$$\mathcal{C} := \{\phi(k,x_0)\} \subset \mathcal{O}_1 \subseteq \mathcal{O} \quad \forall k \in \{0,\ldots,K\}.$$

For the given $\varepsilon > 0$, since $\omega(\cdot)$ is continuous, there exists $\delta_\omega > 0$ such that $r \in \mathcal{C}$ and $|s-r| \leq \delta_\omega$ imply $|\omega(s)-\omega(r)| \leq \varepsilon$. Without loss of generality, we also impose $\delta_\omega \leq \varepsilon$ and $\mathcal{C} + \delta_\omega\overline{\mathcal{B}} \subset \mathcal{O}_1$.

From the continuity of $F(\cdot)$ at $\phi(K,x_0)$, there exists $\delta_K \in (0,\delta_\omega]$ such that for all $z \in \mathcal{O}$ satisfying $|z - \phi(K,x_0)| \leq \delta_K$ we have $F(\phi(K,x_0)) \subseteq F(z) + \delta_\omega\overline{\mathcal{B}}$. Similarly, from the continuity of $F(\cdot)$ at $\phi(K-1,x_0)$, there exists $\delta_{K-1} \in (0,\delta_\omega]$ such that for all $z \in \mathcal{O}$ satisfying $|z-\phi(K-1,x_0)| \leq \delta_{K-1}$ we have $F(\phi(K-1,x_0)) \subseteq F(z)+\delta_K\overline{\mathcal{B}}$. We repeat this procedure until we reach the initial point x_0. From the previous step we will have a $\delta_1 \in (0,\delta_\omega]$. Then, from the continuity of $F(\cdot)$ at x_0, there exists a $\delta_0 \in (0,\delta_\omega]$ such that, for all $z \in \mathcal{O}$,

$$|z - x_0| \leq \delta_0 \quad \Longrightarrow \quad F(x_0) \subseteq F(z) + \delta_1\overline{\mathcal{B}}. \tag{5.3}$$

From (5.3), for any $x \in \{x_0\} + \delta_0\overline{\mathcal{B}}$ there exists a point $\psi(1,x) \in F(x)$ such that

$$|\phi(1,x_0) - \psi(1,x)| \leq \delta_1. \tag{5.4}$$

This follows from (5.3) since $|x - x_0| \leq \delta_0$, so that $\phi(1,x_0) \in F(x_0) \subseteq F(x) + \delta_1\overline{\mathcal{B}}$.

Since (5.4) holds, we see that there exists a point $\psi(2,x) \in F(\psi(1,x))$ such that

$$|\phi(2,x_0) - \psi(2,x)| \leq \delta_2.$$

This follows from (5.4) since $\phi(2,x_0) \in F(\phi(1,x_0)) \subseteq F(\psi(1,x)) + \delta_2\overline{\mathcal{B}}$. That is, for the point $\phi(2,x_0)$, there exists an element in $F(\psi(1,x))$ (which we have called $\psi(2,x)$) that is no more than δ_2 away from $\phi(2,x_0)$.

We can repeat this procedure at each step until we get to $\phi(K+1,x_0)$.

Since, for each $\ell \in \{0,\ldots,K\}$, we imposed $\delta_\ell \leq \delta_\omega$ we see that, with $\psi \in \mathcal{S}(x)$ constructed as above,

$$|\omega(\phi(k,x_0)) - \omega(\psi(k,x))| \leq \varepsilon \quad \forall k \in \{0,\ldots,K+1\}. \qquad ∎$$

We present a lemma regarding sequences of solutions. This lemma is similar to the continuous-time results found in [22, Lemmas 4 and 5], which derived from [5, section 7, Theorem 3].

LEMMA 5.3. *Let $F(\cdot)$ mapping $\mathcal{O}$ to subsets of $\mathbb{R}^n$ satisfy the basic conditions on $\mathcal{O}$. Let $x \in \mathcal{O}$ be given and suppose that all solutions $\phi \in \mathcal{S}(x)$ are defined and belong to $\mathcal{O}$ for all $k \geq 0$. Then each sequence $\{\phi_n\}_{n=1}^{\infty}$ of solutions in $\mathcal{S}(x)$ has a subsequence converging to a function $\phi \in \mathcal{S}(x)$ and the convergence is uniform on each finite time interval.*

Proof. From Claim 2 we know that for each $k \in \mathbb{Z}_{\geq 0}$ the set $F^k(x)$ is a compact set. Since for all n and k, $\phi_n(k,x) \in F^k(x)$, it follows that $\{\phi_n\}_{n=1}^{\infty}$ has a converging subsequence $\{\phi_{1m}\}_{m=1}^{\infty}$ such that $\phi_{1m}(1,x) \to f_1^* =: \phi(1,x)$. Similarly, $\{\phi_{1m}\}_{m=1}^{\infty}$ has a converging subsequence $\{\phi_{2m}\}_{m=1}^{\infty}$ such that $\phi_{2m}(2,x) \to f_2^* =: \phi(2,x)$, and so on. In this way, we construct a subsequence which converges to a solution $\phi \in \mathcal{S}(x)$, and, for a finite time interval, this convergence is uniform. ∎

6. Proof of Theorem 2.7. We demonstrate that robust $\mathcal{KL}$-stability is a necessary and sufficient condition for the existence of a smooth Lyapunov function with respect to (ω_1, ω_2).

6.1. Sufficiency. One of the most useful lemmas regarding comparison functions is frequently referred to as Sontag's lemma on $\mathcal{KL}$-estimates [19, Proposition 7]. This lemma allows us to view asymptotic stability as exponential stability via a suitable nonlinear scaling. The following lemma is a slight refinement of Sontag's original lemma wherein we specify the required regularity property of one of the $\mathcal{K}_\infty$ functions.

LEMMA 6.1. *For each $\beta \in \mathcal{KL}$ and $\lambda > 0$, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(\cdot)$ is Lipschitz on its domain, continuously differentiable on $(0,\infty)$, $\alpha_1(s) \leq s\alpha_1'(s)$ for all $s > 0$, and $\alpha_1(\beta(s,t)) \leq \alpha_2(s)e^{-\lambda t}$ for all $(s,t) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$.*

6.1.1. Bounds. Given $\beta_\sigma \in \mathcal{KL}$ from the assumption of $\mathcal{KL}$-stability with respect to (ω_1, ω_2) for $x^+ \in F_\sigma(x)$, Lemma 6.1 yields $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}_\infty$ such that

$$\hat{\alpha}_1\left(\beta_\sigma(s,k)\right) \leq \hat{\alpha}_2(s)e^{-2k} \quad \forall k \in \mathbb{Z}_{\geq 0},\ \forall s \geq 0. \tag{6.1}$$

For each $x \in \mathcal{G}$ we define the function $V_1 : \mathcal{G} \to \mathbb{R}_{\geq 0}$ by

$$V_1(x) := \sup_{k \in \mathbb{Z}_{\geq 0}, \phi \in \mathcal{S}_\sigma(x)} \hat{\alpha}_1(\omega_1(\phi(k,x)))e^k. \tag{6.2}$$

We claim that

$$V_1(x) = 0 \quad \Longleftrightarrow \quad x \in \mathcal{A}_\sigma = \mathcal{A}. \tag{6.3}$$

To see this, note that if $x \in \mathcal{A}_\sigma$, then, by definition (2.4), $V_1(x) = 0$. Furthermore, $V_1(x) = 0$ implies that

$$0 = \sup_{k \in \mathbb{Z}_{\geq 0}, \phi \in \mathcal{S}_\sigma(x)} \hat{\alpha}_1(\omega_1(\phi(k,x))),$$

and, since $\hat{\alpha}_1 \in \mathcal{K}_\infty$, $x \in \mathcal{A}_\sigma = \mathcal{A}$.

