Further Results on Robustness of (Possibly Discontinuous) Sample and Hold Feedback

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***Abstract*—We demonstrate that sample and hold state feedback control (possibly discontinuous with respect to the state) is robust when the closed loop system possesses an appropriate Lyapunov function. We first show that if a Lyapunov decrease over sampling periods exists for the nominal system, this decrease can be maintained with some degradation relative to a sufficiently small additive perturbation. We then proceed to catalog several applications of this robustness, e.g., robustness to measurement noise, computational delays, or fast actuator dynamics.**

***Index Terms*—Lyapunov models, robustness, sampled-data systems.**

I. Introduction and Background

For over 50 years, control engineers have appreciated that asymptotically stable closed-loop (continuous time or discrete time) systems that use continuous feedback laws exhibit nominal robustness, i.e., small errors lead to convergence to a small neighborhood of the origin. Nominal robustness can be viewed as a consequence of the continuous dependence of solutions of ordinary differential or difference equations with respect to perturbations of the right-hand side. Equivalently, when combined with asymptotic stability, it can be viewed as a consequence of the existence of a smooth Lyapunov function for the right-hand side (see [14]).

Nominal robustness for asymptotically stable closed-loop systems that use discontinuous feedback laws is more subtle. It is equivalent to being able to regularize the differential or difference equation, i.e., the right-hand side is turned into a set-valued map in an appropriate way, without destroying asymptotic stability (see [6] or [19] for continuous time results and [7] or [10] for discrete-time results). However, nominal robustness cannot be guaranteed in general. For example, systems affine in the control that fail Brockett's well-known necessary condition for stabilizability by continuous feedback [3] are not asymptotically stabilizable, nor even practically asymptotically stabilizable with nominal robustness by discontinuous feedback (see [15]).

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Recently, several authors have shown that when (discontinuous) feedback controls are constructed based on locally Lipschitz control Lyapunov functions, nominal robustness is guaranteed by implementing the feedback by means of sufficiently fast (but not arbitrarily fast) sample and hold (see [4], [5], [7], [9], and [17]). By sample and hold we mean that, at each sampling instance, the state is measured and the control for the measured state is then computed and applied as a constant until the next sampling instance. This is sometimes referred to as a zero-order hold. We observe that sample and hold implementation is not a universal panacea for nominal robustness, i.e., robustness is not guaranteed simply by employing a sample and hold strategy. Indeed, consider the system

$$\dot{x} = u, \qquad u = -\text{sign}(x)\sqrt{|x|_{\text{mod}1}} \tag{1}$$

where $\text{sign}(0) = 0$ and $|s|_{\text{mod}1}$ is defined as the difference between $|s|$ and the nonnegative integer $\ell \in \mathbb{Z}_{\geq 0}$ satisfying $\ell < |s|$, i.e., $|s|_{\text{mod}1} = |s| - \ell$. With the control implemented as a sample and hold, with sampling period $T > 0$, we turn our attention to the exact discrete-time model

$$x^+ = x + Tu(x).$$

The origin of the exact discrete-time model, in the absence of any disturbance is (practically) globally asymptotically stable (GAS), with convergence rate in "continuous time" (i.e., $t_k = kT$) uniform in the sampling period T, but the GAS has no robustness. That is, restricting our attention to $x > 0$, for any small $\varepsilon > 0$ the system

$$\dot{x} = u + \varepsilon$$

has equilibria arbitrarily far away from the origin. This can be seen by examining Fig. 1 and noting that the addition of $\varepsilon > 0$ to $x > 0$ will cause the control to be zero near every integer. Implementing the control via a sample and hold does not induce robustness. Indeed, by examining the discrete-time model, we see that every x such that $|x|_{\text{mod}1} = \varepsilon^2$ is an equilibrium point. It is not difficult to show that this example does not admit a continuous Lyapunov function satisfying Assumption 1.

The purpose of this paper is to emphasize that nominal robustness is guaranteed when using sample and hold feedback when-

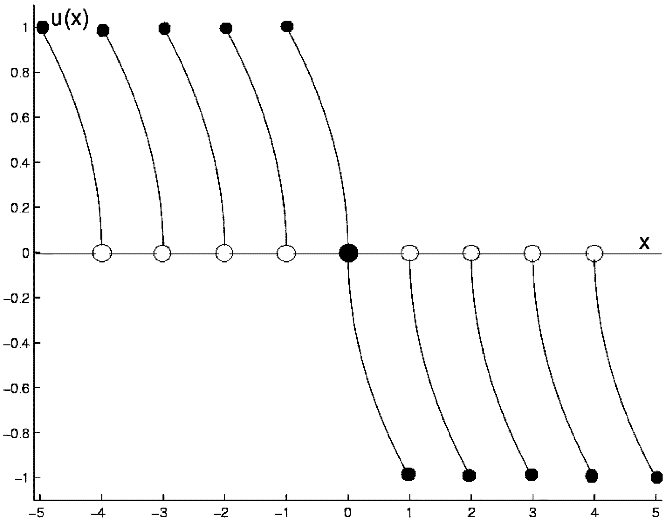

Fig. 1. Control function.

ever there is a *continuous* Lyapunov function that demonstrates asymptotic stability for the system evaluated at samples (in [7] and [10] we have shown that the existence of a smooth Lyapunov function that demonstrates asymptotic stability for the sampled system is necessary for robustness), and to catalog various types of robustness that derive from such Lyapunov functions, e.g., robustness to measurement noise (Section III), to computational delay (Section III-B), to fast actuator dynamics (Section IV), to pulse-width-modulated (PWM)-type actuators (Section V), etc. We do not claim that the applications presented herein comprise a comprehensive list, but, rather, that they are representative of how one may apply the main result (Theorem 1). In this work, we stress that the Lyapunov function does not need to be a control Lyapunov function for the continuous time control system and that it is sufficient for the perturbations to be small in $\mathcal{L}_1$ over each sampling period.