It is easy to see that, for all $x \in \mathcal{G}$,

$$V_1(x) \geq \sup_{\phi \in \mathcal{S}_\sigma(x)} \hat{\alpha}_1(\omega_1(\phi(0,x)))e^0 = \hat{\alpha}_1(\omega_1(x)), \quad \text{and} \tag{6.4}$$

$$\begin{aligned} V_1(x) &\le \sup_{k\in\mathbb{Z}_{\ge 0}} \hat{\alpha}_1(\beta_\sigma(\omega_2(x),k))e^k \\ &\le \sup_{k\in\mathbb{Z}_{\ge 0}} \hat{\alpha}_2(\omega_2(x))e^{-k} = \hat{\alpha}_2(\omega_2(x)). \end{aligned} \tag{6.5}$$

For each $x \in \mathcal{G}$ and $\phi \in \mathcal{S}_\sigma(x)$ we may write

$$\begin{aligned} V_1(\phi(j,x)) &= \sup_{k\in\mathbb{Z}_{\ge 0},\ \psi\in\mathcal{S}_\sigma(\phi(j,x))} \hat{\alpha}_1\left(\omega_1(\psi(k,\phi(j,x)))\right)e^k \\ &\le \sup_{k\in\mathbb{Z}_{\ge j},\ \psi\in\mathcal{S}_\sigma(x)} \hat{\alpha}_1(\omega_1(\psi(k,x)))e^{k-j} \\ &\le \sup_{k\in\mathbb{Z}_{\ge 0},\ \psi\in\mathcal{S}_\sigma(x)} \hat{\alpha}_1(\omega_1(\psi(k,x)))e^k e^{-j} \\ &= V_1(x)e^{-j}\ . \end{aligned}$$

We note that $w \in F_\sigma(x)$ implies the existence of $\phi \in \mathcal{S}_\sigma(x)$ such that $\phi(1,x) = w$. Therefore, we may write

$$\sup_{f\in F_\sigma(x)} V_1(f) \le e^{-1}V_1(x) \quad \forall x \in \mathcal{G}. \tag{6.6}$$

6.1.2. Smoothing V_1. We proceed to smooth the function $V_1(\cdot)$ without destroying the nature of the upper and lower bounds (6.4) and (6.5) and the decrease condition (6.6). To do this, we will appeal to Theorem 3.1, which requires Assumptions 1 and 2 and uses a set $\mathcal{A}$ defined in (3.1) that, as a consequence of (6.3), is the same as the set $\mathcal{A}$ of (2.3). Assumption 1 requires that $V_1(\cdot)$ be upper semicontinuous and that if $V_1(x) > 0$, then $V_1(z) > 0$ for z near x.

$V_1(\cdot)$ *is upper semicontinuous:* We first show that for each $x \in \mathcal{G}\backslash\mathcal{A}$ there exists a solution such that the supremum defining $V_1(\cdot)$ is attained for some solution and over a finite time interval.

CLAIM 7. *Let $x \in \mathcal{G}$ be such that $V_1(x) > 0$. Define*

$$K(x) := -\left\lfloor \ln\left(\frac{V_1(x)}{\hat{\alpha}_2(\omega_2(x))}\right)\right\rfloor + 1. \tag{6.7}$$

Then there exists $\hat{\phi} \in \mathcal{S}_\sigma(x)$ such that, for every $\kappa \ge K(x)$,

$$V_1(x) = \max_{k\in\{0,\dots,\kappa\}} \hat{\alpha}_1\left(\omega_1(\hat{\phi}(k,x))\right)e^k. \tag{6.8}$$

Proof. It is obvious that

$$\sup_{k\in\{0,\dots,\kappa\},\phi\in\mathcal{S}_\sigma(x)} \hat{\alpha}_1\left(\omega_1(\phi(k,x))\right)e^k \le V_1(x). \tag{6.9}$$

We note that, for all $x \in \mathcal{G}\backslash\mathcal{A}$,

$$e^{-\kappa} \le e^{-K(x)} \le \frac{V_1(x)}{\hat{\alpha}_2(\omega_2(x))}e^{-1}.$$

Therefore, with (2.1) and (6.1), we may write

$$\begin{aligned} V_1(x) &= \max\left\{ \sup_{k\in\{0,\ldots,\kappa\},\phi\in\mathcal{S}_\sigma(x)} \hat\alpha_1\left(\omega_1(\phi(k,x))\right)e^k, \right. \\ &\qquad \left. \sup_{k\in\mathbb{Z}_{\geq\kappa},\phi\in\mathcal{S}_\sigma(x)} \hat\alpha_1\left(\omega_1(\phi(k,x))\right)e^k \right\} \\ &\leq \max\left\{ \sup_{k\in\{0,\ldots,\kappa\},\phi\in\mathcal{S}_\sigma(x)} \hat\alpha_1\left(\omega_1(\phi(k,x))\right)e^k, \quad \hat\alpha_2(\omega_2(x))e^{-\kappa} \right\} \\ &\leq \max\left\{ \sup_{k\in\{0,\ldots,\kappa\},\phi\in\mathcal{S}_\sigma(x)} \hat\alpha_1\left(\omega_1(\phi(k,x))\right)e^k, \quad V_1(x)e^{-1} \right\}, \end{aligned}$$

which, together with (6.9), implies

$$\begin{aligned} V_1(x) &= \sup_{k\in\{0,\ldots,\kappa\},\phi\in\mathcal{S}_\sigma(x)} \hat\alpha_1\left(\omega_1(\phi(k,x))\right)e^k \\ &= \sup_{\phi\in\mathcal{S}_\sigma(x)} \max_{k\in\{0,\ldots,\kappa\}} \hat\alpha_1\left(\omega_1(\phi(k,x))\right)e^k, \end{aligned}$$

where we can pass to the "max" since the supremum is taken over a finite number of elements. Now let $\{\phi_\ell\}_{\ell=1}^\infty$ be a maximizing sequence of solutions in $\mathcal{S}_\sigma(x)$; i.e.,

$$V_1(x) = \lim_{\ell\to\infty} \max_{k\in\{0,\ldots,\kappa\}} \hat\alpha_1\left(\omega_1(\phi_\ell(k,x))\right)e^k.$$

Since $F_\sigma(\cdot)$ satisfies the basic conditions (see Claim 6), we may appeal to Lemma 5.3 to see that a subsequence of $\{\phi_\ell(\cdot,x)\}_{\ell=1}^\infty$ converges uniformly on $\{0,\ldots,\kappa\}$ to some solution $\hat\phi \in \mathcal{S}_\sigma(x)$. Since the functions $\hat\alpha_1(\cdot)$ and $\omega_1(\cdot)$ are continuous, we may write

$$V_1(x) = \max_{k\in\{0,\ldots,\kappa\}} \hat\alpha_1\left(\omega_1(\hat\phi(k,x))\right)e^k. \qquad \square$$