II. $\mathcal{L}_1$-Type Robustness

For $V : \mathbb{R}^n \to \mathbb{R}$ and $r, R \in \{-\infty\} \bigcup \mathbb{R}$ such that $r < R$, we define $\mathcal{V}(r, R) := \{x \in \mathbb{R}^n : r \leq V(x) \leq R\}$. We denote $\mathcal{V}(-\infty, R)$ by $\mathcal{V}(R)$. Recall that a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it is continuous, zero at zero, strictly increasing, and unbounded. We denote the Euclidean norm by $|\cdot|$.

We will consider a system of the form

$$\dot{x} = f(x, u), \qquad x \in \mathbb{R}^n, \qquad u \in \mathcal{U} \tag{2}$$

where $\mathcal{U}$ is the set of admissible control inputs. In what follows we will assume we are given positive constants r, R, r_m, and R_m (such that $r < R$, and $R_m < R_m$), and a locally compact metric space $\mathcal{U}$ containing a unique zero element, "0" as well as at least one nonzero element. We assume the existence of a continuous closed loop Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ satisfying the following.

Assumption 1:

1) There exists a function $\rho \in \mathcal{K}_\infty$ such that $|V(x_1) - V(x_2)| \leq \rho(|x_1 - x_2|)$ for all $x_1, x_2 \in \mathcal{V}(R, R + R_m)$.
2) There exists a feedback function $\alpha : \mathbb{R}^n \to \mathcal{U}$ and constants $T, c > 0$ such that the solution $x(t)$ of $\dot{x} = f(x, \alpha(x(0)))$ with $x(0) \in \mathcal{V}(R)$ and $\alpha(\cdot)$ implemented via a sample and hold, satisfies

$$V(x(T)) \leq \max\{V(x(0)) - c, r\}, \text{ and} \tag{3}$$

$$V(x(t)) \leq \max\{V(x(0)), r\} + r_m \qquad \forall\, t \in [0, T]. \tag{4}$$

Remark 1: We note that item 2) in the aforementioned assumption can be extended to cover subsequent sampling periods. That is, since (2) is time-invariant and the solution $x(t)$ is absolutely continuous, the previously mentioned equations actually imply that, for all $i \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} V(x(iT)) &\leq \max\{V(x(0)) - ic, r\}, \text{ and} \\ V(x(t)) &\leq \max\{V(x(iT)), r\} + r_m \\ &\quad \forall\, t \in [iT, (i+1)T]. \end{aligned}$$

Note that item 1) is merely a characterization of the (uniform) continuity of $V(\cdot)$ in the region of interest. We also observe that item 2) requires that the Lyapunov function decrease at each sampling instance and be bounded between samples in a "semiglobal practical" region. The constants r, R, and R_m, along with the function $V(\cdot)$ define this region. We observe that $V(\cdot)$ must be continuous on the set $\{x \in \mathbb{R}^n : r \leq V(x) \leq R + R_m\}$ and if $x(0)$ is such that $V(x(0)) \leq R$ then (3) implies that, eventually (i.e., for sufficiently large $i \in \mathbb{Z}_{\geq 0}$), $V(x(iT)) \leq r$. Finally, the constant $R + r_m$ bounds intersample overshoots of $x(t)$ as seen through $V(\cdot)$ and ensures that such overshoots are not allowed to leave the region $V(x(t)) \leq R + R_m$. This follows from (4) as $V(x(0)) \leq R$ and $r_m < R_m$ so that $V(x(t)) \leq R + R_m$; i.e., $x(t)$ cannot leave the region where our assumptions hold. Finally, note that we have not assumed that $\mathcal{V}(R)$ is compact, i.e., we allow Lyapunov functions for the consideration of closed (not necessarily compact) sets (see [1] and [9]).

Remark 2: We observe that (3) and (4) are given in terms of solutions of the closed-loop nonlinear system, and thus may be difficult to check. However, if one has a continuously differentiable Lyapunov function for the closed-loop system, then (3) and (4) follow. Specifically, if one has a continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\dot{V}(x) = \langle \nabla V(x), f(x, \alpha(x)) \rangle \leq -\gamma(V(x))$$

where $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is continuous and positive definite then, for $x \in \mathcal{V}(r, R)$, $\dot{V}(x) \leq -\gamma(r)$. This clearly implies (3) and (4).

We also require the following assumption on the vector field $f(\cdot, \cdot)$ of (2).

Assumption 2:

1) There exists a constant $M \geq 0$ such that $|f(x, u)| \leq M$, for all $x \in \mathcal{V}(R + R_m)$ and all $u \in \mathcal{U}$.
2) There exist constants $L_{f,x} > 0$ and $L_{f,u} > 0$ such that $|f(x_1, u_1) - f(x_2, u_2)| \leq L_{f,x}|x_1 - x_2| + L_{f,u}|u_1 - u_2|$ for all $x_1, x_2 \in \mathcal{V}(R + R_m)$ and $u_1, u_2 \in \mathcal{U}$.

Remark 3: It has been shown (see [4], [5], [9], and [18]) that Assumption 2 coupled with an asymptotic controllability assumption implies the existence of a function $V(\cdot)$ and a sample and hold controller $\alpha(\cdot)$ such that (3) and (4) are satisfied. We also point out that these results do not apply merely to the case of asymptotic controllability to the origin, but also may be applied to the case of asymptotic controllability to a (not necessarily compact) set.

We now proceed to establish robustness of the assumed stability properties when (2) is subjected to additive perturbations. For simplicity and ease of presentation, we consider only the first sampling period, i.e., $t \in [0, T]$. However, the argument easily extends to subsequent sampling periods $t \in [iT, (i+1)T]$, for all nonnegative integers $i \in \mathbb{Z}_{\geq 0}$.

Theorem 1: Under Assumptions 1 and 2, consider

$$\dot{x}(t) = f(x(t), \alpha(x(0))) + d(t), \qquad t \in [0, T] \tag{5}$$

where $x(0) \in \mathcal{V}(R)$. Let $\sigma \in [0, R_m - r_m)$. If the disturbance $d(\cdot)$ satisfies

$$\max_{t\in[0,T]} \left| \int_0^t d(s)ds \right| \leq \rho^{-1}(\sigma) e^{-L_{f,x}T} \tag{6}$$

then, for all $t \in [0, T]$, the solution $x(t)$ exists and satisfies

$$V(x(T)) \leq \max\{V(x(0)) - c, r\} + \sigma \tag{7}$$

$$V(x(t)) \leq \max\{V(x(0)), r\} + (r_m + \sigma). \tag{8}$$

Remark 4: In Sections III and IV, we use the bound

$$\int_0^T |d(s)|ds \leq \rho^{-1}(\sigma) e^{-L_{f,x}T}. \tag{9}$$

This bound clearly implies the bound assumed in Theorem 1.

Remark 5: We require $\sigma < R_m - r_m$ to ensure that $x(t)$ remains in the region where the assumptions are valid for all $t \in [0, T]$. This follows from (8), i.e,

$$V(x(t)) \leq R + r_m + \sigma < R + R_m \qquad \forall\, t \in [0, T].$$

Remark 6: We note that, for a general sampling period, $t \in [iT, (i+1)T]$, $i \in \mathbb{Z}_{\geq 0}$, (7) and (8) become

$$\begin{aligned} V(x((i+1)T)) &\leq \max\{V(x(0)) - (i+1)c + i\sigma, r\} \\ &\quad + \sigma, \text{ and} \\ V(x(t)) &\leq \max\{V(x(iT)), r\} + (r_m + \sigma) \end{aligned}$$

when the control is implemented via sample and hold and $d(\cdot)$ satisfies (6) for each sampling period. In addition, to repeatedly apply Theorem 1 we require the constraint $\sigma < \min\{R - r, c\}$. This guarantees, via (7), that $V(x(iT)) < R$ for all $i \in \mathbb{Z}_{>0}$, so that the "initial condition" $x(iT) \in \mathcal{V}(R)$.

Proof: Let $w : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfy

$$\dot{w}(t) = f(w(t), \alpha(w(0))) \qquad \forall\, t \in [0, T] \tag{10}$$

with $w(0) = x(0)$ (and, obviously, $\alpha(x(0)) = \alpha(w(0))$). We may write the solutions of (5) and (10) as

$$\begin{aligned} x(t) &= x(0) + \int_0^t f(x(s), \alpha(x(0)))ds + \int_0^t d(s)ds \\ w(t) &= w(0) + \int_0^t f(w(s), \alpha(w(0)))ds. \end{aligned}$$

Therefore

$$|x(t) - w(t)| \leq \int_0^t L_{f,x}|x(s) - w(s)|ds + \left| \int_0^t d(s)ds \right|. \tag{11}$$

Define

$$k(t) := \left| \int_0^t d(s)ds \right|.$$

Note that $k(0) = 0$. Let $\bar{\sigma} := \max_{t\in[0,T]} k(t)$. It follows from (6) that

$$\bar{\sigma} \leq \rho^{-1}(\sigma) e^{-L_{f,x}T}. \tag{12}$$

Applying Gronwall's Lemma, we obtain

$$|x(t) - w(t)| \leq \bar{\sigma} e^{L_{f,x}t} \leq \rho^{-1}(\sigma). \tag{13}$$

Now, from (13) and (3), we get (7)

$$\begin{aligned} V(x(T)) &= V(w(T)) + V(x(T)) - V(w(T)) \\ &\leq \max\{V(w(0)) - c, r\} + \rho\left(|x(T) - w(T)|\right) \\ &\leq \max\{V(w(0)) - c, r\} + \rho\left(\rho^{-1}(\sigma)\right) \\ &= \max\{V(x(0)) - c, r\} + \sigma. \end{aligned}$$

Equation (8) follows via a similar argument, appealing this time to (4)

$$\begin{aligned} V(x(t)) &= V(w(t)) + V(x(t)) - V(w(t)) \\ &\leq \max\{V(w(0)), r\} + r_m + \rho\left(|x(t) - w(t)|\right) \\ &\leq \max\{V(x(0)), r\} + r_m + \sigma. \end{aligned}$$

■

III. Robustness to Measurement Noise

In this section, we will demonstrate that, under Assumptions 1 and 2, a sample and hold controller will be robust with respect to small measurement errors. That is, if we implement the control using a corrupted measurement $x + \eta$ rather than with the true state x, the trajectory of the controlled system will still be such that the closed loop Lyapunov function decreases at sampling instances. Similar results are established in [4] and [18]. The important observation here is that, while the measurement noise may be persistent, nothing more than measurable, and unknown (but bounded), since we implement the control via a sample and hold procedure, it is only the noise values at the sampling instances that are important.

Proposition 1: Let $\sigma \in [0, \min\{R_m - r_m, R - r, (c/4)\})$ and

$$N \in \left[0, \min\left\{ \frac{\rho^{-1}(\sigma)}{(2 + L_{f,x}T)} e^{-L_{f,x}T},\ \rho^{-1}(R - r - \sigma) \right\} \right). \tag{14}$$

Under Assumptions 1 and 2, consider the system

$$\dot{x}(t) = f(x(t), \alpha(x(iT) + \eta(iT))), \qquad x(0) \in \mathcal{V}(R - 2\rho(N)) \tag{15}$$

where $\eta(iT)$ represents samples of a bounded noise function $\eta : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$. If $|\eta(t)| \leq N$ then, for all $t \geq 0$ the solution $x(t)$ exists and satisfies

$$V(x(iT)) \leq \max\left\{V(x(0)) - \frac{(3i-2)c}{4}, \; r + \rho(N) + \sigma\right\}, \text{ and} \tag{16}$$

$$V(x(t)) \leq \max\{V(x(iT)) + \rho(N), r\} + (r_m + \sigma + \rho(N)). \tag{17}$$

where (17) holds for all $t \in [iT, (i+1)T]$.