We now prove that $V_1(\cdot)$ is upper semicontinuous. In order to obtain a contradiction, suppose that there exist $x\in\mathcal{G}$ and a sequence $\{x_\ell\}_{\ell=1}^\infty$ of points in $\mathcal{G}$ converging to $x\in\mathcal{G}$ such that

$$\limsup_{\ell\to\infty} V_1(x_\ell) > V_1(x) \geq 0.$$

Without loss of generality, for all ℓ and some $\eta > 0$

$$V_1(x_\ell) \geq \eta. \tag{6.10}$$

Define $\kappa := \sup_\ell K(x_\ell)$. From (6.10), the continuity of $\hat\alpha_2\circ\omega_2(\cdot)$, and the definition of $K(\cdot)$ in (6.7), we see that $\kappa < \infty$. Let $\hat\phi \in \mathcal{S}_\sigma(x_\ell)$ come from Claim 7 so that

$$V_1(x_\ell) = \max_{k\in\{0,\ldots,\kappa\}} \hat\alpha_1\left(\omega_1(\hat\phi(k,x_\ell))\right)e^k.$$

Let $\varepsilon > 0$. Since $F_\sigma(\cdot)$ satisfies the basic conditions, we appeal to Lemma 5.1 with the triple (κ,ε,x) and the continuity of $\hat\alpha_1\circ\omega_1(\cdot)$ to assert the existence of ℓ_ε so that, for all $\ell\geq\ell_\varepsilon$, there exists $\psi_\ell\in\mathcal{S}_\sigma(x)$ such that

$$\begin{aligned} V_1(x_\ell) = \max_{k\in\{0,\ldots,\kappa\}} \hat\alpha_1\left(\omega_1(\hat\phi(k,x_\ell))\right)e^k &\leq \varepsilon + \max_{k\in\{0,\ldots,\kappa\}} \hat\alpha_1\left(\omega_1(\psi_\ell(k,x))\right)e^k \\ &\leq \varepsilon + V_1(x). \end{aligned}$$

This implies $\limsup_{\ell\to\infty} V_1(x_\ell) \le V_1(x)$, which is a contradiction. In addition, it also establishes continuity of $V_1(\cdot)$ at each point $x \in \{\xi \in \mathcal{G} : V_1(\xi) = 0\}$ since, for each such x, we may write

$$0 \le \limsup_{z\to x} V_1(z) \le V_1(x) = 0.$$

Next we establish item 1 of Assumption 1.

CLAIM 8. *If $V_1(x) > 0$, then there exists $\delta > 0$ such that $|z - x| < \delta$ implies $V_1(z) > 0$.*

With this claim, we see that the set $\mathcal{G}\backslash\mathcal{A}$ is open.

Proof. If $x \in \mathcal{G}\backslash\mathcal{A}$ is such that $\omega_1(x) > 0$, then the result follows from the continuity of $\omega_1(\cdot)$, the lower bound (6.4), and the fact that $\hat{\alpha}_1 \in \mathcal{K}_\infty$. So we just need to consider points $x \in \mathcal{G}\backslash\mathcal{A}$ such that $\omega_1(x) = 0$. We first assert that

$$\sup_{f\in F(x)} V_1(f) > 0. \tag{6.11}$$

If this were not the case, then with the lower bound (6.4) and the fact that $\hat{\alpha}_1 \in \mathcal{K}_\infty$ we would have $\max_{f\in F(x)} \omega_1(f) = 0$. Furthermore, with the decrease condition (6.6) and $F(x) \subseteq F_\sigma(x)$ we would have, for all $f \in F(x)$,

$$\sup_{g\in F(f)} V_1(g) \le e^{-1} V_1(f) = 0.$$

Iterating and using the condition $\omega_1(x) = 0$, we would have

$$\sup_{k\in\mathbb{Z}_{\ge 0},\phi\in\mathcal{S}(x)} \omega_1(\phi(k,x)) = 0;$$

i.e., $x \in \mathcal{A}$, which is a contradiction.

We also have, according to the definition of robust $\mathcal{KL}$ stability, $\sigma(x) > 0$ for $x \in \mathcal{G}\backslash\mathcal{A}$. Using the continuity of $\sigma(\cdot)$, there exists $\delta > 0$ such that $\delta \le \min_{q\in\delta\overline{\mathcal{B}}} \sigma(x+q)$. It follows that

$$0 \in \bigcap_{z\in\delta\overline{\mathcal{B}}} \{z\} + \sigma(x+z)\overline{\mathcal{B}}$$

and thus, for any $z \in \delta\overline{\mathcal{B}}$, we see that $F(x) \subseteq F(x + z + \sigma(x+z)\overline{\mathcal{B}})$. Now, using (6.6) and (6.11), we have, for $z \in \delta\overline{\mathcal{B}}$,

$$e^{-1} V_1(x+z) \ge \sup_{f\in F_\sigma(x+z)} V_1(f) \ge \sup_{f\in F(x)} V_1(f) > 0,$$

which establishes the claim. □

Finally, we need to construct a function $\sigma_2 : \mathcal{G}\backslash\mathcal{A} \to \mathbb{R}_{>0}$ satisfying Assumption 2 and such that the smooth function $V(\cdot)$ of Theorem 3.1 retains bounds like (6.4) and (6.5) and the decrease condition (6.6).

Construction of σ_2:

CLAIM 9. *There exists a smooth function $\sigma_1 : \mathcal{G}\backslash\mathcal{A} \to \mathbb{R}_{>0}$ such that, for all $x \in \mathcal{G}\backslash\mathcal{A}$,*

$$\sigma_1(x) \le \sigma(x), \qquad \text{and} \tag{6.12}$$

$$\sup_{f\in F_{\sigma_1}(x), f\in\mathcal{G}\backslash\mathcal{A}} V_1(f) \le e^{-1} \inf_{z\in\sigma_1(x)\overline{\mathcal{B}}} V_1(x+z)\ . \tag{6.13}$$

Proof. We define a function $\sigma_1 : \mathcal{G}\backslash\mathcal{A} \to \mathbb{R}_{>0}$ by associating to each $x \in \mathcal{G}\backslash\mathcal{A}$ one-half the supremum over all values $\tilde{\sigma}_1$ satisfying

$$2\tilde{\sigma}_1 \leq \min_{q\in\tilde{\sigma}_1\overline{\mathcal{B}}} \sigma(x+q). \tag{6.14}$$

The existence of $\sigma_1(\cdot)$ follows from continuity of $\sigma(\cdot)$ and the fact that $\sigma(x) > 0$ for all $x \in \mathcal{G}\backslash\mathcal{A}$. These properties for $\sigma(\cdot)$ also guarantee that $\sigma_1(\cdot)$ is bounded away from zero on compact subsets of $\mathcal{G}\backslash\mathcal{A}$.