We observe that the result of Theorem 1 holds for a fixed sampling period T and fixed constants R, c, etc. However, we observe that, in general, the decrease given by c will be dependent on the size of the sampling period. In particular, as $T \to 0$, $c \to 0$. As a consequence, in the aforementioned proposition, we note that $\sigma \to 0$ which then implies that $N \to 0$. Thus, we observe the result from [18] that fast sampling reduces the allowable measurement error.

Proof: We construct a fictitious noise function $\eta_L(\cdot)$ that is globally Lipschitz and matches $\eta(\cdot)$ at sampling instances. Consequently, the solutions under $\eta_L(\cdot)$ are the same as the solutions under $\eta(\cdot)$. Let $\eta_L(\cdot)$ be the linear interpolation between each noise sample. Then, with the noise bound N of (14) and the sampling period T, $\eta_L(\cdot)$ is bounded by N and has Lipschitz constant less than or equal to $2N/T$. We remark that any representation for $\eta_L(\cdot)$ such that $\eta_L(\cdot)$ is globally Lipschitz and matches $\eta(\cdot)$ at sampling instances is sufficient. We also note that, in what follows, precise knowledge of this signal is unnecessary. That is, we have not assumed that we know this function, only that it is bounded and we know an upper bound on its Lipschitz constant. We perform a coordinate change, $z = x + \eta_L$, in order to write

$$\dot{z} = f(z - \eta_L, \alpha(z_i)) + \dot{\eta}_L. \tag{18}$$

We rewrite the system as

$$\begin{aligned}\dot{z} &= f(z - \eta_L, \alpha(z_i)) + f(z, \alpha(z_i)) - f(z, \alpha(z_i)) + \dot{\eta}_L \\ &= f(z, \alpha(z_i)) + d\end{aligned} \tag{19}$$

where we have defined

$$d := f(z - \eta_L, \alpha(z_i)) - f(z, \alpha(z_i)) + \dot{\eta}_L. \tag{20}$$

We note that, if $z \in \mathcal{V}(R + R_m - \rho(N))$ then $z - \eta_L \in \mathcal{V}(R + R_m)$ so that

$$|d| \leq \frac{N}{T}(L_{f,x}T + 2). \tag{21}$$

With the first bound shown previously in the minimum defining N we see that

$$\int_0^T |d(\tau)| d\tau \leq \rho^{-1}(\sigma) e^{-L_{f,x}T}. \tag{22}$$

Therefore, we apply Theorem 1 to (19) to get

$$V(z(T)) \leq \max\{V(z(0)) - c, r\} + \sigma \tag{23}$$

$$V(z(t)) \leq \max\{V(z(0)), r\} + r_m + \sigma \quad \forall\, t \in [0, T]. \tag{24}$$

We first ensure that the trajectory $z(t)$ remains in the region where our assumptions hold. Note that, with $z(0) \in \mathcal{V}(R - \rho(N))$ and $\sigma < R_m - r_m$, the first term in the maximum in (24) becomes

$$V(z(t)) \leq V(z(0)) + r_m + \sigma < R - \rho(N) + R_m. \tag{25}$$

We further note that the bound $N \leq \rho^{-1}(R - r - \sigma)$ implies $r < R - \rho(N)$ so that the second term in the maximum in (24) yields

$$V(z(t)) \leq r + r_m + \sigma < R - \rho(N) + R_m. \tag{26}$$

Therefore, the bound on d from (21) holds.

Remark 7: The astute reader will have noticed that the disturbance $d(t)$ as defined in (20) also depends on the system state $z(t)$, and Assumption 1 holds only when $z(t) \in \mathcal{V}(R + R_m)$ for $t \in [0, T]$ which looks like the conclusion (8) of Theorem 1 . Furthermore, our ability to use the Lipschitz constant on $f(\cdot, u)$ relied on $z(t) \in \mathcal{V}(R + R_m - \rho(N))$. Nevertheless, Theorem 1 can still be used to get the final robustness arguments. This can be argued as follows: let $\hat{t} < T$ be the maximal time such that $z(\hat{t}) \in \mathcal{V}(R + R_m - \rho(N))$. That is, $z(t) \in \mathcal{V}(R + R_m - \rho(N))$ for $t \in [0, \hat{t}]$ and $z(t) \notin \mathcal{V}(R + R_m - \rho(N))$ immediately after $\hat{t}$. We know that (8) holds for all $t \in [0, \hat{t}]$. Specifically, we may write, for all $t \in [0, \hat{t}]$

$$\begin{aligned}V(z(t)) &\leq \max\{V(z(0)), r\} + r_m + \sigma \\ &< \max\{V(z(0)), r\} + R_m.\end{aligned}$$

Using $z(0) \in \mathcal{V}(R - \rho(N))$ and $r < R - \rho(N) - \sigma$ we see that $V(z(t)) < R + R_m - \rho(N)$. Therefore, $z(\hat{t})$ is strictly contained in $\mathcal{V}(R + R_m - \rho(N))$ (i.e., $z(\hat{t})$ is not on the boundary of $\mathcal{V}(R + R_m - \rho(N))$). However, since $z(\cdot)$ is absolutely continuous and $V(\cdot)$ is continuous, this contradicts $\hat{t}$ being the maximal time such that $z(\hat{t}) \in \mathcal{V}(R + R_m - \rho(N))$. Consequently, $z(t) \in \mathcal{V}(R + R_m - \rho(N))$ for all $t \in [0, T]$.

We note that, for the first bound in (23) we have

$$V(z(T)) \leq V(z(0)) - c + \sigma < R - \rho(N) \tag{27}$$

and the second bound similarly yields

$$V(z(T)) \leq r + \sigma < R - \rho(N) \tag{28}$$

so that we may repeat the above arguments for $t \in [T, 2T]$, $t \in [2T, 3T]$, and so on.

Let $i \in \mathbb{Z}_{\geq 0}$. If (21) holds for all sampling periods (which it does by construction) then (23) can be written as

$$V(z(iT)) \leq \max\{V(z(0)) - i(c - \sigma), r + \sigma\} \tag{29}$$

so that, with the constraint $\sigma < (c/4)$, we see that $V(z(iT))$ is decreasing with i, i.e., the value of V evaluated along the trajectory $z(t)$ decreases at each sampling instance until it lies below the value $r + \sigma$.

We rewrite (29) as

$$V(z(iT)) \leq \max\{V(x(0)) + \rho(N) - i(c - \sigma), r + \sigma\} \tag{30}$$

and, from (27) and (28), we see that $x(iT)$ and $z(iT)$ are both in $\mathcal{V}(R+R_m)$ (in fact, they are both in $\mathcal{V}(R)$) so we may write

$$V(x(iT)) \leq \max\{V(x(0))+2\rho(N)-i(c-\sigma), r+\rho(N)+\sigma\}. \quad (31)$$

We note that, the first bound on N implies that $N < \rho^{-1}(\sigma)$. So, in the case of the first bound obtaining, using $\sigma < (c/4)$ we have

$$\begin{aligned} V(x(iT)) &< V(x(0)) + 2\sigma - i(c-\sigma) \\ &\leq V(x(0)) - \frac{(3i-2)c}{4}. \end{aligned} \quad (32)$$

In other words, V evaluated along $x(t)$ decreases at sampling instances until it lies below the value $r+\rho(N)+\sigma$.

Similarly, with the bounds (25) and (26), from (24) we obtain

$$V(x(t)) \leq \max\{V(x(0))+\rho(N), r\} + r_m + \sigma + \rho(N). \quad (33)$$

■

A. Estimation Robustness

One obvious type of measurement noise arises when a state observer is employed to estimate the actual state. Let the observed state be denoted by $\hat{x}_i$ and the actual state by x_i. Under the assumption that the difference between the observed state and the actual state is sufficiently small at sampling periods, we will show that stability is maintained in spite of small estimation errors. See [16] for an illustration of the use of this concept under a weak observability assumption.

Proposition 1: Let $\sigma \in [0, \min\{R_m - r_m, R-r, (c/4)\})$ and

$$N \in \left[\min\left\{\frac{\rho^{-1}(\sigma)}{(2+L_{f,x}T)}e^{-L_{f,x}T}, \qquad \rho^{-1}(R-r-\sigma)\right\}\right). \quad (34)$$

Under Assumptions 1 and 2, consider

$$\dot{x}(t) = f(x(t), \alpha(\hat{x}(iT))), \qquad x(0) \in \mathcal{V}(R-2\rho(N)). \quad (35)$$

If $|\hat{x}(iT) - x(iT)| \leq N$ then, for all $t \geq 0$ the solution $x(t)$ exists and satisfies (16) and (17).

Proof: Again, we note that, due to the use of a sample and hold control, it is only the estimation error at sampling instances that affects the system. Denote the estimation error at the sampling instance by $\eta(iT) := \hat{x}(iT) - x(iT)$. We may rewrite (35) as

$$\begin{aligned} \dot{x} &= f(x, \alpha(\hat{x}(iT))) = f(x, \alpha(\hat{x}(iT) + x(iT) - x(iT))) \\ &= f(x, \alpha(x(iT) + \eta(iT))). \end{aligned}$$

With the bound (34) on the estimation error, we may repeat the argument of the proof of Proposition 1. ■

B. Computational Delay

Another type of error that can be seen to be equivalent to measurement noise is the error due to computational delay. Suppose a small delay constant $\delta > 0$. Then, the delay can be considered as measurement noise if we suppose that the measurement is taken at time $t = iT - \delta$ and then implement the control (from the measurement $x(iT-\delta)$) at time iT.

Proposition 3: Let $\sigma \in [0, \min\{R_m - r_m, R-r, (c/4)\})$ and

$$\delta \in \left[0, \min\left\{\frac{\rho^{-1}(\sigma)}{M(2+L_{f,x}T)}e^{-L_{f,x}T}, \frac{\rho^{-1}(R-r-\sigma)}{M}\right\}\right). \quad (36)$$

Under Assumptions 1 and 2, consider

$$\dot{x} = f(x, \alpha(x(iT-\delta))), \qquad x(0) \in \mathcal{V}(R - 2\rho(M\delta)). \quad (37)$$

Then, for all $t \geq 0$ the solution $x(t)$ exists and satisfies (16) and (17).

In the case of linear systems, [20] presented a checkable condition for a prespecified delay in terms of the existence of a right coprime factorization of a certain constructed matrix.

Proof: Similar to Section III-A, we can rewrite (37) as

$$\begin{aligned} \dot{x} &= f(x, \alpha(x(iT-\delta))) \\ &= f(x, \alpha(x(iT-\delta) + x(iT) - x(iT))) \\ &= f(x, \alpha(x(iT) + \eta(iT))) \end{aligned}$$

where $\eta(iT) := x(iT-\delta) - x(iT)$. From item 1 of Assumption 2, we know that $|\eta(iT)| = |x(iT-\delta) - x(iT)| \leq M\delta$. Therefore, with the bound (36) on the delay, we can again follow the proof of Proposition 1. ■

IV. Fast Actuator Dynamics

In this section, we demonstrate that the assumed stability properties are preserved when the sample and hold control is applied through an actuator with sufficiently fast dynamics. When the controller is implemented continuously (i.e., not via a sample and hold procedure), this result is well-known in the singular perturbation literature (see [11] or [13]) and can be applied in the case of output feedback (see [2]). Suppose that the actuator dynamics are written in the following singularly perturbed form:

$$\epsilon \dot{x}_a = f_a(x_a, u_a) \quad (38)$$

$$y_a = h_a(x_a, u_a) \quad (39)$$

where $x_a \in \mathbb{R}^p$, $\epsilon > 0$ is a small positive constant, $f_a : \mathbb{R}^p \times \mathcal{U} \to \mathbb{R}^p$ is locally Lipschitz in its first argument and measurable in its second argument, and $h_a : \mathbb{R}^p \times \mathcal{U} \to \mathcal{U}$ is locally Lipschitz in its first argument uniformly in $u_a \in \mathcal{U}$. The interconnection between (2) and the actuator (38) and (39) is given by $u_a = \alpha(x(iT))$ and $u = y_a$.