It follows from (6.14) that

$$\sigma_1(x) \leq \sigma(x) \quad \forall x \in \mathcal{G}\backslash\mathcal{A}, \tag{6.15}$$

i.e., (6.12) holds. We further see that

$$\sigma_1(x)\overline{\mathcal{B}} \subseteq \bigcap_{z\in\sigma_1(x)\overline{\mathcal{B}}} \{z\} + \sigma(x+z)\overline{\mathcal{B}},$$

so that

$$F(x+\sigma_1(x)\overline{\mathcal{B}}) \subseteq \bigcap_{z\in\sigma_1(x)\overline{\mathcal{B}}} F(x+z+\sigma(x+z)\overline{\mathcal{B}}). \tag{6.16}$$

With the definition of $F_\sigma(\cdot)$, (6.15), and (6.16) we see that, for any $z \in \sigma_1(x)\overline{\mathcal{B}}$,

$$F_{\sigma_1}(x) \subseteq F_\sigma(x+z). \tag{6.17}$$

From the decrease condition (6.6), we have

$$e^{-1}V_1(x+z) \geq \sup_{f\in F_\sigma(x+z)} V_1(f).$$

Taking the infimum on both sides and appealing to (6.17) we have

$$\inf_{z\in\sigma_1(x)\overline{\mathcal{B}}} e^{-1}V_1(x+z) \geq \inf_{z\in\sigma_1(x)\overline{\mathcal{B}}} \left[\sup_{f\in F_{\sigma(x+z)}(x+z)} V_1(f) \right] \geq \sup_{f\in F_{\sigma_1(x)}(x)} V_1(f).$$

It is clear that this inequality and (6.15) hold for any function smaller than $\sigma_1(\cdot)$, and so we can smooth $\sigma_1(\cdot)$ using Lemma 3.3 to prove the claim. ∎

Let the function $\sigma_a : \mathcal{G}\backslash\mathcal{A} \to \mathbb{R}_{\geq 0}$ be given by

$$\sigma_a(x) := \min\left\{1, \frac{1}{2}\sup\left\{\eta \ : \ |z-x| \leq \eta \ \Rightarrow \ |\omega_2(z)-\omega_2(x)| \leq \frac{1}{2}\omega_2(x)\right\}\right\} \tag{6.18}$$

and the function $\sigma_b : \mathcal{G}\backslash\mathcal{A} \to \mathbb{R}_{\geq 0}$ be given by

$$\sigma_b(x) := \min\left\{1, \frac{1}{2}\sup\left\{\eta \ : \ V_1(x+\eta\overline{\mathcal{B}}) \geq \hat{\alpha}_1\left(\frac{1}{2}\omega_1(x)\right)\right\}\right\}. \tag{6.19}$$

We then define, for each $x \in \mathcal{G}\backslash\mathcal{A}$,

$$\bar{\sigma}_2(x) := \min\{\sigma_a(x), \sigma_b(x), \sigma_1(x), |x|_{\mathcal{A}}\}. \tag{6.20}$$

Before proceeding, we demonstrate that the function $\bar{\sigma}_2(\cdot)$ is bounded away from zero on compact subsets of $\mathcal{G}\backslash\mathcal{A}$. Since $\sigma_1(\cdot)$ and $|\cdot|_{\mathcal{A}}$ are continuous and positive on $\mathcal{G}\backslash\mathcal{A}$, we need to establish this property only for $\sigma_a(\cdot)$ and $\sigma_b(\cdot)$.

We first note that $\omega_2(x) > 0$ for $x \in \mathcal{G}\backslash\mathcal{A}$. Suppose not. If $x \notin \mathcal{A}$, then there exists a solution $\phi \in \mathcal{S}_\sigma(x)$ and a time $k \in \mathbb{Z}_{\geq 0}$ such that $\omega_1(\phi(k, x)) > 0$. However, from the stability estimate we see that $0 < \omega_1(\phi(k, x)) \leq \beta_\sigma(\omega_2(x), 0) = 0$, which is a contradiction.

First we demonstrate that $\sigma_a(\cdot)$ is bounded away from zero on compact subsets of $\mathcal{G}\backslash\mathcal{A}$. Suppose not. Then there exists a compact set $D \subset \mathcal{G}\backslash\mathcal{A}$, a sequence $\{x_i\}_{i=1}^{\infty}$ in D, and a sequence $z_i \in \{x_i\} + \sigma_a(x_i)\overline{\mathcal{B}} \subset \{x_i\} + \frac{1}{i}\overline{\mathcal{B}}$ such that

$$|\omega_2(z_i) - \omega_2(x_i)| > \frac{1}{2}\omega_2(x_i). \tag{6.21}$$

The sequence x_i has an accumulation point $x^* \in D$. Now, since $x^* \in D$, we have $\omega_2(x^*) > 0$. Since $\omega_2(\cdot)$ is continuous we have, as $i \to \infty$, $|\omega_2(z_i) - \omega_2(x_i)| \to 0$ while $\frac{1}{2}\omega_2(x_i) \to \frac{1}{2}\omega_2(x^*) > 0$. This contradicts (6.21).

Next we demonstrate that $\sigma_b(\cdot)$ is bounded away from zero on compact subsets of $\mathcal{G}\backslash\mathcal{A}$. Suppose not. Then there exist a compact set $D \subset \mathcal{G}\backslash\mathcal{A}$, a sequence $x_i \in D$, and a sequence z_i with $|x_i - z_i| \leq 1/i$ such that

$$V_1(z_i) < \hat{\alpha}_1\left(\frac{1}{2}\omega_1(x_i)\right). \tag{6.22}$$

The sequence x_i has an accumulation point $x^* \in D$. Henceforth we use x_i to denote the converging subsequence, and z_i the associated subsequence. Suppose $\omega_1(x^*) > 0$. Using the continuity of $\omega_1(\cdot)$, there exists i^* such that $\omega_1(z_i) \geq \frac{1}{2}\omega_1(x_i)$ for all $i \geq i^*$ and thus

$$V_1(z_i) \geq \hat{\alpha}_1\left(\omega_1(z_i)\right) \geq \hat{\alpha}_1\left(\frac{1}{2}\omega_1(x_i)\right),$$

which contradicts (6.22).

Alternatively, suppose $\omega_1(x^*) = 0$. We make the following claim.

CLAIM 10. *There exists $c > 0$ such that*

$$V_1(x^* + \sigma_1(x^*)\overline{\mathcal{B}}) \geq c. \tag{6.23}$$

Proof. Suppose not. Then, for all $c > 0$

$$\inf_{z\in\sigma_1(x^*)\overline{\mathcal{B}}} V_1(x^* + z) < c.$$

We note that this implies that if $f \in F(x^*)$, then $f \in \mathcal{A}$. Suppose not. Then, since $x^* \in D \subset \mathcal{G}\backslash\mathcal{A}$, (6.13) implies that $\sup_{f\in F(x^*)} V_1(f) = 0$. However, appealing to (6.3), we see that $V_1(f) = 0$ if and only if $f \in \mathcal{A}$.