Recall that a function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if, for each $t \geq 0$, $\beta(\cdot, t)$ is nondecreasing and $\lim_{s \to 0^+} \beta(s,t) = 0$, and, for each $s \geq 0$, $\beta(s, \cdot)$ is nonincreasing and $\lim_{t\to\infty} \beta(s,t) = 0$. We characterize the fast actuator (38) and (39) by the following assumption.

Assumption 3:

1) For each $u_a \in \mathcal{U}$, $f_a(x_a, u_a) = 0$ has a unique solution $\bar{x}_a(u_a)$ such that $\bar{y}_a := h_a(\bar{x}_a, u_a) = u_a$ and there exists a compact set $X_a \subseteq \mathbb{R}^p$ such that $\bar{x}_a(u_a) \in X_a$ for all $u_a \in \mathcal{U}$.
2) Let $x_a(t)$ denote the solution to (38) with a constant input $u_a \in \mathcal{U}$, and let $r_a \in \mathbb{R}_{>0}$ be some positive constant. There exists a $\mathcal{KL}$ function $\beta(\cdot,\cdot)$, independent of $\epsilon > 0$, such that, $|x_a(t) - \bar{x}_a(u_a)| \le \beta(|x_a(0) - \bar{x}_a(u_a)|, t/\epsilon)$ when $x_a(0) \in B(X_a, r_a) := \{\zeta_a \in \mathbb{R}^p : |\zeta_a|_{X_a} \le r_a\}$.

Remark 8: The first item of Assumption 1 refers to the ability of the actuator to accurately reproduce the reference input. The second item assumes convergence and characterizes the speed with which the actuator moves from its initial state to that which reproduces the reference input, insisting that it is uniform in the time scale $\tau = (t/\varepsilon)$ and with respect to u_a.

We denote the diameter of X_a (plus a small margin) by

$$K := \max\{|a-b| : a, b \in X_a\} + r_a$$

and let L_h be the Lipschitz constant for $h_a(\cdot, u_a)$ on $B(X_a, r_a)$ uniformly in $u_a \in \mathcal{U}$. Let $\widetilde{T} > 0$ be such that

$$\beta(K, \widetilde{T}) \le \frac{\rho^{-1}(\sigma)}{2TL_{f,u}L_h} e^{-L_{f,x}T}.$$

We note that such a $\widetilde{T} > 0$ exists by virtue of $\beta \in \mathcal{KL}$.

Proposition 4: Let $\sigma \in [0, R_m - r_m)$. Under Assumptions 1–3, consider (2), (38) and (39) where the actuator input is $\alpha(x(0))$ $(x(0) \in \mathcal{V}(R))$ and initial condition $x_a(0) \in B(X_a, r_a)$. If

$$\epsilon \le \epsilon^* := \frac{\rho^{-1}(\sigma)}{2\widetilde{T}\beta(K,0)L_{f,u}L_h} e^{-L_{f,x}T} \tag{40}$$

then, for all $t \in [0, T]$ the solution $x(t)$ exists and satisfies (7) and (8).

Remark 9: We note that, with the additional constraint that $\epsilon > 0$ guarantees $\beta(K, T/\epsilon) \le r_a$ we have that $x(T) \in B(X_a, r_a)$. If we further constrain σ by $\sigma \in [0, \min\{R_m - r_m, R - r, c\})$, then Proposition 4 can be applied for subsequent intervals $[iT, (i+1)T]$, where $i \in \{1, 2, \ldots\}$.

Proof: We write the controlled system as

$$\begin{aligned}\dot{x}(t) &= f(x(t), y_a(t)) \\ &= f(x(t), \bar{y}_a) + [f(x(t), y_a(t)) - f(x(t), \bar{y}_a)] \\ &= f(x(t), \alpha(x(0))) + d(t)\end{aligned}$$

where $d(t) = f(x(t), y_a(t)) - f(x(t), \bar{y}_a)$.

Using the Lipschitz constant for $h_a(\cdot, u_a)$ and the Lipschitz constant for $f(x, \cdot)$ from item 2) of Assumption 2, we see that

$$\begin{aligned}|d(t)| &\le L_{f,u}L_h\beta\left(|x_a(0) - \bar{x}_a|, \frac{t}{\epsilon}\right) \\ &\le L_{f,u}L_h\beta\left(K, \frac{t}{\epsilon}\right).\end{aligned}$$

Then

$$\int_0^T |d(s)|ds \le L_{f,u}L_h \int_0^T \beta\left(K, \frac{s}{\epsilon}\right) ds. \tag{41}$$

Now, if the right-hand side can be made arbitrarily small by choosing ϵ sufficiently small, then Theorem 1 establishes robustness with respect to fast actuator dynamics.

For simplicity, let

$$\bar{\sigma} := \frac{\rho^{-1}(\sigma)}{L_{f,u}L_h} e^{-L_{f,x}T}.$$

We define

$$\tilde{t} = \frac{\bar{\sigma}}{2\beta(K,0)}$$

and note that $\widetilde{T} > 0$ is then such that $\beta(K, \widetilde{T}) \le (\bar{\sigma}/2T)$.