From (6.6), we see that if $f \in \mathcal{A}$, then

$$\sup_{f_1\in F(f)} V_1(f_1) \leq \sup_{f_1\in F_\sigma(f)} V_1(f_1) \leq e^{-1}V_1(f) = 0.$$

In other words, any solution starting from a point $f \in F(x^*)$ is such that $V_1(\cdot)$ remains identically zero and, from (6.4), we see that $\omega_1(\cdot)$ also remains identically zero. Furthermore, with $\omega_1(x^*) = 0$, it follows that any solution starting at x^* is such that $\omega_1(\cdot)$ remains identically zero so that $x^* \in \mathcal{A}$, which contradicts $x^* \in \mathcal{G}\backslash\mathcal{A}$. ∎

For sufficiently large i we have $z_i \in \{x^*\} + \sigma_1(x^*)\overline{\mathcal{B}}$ and, since $\omega_1(x^*) = 0$, we have $\hat{\alpha}_1\left(\frac{1}{2}\omega_1(x_i)\right) \leq c$ so that, with (6.23),

$$V(z_i) \geq \hat{\alpha}_1\left(\frac{1}{2}\omega_1(x_i)\right),$$

which contradicts (6.22).

Let the function $\sigma_2 : \mathcal{G}\backslash\mathcal{A} \to \mathbb{R}_{>0}$ come from the application of Lemma 3.3 to the function $\bar{\sigma}_2(\cdot)$ defined in (6.20) so that $\sigma_2(\cdot)$ is smooth and positive on $\mathcal{G}\backslash\mathcal{A}$. We see that $\sigma_2(\cdot)$ is such that, for a sequence of points $x_i \in \mathcal{G}\backslash\mathcal{A}$ such that $x_i \to x^* \in \mathcal{A}$, we have $\sigma_2(x_i) \to 0$, since $\sigma_2(x) \leq |x|_{\mathcal{A}}$. We also note that $x \in \mathcal{G}\backslash\mathcal{A}$ implies $\{x\} + \sigma_2(x)\overline{\mathcal{B}} \subset \mathcal{G}$. This follows from the fact that $x \in \mathcal{G}\backslash\mathcal{A}$ implies $\{x\} + \sigma_1(x)\overline{\mathcal{B}} \subset \mathcal{G}$ (which stems from the fact that $\sigma_2(x) \leq \sigma_1(x) \leq \sigma(x)$ and the definition of robust $\mathcal{KL}$ stability). Consequently, $\sigma_2(\cdot)$ satisfies Assumption 2.

Let $\psi : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ be smooth, vanish outside $\mathbb{R}^n\backslash\overline{\mathcal{B}}$, and satisfy $\int \psi(\xi)d\xi = 1$. For $x \in \mathcal{G}\backslash\mathcal{A}$ we define

$$V_s(x) := \int V_1(x + \sigma_2(x)\xi)\psi(\xi)\, d\xi.$$

For $x \in \mathcal{A}$ we let $V_s(x) = 0$. That $V_s(\cdot)$ is smooth on $\mathcal{G}\backslash\mathcal{A}$ and continuous on $\mathcal{G}$ follows from Theorem 3.1.

Using (6.18) and $\sigma_2(x) \leq \sigma_a(x)$ for all $x \in \mathcal{G}\backslash\mathcal{A}$, we see that

$$V_s(x) \leq \max_{z \in \{x\}+\sigma_2(x)\overline{\mathcal{B}}} \hat{\alpha}_2(\omega_2(z)) \leq \hat{\alpha}_2\left(\frac{3}{2}\omega_2(x)\right).$$

From (6.19) and $\sigma_2(x) \leq \sigma_b(x)$ for all $x \in \mathcal{G}\backslash\mathcal{A}$, we have

$$V_s(x) \geq \hat{\alpha}_1\left(\frac{1}{2}\omega_1(x)\right).$$

We now check that an appropriate decrease condition holds for $V_s(\cdot)$. Suppose $x \notin \mathcal{A}$ and $f \in F(x)$ is such that $f \in \mathcal{A}$. Then it is obvious that $V_s(f) \leq e^{-1}V_s(x)$. If $x \in \mathcal{A}$, by definition of $\mathcal{A}$ this implies that $f \in \mathcal{A}$ for all $f \in F(x)$. Therefore, $V_s(f) = 0 = e^{-1}V_s(x)$. It remains to check the decrease condition for $x, f \notin \mathcal{A}$.

Making use of (6.6), the result of Claim 9, and the fact that $\sigma_2(\cdot) \leq \sigma_1(\cdot)$, we may write

$$\begin{aligned} \max_{f \in F(x)} V_s(f) &= \max_{f \in F(x)} \int V_1(f + \sigma_2(f)\xi)\psi(\xi)\, d\xi \;\leq\; \int \max_{f \in F_{\sigma_2}(x)} V_1(f)\psi(\xi)\, d\xi \\ &\leq e^{-1} \int \min_{z \in \sigma_2(x)\overline{\mathcal{B}}} V_1(x+z)\psi(\xi)\, d\xi \;\leq\; e^{-1} \int V_1(x + \sigma_2(x)\xi)\psi(\xi)\, d\xi \\ &= e^{-1}V_s(x). \end{aligned} \tag{6.24}$$

Let $\rho \in \mathcal{K}_\infty$ come from Lemma 3.2 and define $V(x) := \rho \circ V_s(x)$. Then we may write

$$V(x) \leq \rho \circ \hat{\alpha}_2\left(\frac{3}{2}\omega_2(x)\right) =: \alpha_2(\omega_2(x)), \qquad \text{and}$$

$$V(x) \geq \rho \circ \hat{\alpha}_1\left(\frac{1}{2}\omega_1(x)\right) =: \alpha_1(\omega_1(x)).$$

Since $\rho \in \mathcal{K}_\infty$ is convex, $\rho(e^{-1}s) \le e^{-1}\rho(s)$ for all $s \in \mathbb{R}_{\ge 0}$. Consequently, following (6.24), we may write

$$\max_{f\in F(x)} V(f) = \rho\left(\max_{f\in F(x)} V_s(f)\right) \le \rho(V_s(x)e^{-1}) \le \rho(V_s(x))e^{-1}$$
$$= V(x)e^{-1}. \qquad \square$$

6.2. Necessity. We note that in order to demonstrate that robust $\mathcal{KL}$-stability follows from a smooth Lyapunov function, we actually only make use of a continuous Lyapunov function. Furthermore, the upper semicontinuity of the set-valued map $F(\cdot)$ is not used. This, then, is the result of Theorem 2.8 as well as the necessity of Theorem 2.7

We assume we have a continuous function $V : \mathcal{G} \to \mathbb{R}_{\ge 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that, for every $x \in \mathcal{G}$,

$$\alpha_1(\omega_1(x)) \le V(x) \le \alpha_2(\omega_2(x)), \tag{6.25}$$
$$\max_{f\in F(x)} V(f) \le e^{-1}V(x), \tag{6.26}$$

and $V(x) = 0$ if and only if $x \in \mathcal{A}$. Since $V(\cdot)$ is continuous on $\mathcal{G}$ and bounded away from zero on compact subsets of $\mathcal{G}\backslash\mathcal{A}$, we see that $\mathcal{G}\backslash\mathcal{A}$ is open.

Let $\varepsilon > 0$ satisfy $(1+\varepsilon)^2e^{-1} < 1$. Since $V(\cdot)$ is continuous, for each $x \in \mathcal{G}\backslash\mathcal{A}$ there exists $\tilde{\delta}_1 > 0$ such that for all $\xi \in \mathcal{G}$

$$|x - \xi| \le \tilde{\delta}_1 \quad \Longrightarrow \quad |V(x) - V(\xi)| \le \varepsilon V(x). \tag{6.27}$$

For each $x \in \mathcal{G}\backslash\mathcal{A}$ let $\delta_1(x)$ be one-half the supremum over all $\tilde{\delta}_1 \le 1$ such that (6.27) holds and, for $x \in \mathcal{A}$, let $\delta_1(x) = 0$. Then $\delta_1(\cdot)$ is bounded away from zero on compact subsets of $\mathcal{G}\backslash\mathcal{A}$. Suppose not. Then there exists a compact set $D \subset \mathcal{G}\backslash\mathcal{A}$, a sequence of points $x_i \in D$, and an accumulation point $x^* \in D$ such that $\delta_1(x_i) \to 0$ as $x_i \to x^*$ and

$$|V(z_i) - V(x_i)| > \varepsilon V(x_i), \tag{6.28}$$

where $z_i \in \{x_i\} + \delta(x_i)\overline{\mathcal{B}}$. Since $\delta(x_i) \to 0$ we may pick a subsequence (which we do not relabel) such that $\delta(x_i) < \frac{1}{i}$, which implies that $|z_i - x_i| \le \frac{1}{i}$. Since $x^* \in D$, $V(x^*) > 0$. With the continuity of $V(\cdot)$, as $i \to \infty$, $|V(z_i) - V(x_i)| \to 0$ while

$$V(x_i) \to V(x^*) > 0,$$

which contradicts (6.28).

For every $x \in \mathcal{G}$ let $\delta_2(x)$ be the supremum over all $\hat{\delta} \le 1$ such that $\{x\} + 2\hat{\delta}\overline{\mathcal{B}} \subset \mathcal{G}$. Since $\mathcal{G}$ is open, $\hat{\delta}$ always exists and satisfies $\hat{\delta} > 0$. Moreover $\delta_2(\cdot)$ is bounded away from zero on compact subsets of $\mathcal{G}$. Suppose not. Then there exists $D \subset \mathcal{G}$ compact and a sequence $\{x_i\}_{i=1}^\infty$ such that $x_i \in D$ and the sequence has an accumulation point $x^* \in D$ such that $\delta_2(x_i) \to 0$, and for each i there exists $z_i \in \{x_i\} + \frac{1}{2}\delta_2(x_i)\overline{\mathcal{B}}$ so that $z_i \notin \mathcal{G}$. We pick a subsequence (without relabeling) such that $\delta(x_i) < \frac{1}{2i}$. Since $\mathcal{G}$ is open and $D \subset \mathcal{G}$ is compact, x^* is in the interior of $\mathcal{G}$, and consequently, for i sufficiently large, $\{x^*\} + \frac{1}{i}\overline{\mathcal{B}} \subset \mathcal{G}$. Therefore, again for i sufficiently large, we see that

$$z_i \in \{x_i\} + \frac{1}{2i}\overline{\mathcal{B}} \subset \{x^*\} + \frac{1}{i}\overline{\mathcal{B}},$$

which is a contradiction.

For each $x \in \mathcal{G}$ we define

$$\delta(x) := \min\{\delta_1(x), \delta_2(x), |x|_{\mathcal{A}}\}$$

and note that $x \in \mathcal{A}$ implies $\delta(x) = 0$. That $\delta(\cdot)$ is bounded away from zero on compact subsets of $\mathcal{G}\backslash\mathcal{A}$ follows from the fact that $\delta_2(x)$ is bounded away from zero on compact subsets of $\mathcal{G}$ and that $\delta_1(x)$ and $|x|_{\mathcal{A}}$ are bounded away from zero on compact subsets of $\mathcal{G}\backslash\mathcal{A}$.

Let the function $\sigma : \mathcal{G} \to \mathbb{R}_{\geq 0}$ come from the application of Lemma 3.3 to the function $\delta(\cdot)$ on $\mathcal{G}\backslash\mathcal{A}$ and define $\sigma(x) = 0$ for $x \in \mathcal{A}$. The restriction of $\sigma(\cdot)$ to $\mathcal{G}\backslash\mathcal{A}$ is smooth and, for all $x \in \mathcal{G}\backslash\mathcal{A}$, $\sigma(x) > 0$, satisfying item 2 in Definition 2.3. Since $\delta(x) \leq |x|_{\mathcal{A}}$ for all $x \in \mathcal{G}$, it follows that $\sigma(\cdot)$ is continuous on $\mathcal{G}$. Since $\delta(x) \leq \delta_2(x)$, $\{x\} + \sigma(x)\overline{\mathcal{B}} \subset \mathcal{G}$ for all $x \in \mathcal{G}$, satisfying item 1 of Definition 2.3. It remains to check items 3 and 4 of Definition 2.3.

Since $\sigma(x) \leq \delta_1(x)$ for all $x \in \mathcal{G}\backslash\mathcal{A}$, we may write

$$V(\xi + \sigma(\xi)\overline{\mathcal{B}}) \leq (1+\varepsilon)V(\xi) \ . \tag{6.29}$$

Then, using the definition of $F_\sigma(x)$ in (2.2), the bound (6.29), the decrease (6.26), and (6.27) we may write, for $x \in \mathcal{G}\backslash\mathcal{A}$,

$$\begin{aligned} \max_{\substack{f \in F_\sigma(x) \\ f \in \mathcal{G}\backslash\mathcal{A}}} V(f) \ &\leq \ (1+\varepsilon) \max_{\substack{f \in F(x+\sigma(x)\overline{\mathcal{B}}) \\ f \in \mathcal{G}\backslash\mathcal{A}}} V(f) \ \leq \ (1+\varepsilon)e^{-1} \max_{z \in \{x\}+\sigma(x)\overline{\mathcal{B}}} V(z) \\ &\leq \ (1+\varepsilon)^2 e^{-1} V(x). \end{aligned} \tag{6.30}$$

We can see that (6.30) holds for all $x \in \mathcal{G}$ by considering the two remaining cases. First, suppose $x \in \mathcal{A}$; then, by the definition of $\mathcal{A}$ and the fact that $\sigma(x) = 0$ for $x \in \mathcal{A}$, $f \in \mathcal{A}$, (6.30) is trivially satisfied. Second, suppose $x \notin \mathcal{A}$ and $f \in \mathcal{A}$, then (6.30) is again satisfied since $0 \leq (1+\varepsilon)^2 e^{-1} V(x)$. Let $\lambda := (1+\varepsilon)^2 e^{-1} < 1$. From (6.30), we see that, for any $x \in \mathcal{G}$ and $\phi \in \mathcal{S}_\sigma(x)$, we may write

$$V(\phi(k,x)) \leq \lambda^k V(x) \quad \forall k \in \mathbb{Z}_{\geq 0}.$$

If $x \in \mathcal{A}$, then $V(x) = 0$ and, using (6.25), we may write, for any $\phi \in \mathcal{S}_\sigma(x)$,

$$\alpha_1(\omega_1(\phi(k,x))) \leq V(\phi(k,x)) \leq V(x)\lambda^k = 0 \quad \forall k \in \mathbb{Z}_{\geq 0},$$

which implies that $x \in \mathcal{A}_\sigma$; i.e.,

$$\sup_{k \in \mathbb{Z}_{\geq 0}, \phi \in \mathcal{S}_\sigma(x)} \omega_1(\phi(k,x)) = 0 \quad \forall x \in \mathcal{A}.$$

Furthermore, if $x \in \mathcal{A}_\sigma$, then $x \in \mathcal{A}$ as a consequence of $\mathcal{S}(x) \subseteq \mathcal{S}_\sigma(x)$. Consequently, $\mathcal{A} = \mathcal{A}_\sigma$ and item 3 of Definition 2.3 is satisfied.