Since $\epsilon^* = (\tilde{t}/\widetilde{T})$, we may write

$$\begin{aligned}\int_0^T \beta\left(K, \frac{s}{\epsilon}\right) ds &= \int_0^{\tilde{t}} \beta\left(K, \frac{s}{\epsilon}\right) ds + \int_{\tilde{t}}^T \beta\left(K, \frac{s}{\epsilon}\right) ds \\ &\le \int_0^{\tilde{t}} \beta(K,0) ds + \int_{\tilde{t}}^T \beta\left(K, \frac{\tilde{t}}{\epsilon}\right) ds \\ &\le \beta(K,0)\tilde{t} + \beta\left(K, \frac{\tilde{t}}{\epsilon}\right)(T - \tilde{t}) \\ &\le \frac{\bar{\sigma}}{2} + \beta(K, \widetilde{T})T \\ &\le \bar{\sigma}.\end{aligned}$$

In other words

$$\begin{aligned}\int_0^T |d(s)|ds &\le L_{f,u}L_h \int_0^T \beta\left(K, \frac{s}{\epsilon}\right) ds \\ &\le L_{f,u}L_h\bar{\sigma} = \rho^{-1}(\sigma)e^{-L_{f,x}T}\end{aligned}$$

allowing the application of Theorem 1. ■

A. Integrator Backstepping

Consider

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= u.\end{aligned}$$

Suppose $x_2 = \alpha_1(x_1(0))$ is designed to satisfy item 2 of Assumption 1. That is, we sample the state x_1 and design a virtual sample and hold control using x_2. Then, for $t \in [0, T]$, we may write

$$\begin{aligned}\dot{x}_1(t) &= f_1(x_1(t), x_2(t) + \alpha_1(x_1(0)) - \alpha(x_1(0))) \\ &= f_1(x_1(t), \alpha_1(x_1(0))) + d(t)\end{aligned}$$

where

$$\begin{aligned}|d(t)| &= |f_1(x_1(t), x_2(t) + \alpha_1(x_1(0)) - \alpha_1(x_1(0))) \\ &\qquad - f_1(x_1(t), \alpha_1(x_1(0)))| \\ &\le L_{f_1,u}|x_2(t) - \alpha_1(x_1(0))|.\end{aligned}$$

Assuming we can choose u sufficiently large, we can force

$$|x_2(t) - \alpha_1(x_1(0))| \leq \beta\left(|x_2(0) - \alpha_1(x_1(0))|, \frac{t}{\epsilon}\right) \quad \forall\, t \in [0,T] \tag{42}$$

allowing the application of Proposition 4. If we select, for example

$$u(t) = -\frac{1}{\epsilon}(x_2(t) - \alpha_1(x_1(0))), \qquad t \in [0,T]$$

this guarantees

$$|x_2(t) - \alpha_1(x_1(0))| \leq |x_2(0) - \alpha_1(x_1(0))|e^{-t/\epsilon} \quad \forall\, t \in [0,T].$$

In fact, if we can guarantee equation (42) through a chain of integrators, or even a dynamical system, we may apply Proposition 4. As with Proposition 4, following Remark 9, this may be extended to cover all $t \geq 0$.

V. Averaged (PWM-Type) Actuator

Finally, we consider the case when the control is through an averaged PWM-type actuator modeled by

$$\begin{aligned} u(t) &= p_{T_P}(t, u_a) \\ u_a &= \alpha(x(iT)) = \alpha(x_i). \end{aligned} \tag{43}$$

We characterize the averaging actuator by the following assumption.

Assumption 4: The function $p_{T_P}(\cdot, u_a)$ is periodic with period $T_P > 0$ for each fixed $u_a \in \mathcal{U}$ and for each $x \in \mathcal{V}(R+R_m)$ and $u_a \in \mathcal{U}$

$$\frac{1}{T_P}\int_0^{T_P} f(x, p_{T_P}(t, u_a))dt = f(x, u_a).$$

Remark 10: This assumption excludes systems such as $\dot{x} = -x + 2xu^2$. It can be seen that, $u = 0$ is a stabilizing control whereas an alternating PWM-type control obtained by varying $u(t)$ between 1 and -1 is not. A class of systems for which Assumption 4 trivially holds is the class of input-affine systems with the usual characterization of a PWM actuator, i.e.,

$$\frac{1}{T_P}\int_0^{T_P} p_{T_P}(t, u_a)dt = u_a.$$

Our claim is that, if the PWM period T_P is sufficiently small, the closed-loop system (44) is stable in the sense that the decrease and the boundedness of $V(\cdot)$ is maintained from the nominal design, i.e., that (7) and (8) obtain. In [12], consideration was given to linear plants where stabilization of the sampled-data plant is guaranteed if a certain matrix, dependent on the switching period T_p, is nonsingular. The authors observed that this condition can be difficult to check (see [12, Th. 3.1]).

We denote the diameter of the input space $\mathcal{U}$ by

$$D := \max\{|u_1 - u_2| : u_1, u_2 \in \mathcal{U}\}.$$

Proposition 5: Let $\sigma \in [0, R_m - r_m)$. Under Assumptions 1, 2, and 4, consider the system

$$\dot{x}(t) = f\left(x(t), p_{T_p}(t, \alpha(x(0)))\right), \qquad x(0) \in \mathcal{V}(R). \tag{44}$$

If

$$T_P \leq \min\left\{\frac{\rho^{-1}(\sigma)}{4L_{f,x}MT}e^{-L_{f,x}T}, \frac{\rho^{-1}(\sigma)}{2L_{f,u}D}e^{-L_{f,x}T}\right\} \tag{45}$$

then, for all $t \in [0,T]$ the solution $x(t)$ exists and satisfies (7) and (8).

Remark 11: Similar to the observation in Remark 9, if we further constrain σ by $\sigma \in [0,\ \min\{R_m - r_m,\ R - r,\ c\})$ then we can apply Proposition 5 to subsequent sampling periods (i.e., $[T, 2T]$, $[2T, 3T]$, etc.).