Now, using the upper and lower $\mathcal{K}_\infty$ bounds (6.25) on the Lyapunov function we may write, for all $x \in \mathcal{G}$, $\phi \in \mathcal{S}(x)$, and $k \in \mathbb{Z}_{\geq 0}$,

$$\alpha_1(\omega_1(\phi(k,x))) \leq V(\phi(k,x)) \leq \lambda^k V(x) \leq \lambda^k \alpha_2(\omega_2(x)).$$

Inverting $\alpha_1(\cdot)$ we obtain

$$\omega_1(\phi(k,x)) \leq \alpha_1^{-1}\left(\alpha_2(\omega_2(x))\lambda^k\right) =: \beta_\sigma(\omega_2(x), k);$$

i.e., $x^+ \in F_\sigma(x)$ is $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$, satisfying item 4 of Definition 2.3. Therefore $x^+ \in F(x)$ is robustly $\mathcal{KL}$-stable with respect to (ω_1, ω_2) on $\mathcal{G}$. ∎

7. Proof of Theorem 2.10. If we can demonstrate that there exists a continuous Lyapunov function, the result of Theorem 2.8 yields that the $\mathcal{KL}$-stability is robust. Toward this end, we will define a (Lyapunov) function that is similar to (6.2), with the only difference being that the solution set under consideration in (6.2) is for the perturbed difference inclusion $x^+ \in F_\sigma(x)$. Here, however, we are not assuming robust $\mathcal{KL}$-stability. Rather, we are assuming $\mathcal{KL}$-stability of $x^+ \in F(x)$ and continuity of $F(\cdot)$ on $\mathcal{G}\backslash\mathcal{A}$.

In particular, we apply Lemma 6.1, with $\lambda = 2$, to the function $\beta \in \mathcal{KL}$ defining the stability estimate in order to obtain functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(\beta(s,k)) \leq \alpha_2(s)e^{-2k}$ for all $(s,k) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. We define our Lyapunov function as

$$V(x) := \sup_{k \in \mathbb{Z}_{\geq 0}, \phi \in \mathcal{S}(x)} \alpha_1(\omega_1(\phi(k,x)))e^k \quad \forall x \in \mathcal{G}. \tag{7.1}$$

We can then obtain appropriate upper and lower bounds, the required decrease condition, upper semicontinuity of $V(\cdot)$ on $\mathcal{G}$, and continuity of $V(\cdot)$ on $\mathcal{A}$ by following the proof given in section 6.1. We note that the result of Claim 7 also holds. Therefore, in order to appeal to Theorem 2.8 it remains to show that $V(\cdot)$ as defined by (7.1) is lower semicontinuous on $\mathcal{G}\backslash\mathcal{A}$.

Lower semicontinuity of $V(\cdot)$ on $\mathcal{G}\backslash\mathcal{A}$: Let $x \in \mathcal{G}\backslash\mathcal{A}$. Appealing to Claim 7, there exists $\hat{\phi} \in \mathcal{S}(x)$ and $K(x) \in \mathbb{Z}_{\geq 0}$ such that

$$V(x) = \max_{k \in \{0,\ldots,K(x)\}} \alpha_1\left(\omega_1(\hat{\phi}(k,x))\right)e^k.$$

Let $\kappa \in \{0,\ldots,K(x)-1\}$ be the smallest integer such that $\hat{\phi}(\kappa+1,x) \in \mathcal{A}$ or, if $\hat{\phi}(k,x) \in \mathcal{G}\backslash\mathcal{A}$ for all $k \in \{0,\ldots,K(x)\}$, let $\kappa = K(x)$. We see that

$$\max_{k \in \{0,\ldots,K(x)\}} \alpha_1\left(\omega_1(\hat{\phi}(k,x))\right)e^k = \max_{k \in \{0,\ldots,\kappa\}} \alpha_1\left(\omega_1(\hat{\phi}(k,x))\right)e^k$$

since, if $K(x) \neq \kappa$, then $\hat{\phi}(k,x) \in \mathcal{A}$ for $k \in \{\kappa,\ldots,K(x)\}$, which implies that, for those k, $\omega_1(\hat{\phi}(k,x)) = 0$.

Since $F(\cdot)$ is continuous on $\mathcal{G}\backslash\mathcal{A}$ and since $\hat{\phi}(k,x) \in \mathcal{G}\backslash\mathcal{A}$ for all $k \in \{0,\ldots,\kappa\}$, given any $\varepsilon > 0$, Lemma 5.2 yields a $\delta > 0$ such that, for any $\bar{x} \in \{x\} + \delta\overline{\mathcal{B}}$, there exists a solution $\psi \in \mathcal{S}(\bar{x})$ such that we may write

$$\begin{aligned} V(x) &= \max_{k \in \{0,\ldots,\kappa\}} \alpha_1\left(\omega_1(\hat{\phi}(k,x))\right)e^k \\ &\leq \max_{k \in \{0,\ldots,\kappa\}} \alpha_1\left(\omega_1(\psi(k,\bar{x}))\right)e^k \\ &\quad + \max_{k \in \{0,\ldots,\kappa\}} \left|\alpha_1\left(\omega_1(\hat{\phi}(k,x))\right) - \alpha_1\left(\omega_1(\psi(k,\bar{x}))\right)\right|e^k \\ &\leq \sup_{k \in \mathbb{Z}_{\geq 0}} \alpha_1\left(\omega_1(\psi(k,\bar{x}))\right)e^k + \varepsilon \\ &\leq V(\bar{x}) + \varepsilon. \end{aligned}$$

Therefore, $V(\cdot)$ is lower semicontinuous at x; i.e., $\liminf_{z \to x} V(z) \geq V(x)$. □

8. Proof of Claim 1. In order to simplify the presentation, we define

$$V^+(x) := \sup_{f \in F(x)} V(f) \quad \forall x \in \mathcal{G}.$$

We will use W_1^+ and W^+ for the same purpose.