It may be interesting to note that, even though the actual closed-loop system is time-varying due to the PWM actuator, we consider a time-invariant system plus a time-varying disturbance (to account for the actuator) which satisfies the conditions of Theorem 1.

Proof: Let $u_a = \alpha(x(0))$. We write the closed-loop system, using the averaged actuator (43), as

$$\begin{aligned} \dot{x}(t) &= f(x(t), p_{T_P}(t, u_a)) \\ &= f(x(t), u_a) + [f(x(t), p_{T_P}(t, u_a)) - f(x(t), u_a)] \\ &= f(x(t), u_a) + d(t) \end{aligned} \tag{46}$$

where $d(t)$ is defined by the bracketed term.

Let $\bar{t} \in (0, T]$ be such that $x(t) \in \mathcal{V}(R + R_m)$ for $t \in [0, \bar{t}]$. (Such a $\bar{t}$ always exists since $x(0) \in \mathcal{V}(R)$, the solution is absolutely continuous with respect to time, and $V(\cdot)$ is continuous. However, we will ultimately show that we can pick $\bar{t} = T$.) Let N be the largest integer satisfying $T_P N \leq \bar{t}$

$$\begin{aligned} &\left|\int_0^{\bar{t}} [f(x(s), p_{T_P}(s, u_a)) - f(x(s), u_a)]ds\right| \\ &\quad \leq \left|\int_0^{T_P N} [f(x(s), p_{T_P}(s, u_a)) - f(x(s), u_a)]ds\right| \\ &\qquad + \int_{T_P N}^{\bar{t}} \Big|f(x(s), p_{T_P}(s, u_a)) - f(x(s), u_a)\Big|ds. \end{aligned} \tag{47}$$

Now, we show that each term of right-hand side in (47) is less than $(1/2)\rho^{-1}(\sigma)e^{-L_{f,x}T}$. The second term of (47) is easily shown to be less than this quantity as follows:

$$\begin{aligned} &\int_{T_P N}^{\bar{t}} |f(x(s), p_{T_P}(s, u_a)) - f(x(s), u_a)|\, ds \\ &\quad \leq \int_{T_P N}^{\bar{t}} L_{f,u}|p_{T_P}(s, u_a) - u_a|ds \\ &\quad \leq (\bar{t} - T_P N) \cdot L_{f,u}D \leq T_P L_{f,u}D \leq \frac{1}{2}\rho^{-1}(\sigma)e^{-L_{f,x}T} \end{aligned} \tag{48}$$

where the last two inequalities follow from $\bar{t} < T_P(N+1)$ and the second bound in (45). On the other hand, we may write

$$\left| \int_0^{T_P N} [f(x(s), p_{T_P}(s, u_a)) - f(x(s), u_a)] ds \right| \\ \le \sum_{j=0}^{N-1} \left| \int_{jT_P}^{(j+1)T_P} [f(x(s), p_{T_P}(s, u_a)) - f(x(s), u_a)] ds \right|$$

which, by Assumption 4

$$= \sum_{j=0}^{N-1} \left| \int_{jT_P}^{(j+1)T_P} [f(x(s), p_{T_P}(s, u_a)) - f(x(jT_P), p_{T_P}(s, u_a))] ds - \int_{jT_P}^{(j+1)T_P} [f(x(s), u_a) - f(x(jT_P), u_a)] ds \right|$$

and by Assumption 2

$$\le \sum_{j=0}^{N-1} 2 \cdot \int_{jT_P}^{(j+1)T_P} L_{f,x} |x(s) - x(jT_P)| ds \\ \le NT_P \cdot 2L_{f,x} \cdot MT_P \le 2L_{f,x} MTT_P \\ \le \frac{1}{2} \rho^{-1}(\sigma) e^{-L_{f,x}T} \tag{49}$$

in which the inequality $T_P N \le \bar{t} \le T$ is used.

Therefore, using (47)–(49), and the definition of $d(\cdot)$, we see that

$$\left| \int_0^{\bar{t}} d(\tau) d\tau \right| \le \rho^{-1}(\sigma) e^{-L_{f,x}T}. \tag{50}$$

We apply Theorem 1 to (46) for $t \in [0, \bar{t}]$. Similar to the argument of Remark 7, we conclude that for any $\bar{t} \in (0, T]$, $x(t) \in \mathcal{V}(R + R_m)$ for $t \in [0, \bar{t}]$ and (47) is less than $\rho^{-1}(\sigma) e^{-L_{f,x}T}$. In other words

$$\max_{t \in [0,T]} \left| \int_0^T d(\tau) d\tau \right| \le \rho^{-1}(\sigma) e^{-L_{f,x}T}.$$

Consequently, Theorem 1 implies that, for all $t \in [0, T]$ the solution $x(t)$ exists and satisfies(7) and (8). ∎

VI. Conclusion

In this paper, we have provided a framework for examining robustness issues surrounding the use of feedbacks discontinuous in the state. In particular, we have shown that, under mild technical assumptions, robustness is guaranteed when the feedback is implemented as a sample and hold (or zero-order hold) and one can demonstrate a continuous closed-loop Lyapunov function that is bounded during intersample periods and decreases at sampling instances. Under these assumptions, we demonstrated that when the vector field is subject to small (in an $\mathcal{L}_1$ sense) disturbances, stability is maintained in the sense that the closed-loop Lyapunov function continues to decrease, albeit at a possibly slower rate. We then proceeded to show that this type of robustness implies robustness to such disturbances as measurement noise and delays arising from the computation of the control. This result also implies that the sample and hold controller, again possessing an appropriate Lyapunov function, can be implemented via an averaging actuator or through a fast actuator. We have not furnished a comprehensive list of applications of our main result and other robustness results may be possible, e.g., robustness to quantization error or stability when the controller is obtained through an emulation design.

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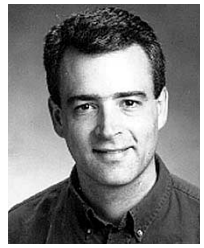

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