Let $g \in \mathcal{K}_\infty$ be such that $g'(\cdot)$ is nondecreasing, $g'(s) \geq 1$ for all $s \geq 0$, and such that there exists $\gamma \in \mathcal{K}_\infty$ such that

$$\alpha(s)\,(\exp(g(s)) - 1) \geq \gamma(s) \quad \forall s \geq 0.$$

We define $\rho(s) := \exp(g(s)) - 1$ and note that

$$\rho'(s) = (\rho(s) + 1)\,g'(s) \geq \rho(s)\ .$$

The equality shows that $\rho'(\cdot)$ is nondecreasing, so that, by the mean value theorem,

$$\rho(V^+(x)) - \rho(V(x)) \leq \rho'(V^+(x))[V^+(x) - V(x)] \quad \forall x \in \mathcal{G}.$$

For all $x \in \mathcal{G}$ we define $W_1(x) := \rho(V(x))$. It is obvious that $W_1(x) = 0$ if and only if $x \in \mathcal{A}$. For all $s \geq 0$ let $\mu(s) := \frac{1}{2}\min\left\{s, \gamma \circ \rho^{-1}(s)\right\}$ so that $\mu \in \mathcal{K}_\infty$. With this definition we observe that $(\mathrm{Id} - \mu) \in \mathcal{K}_\infty$. Now, for every $x \in \mathcal{G}$ either $\rho(V^+(x)) \leq \frac{1}{2}\rho(V(x))$ or $\rho(V^+(x)) \geq \frac{1}{2}\rho(V(x))$. In the first case

$$W_1^+(x) := \rho(V^+(x)) \leq \frac{1}{2}\rho(V(x)) = \frac{1}{2}W_1(x) \leq W_1(x) - \mu(W_1(x)), \tag{8.1}$$

while, in the latter case, we may write

$$\begin{aligned} W_1^+(x) - W_1(x) &= \rho(V^+(x)) - \rho(V(x)) \leq \rho'(V^+(x))\left[V^+(x) - V(x)\right] \\ &\leq -\rho'(V^+(x))\alpha(V(x)) \leq -\rho(V^+(x))\alpha(V(x)) \\ &\leq -\frac{1}{2}\rho(V(x))\alpha(V(x)) \leq -\frac{1}{2}\gamma(V(x)) \\ &= -\frac{1}{2}\gamma \circ \rho^{-1}(W_1(x)) \leq -\mu(W_1(x))\ . \end{aligned} \tag{8.2}$$

Combining (8.1) and (8.2) we have $W_1^+(x) \leq W_1(x) - \mu(W_1(x))$ for all $x \in \mathcal{G}$.

We require the following lemma, which appeared as [9, Lemma 2.4].

Lemma 8.1. *If $\ell > 1$ and $\varphi \in \mathcal{K}_\infty$ satisfies $(\varphi - \mathrm{Id}) \in \mathcal{K}_\infty$, then there exists $\tilde{\alpha} \in \mathcal{K}_\infty$ such that $\tilde{\alpha} \circ \varphi(s) = \ell\tilde{\alpha}(s)$ for all $s \geq 0$.*

Define $\varphi \in \mathcal{K}_\infty$ as $\varphi(s) := (\mathrm{Id} - \mu)^{-1}(s)$ for all $s \geq 0$. We note that $\varphi(\cdot)$ is well defined by virtue of $(\mathrm{Id} - \mu) \in \mathcal{K}_\infty$. From the definition of $\varphi(\cdot)$ we see that, for all $s \geq 0$, $s - \varphi^{-1}(s) = \mu(s)$ or, equivalently, $\varphi(s) - s = \mu \circ \varphi(s)$. Therefore, $(\varphi - \mathrm{Id}) \in \mathcal{K}_\infty$. Let $\ell = e^1 > 1$ and let $\tilde{\alpha} \in \mathcal{K}_\infty$ come from Lemma 8.1. For all $x \in \mathcal{G}$ we define $W(x) := \tilde{\alpha}(W_1(x))$. We may then write

$$\begin{aligned} W^+(x) &= \tilde{\alpha}(W_1^+(x)) \leq \tilde{\alpha}\left(W_1(x) - \mu(W_1(x))\right) \\ &= \tilde{\alpha}(\varphi^{-1}(W_1(x))) = e^{-1}\tilde{\alpha}(W_1(x)) = e^{-1}W(x) \quad \forall x \in \mathcal{G}. \end{aligned}$$

Finally, we define the functions $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}_\infty$ by $\hat{\alpha}_1 := \tilde{\alpha} \circ \rho \circ \alpha_1$ and $\hat{\alpha}_2 := \tilde{\alpha} \circ \rho \circ \alpha_2$ so that (2.9) holds. ☐

REFERENCES

[1] R. P. Agarwal, *Difference Equations and Inequalities: Theory, Methods, and Applications*, 2nd ed., Marcel Dekker, New York, 2000.

[2] J.-P. Aubin and A. Cellina, *Differential Inclusions: Set-Valued Maps and Viability Theory*, Springer-Verlag, New York, 1984.

[3] J.-P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser Boston, Boston, 1990.
[4] F. H. Clarke, Y. S. Ledyaev, and R. J. Stern, *Asymptotic stability and smooth Lyapunov functions*, J. Differential Equations, 149 (1998), pp. 69–114.
[5] A. F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic, Norwell, MA, 1988.
[6] G. Grimm, M. Messina, A. R. Teel, and S. Tuna, *Examples when model predictive control is nonrobust*, Automatica, submitted.
[7] F. C. Hoppensteadt, *Singular perturbations on the infinite interval*, Trans. Amer. Math. Soc., 123 (1966), pp. 521–535.
[8] Z. Jiang and Y. Wang, *A converse Lyapunov theorem for discrete-time systems with disturbances*, Systems Control Lett., 45 (2002), pp. 49–58.
[9] C. M. Kellett, *Advances in Converse and Control Lyapunov Functions*, Ph.D. thesis, University of California, Santa Barbara, 2002.
[10] C. M. Kellett and A. R. Teel, *Results on converse Lyapunov theorems for difference inclusions*, in Proceedings of the 42nd IEEE Conference on Decision and Control, Maui, HI, 2003.
[11] C. M. Kellett and A. R. Teel, *Smooth Lyapunov functions and robustness of stability for difference inclusions*, Systems Control Lett., 52 (2004), pp. 395–405.
[12] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer Academic, Dordrecht, The Netherlands, 1991.
[13] J. Kurzweil, *On the inversion of Ljapunov's second theorem on stability of motion*, Amer. Math. Soc. Transl., Ser. 2, 24 (1956), pp. 19–77.
[14] V. Lakshmikantham and L. Salvadori, *On Massera type converse theorem in terms of two different measures*, Boll. Unione Mat. Ital., 13 (1976), pp. 293–301.
[15] Y. Lin, E. D. Sontag, and Y. Wang, *A smooth converse Lyapunov theorem for robust stability*, SIAM J. Control Optim., 34 (1996), pp. 124–160.
[16] D. Mayne, J. Rawlings, C. Rao, and P. Scokaert, *Constrained model predictive control: Stability and optimality*, Automatica, 36 (2000), pp. 789–814.
[17] A. A. Movchan, *Stability of processes with respect to two metrics*, J. Appl. Math. Mech., 24 (1960), pp. 1475–1740 (translation of Prikl. Mat. Mekh.).
[18] D. Nešić, A. R. Teel, and P. V. Kokotović, *Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations*, Systems Control Lett., 38 (1999), pp. 259–270.
[19] E. D. Sontag, *Comments on integral variants of ISS*, Systems Control Lett., 34 (1998), pp. 93–100.
[20] E. D. Sontag and Y. Wang, *Lyapunov characterizations of input to output stability*, SIAM J. Control Optim., 39 (2001), pp. 226–249.
[21] A. Stuart and A. Humphries, *Dynamical Systems and Numerical Analysis*, Cambridge University Press, Cambridge, UK, 1996.
[22] A. R. Teel and L. Praly, *A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions*, ESAIM Control Optim. Calc. Var., 5 (2000), pp. 313–367.
[23] F. W. Wilson, *Smoothing derivatives of functions and applications*, Trans. Amer. Math. Soc., 139 (1969), pp. 413–428.