49th IEEE Conference on Decision and Control
December 15-17, 2010
Hilton Atlanta Hotel, Atlanta, GA, USA

A Bounded Real Lemma for Nonlinear $\mathcal{L}_2$-Gain

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***Abstract*— The well-known Bounded Real Lemma plays a key role in $\mathcal{L}_2$-gain based control methods such as H_∞ control for both linear and nonlinear systems. Recently, conventional (linear) $\mathcal{L}_2$-gain concepts have been generalized to the nonlinear $\mathcal{L}_2$-gain framework to encompass a larger class of systems. A bounded real lemma has been developed corresponding to this generalized notion of nonlinear $\mathcal{L}_2$-gain.**

***Index Terms*— $\mathcal{L}_2$-gain, dissipative systems, bounded real lemma**

I. INTRODUCTION

System analysis and controller synthesis based on the notion of finite $\mathcal{L}_2$-gain have been extensively studied [1], [7], [14] for both linear and nonlinear systems. Deep internal information of the dynamics can be revealed by investigating the input-output (generalized) energy transfer properties, which makes finite $\mathcal{L}_2$-gain an important tool in system analysis, especially in the consideration of stability properties. This notion has been shown to be equivalent to other input-output characterizations such as dissipative systems theory [15] and input to state stability (ISS) [13], [6]. While as a system design performance objective, it leads to the intensively studied H_∞ control methods [1], [5], [7], which minimizes the $\mathcal{L}_2$-gain from the external disturbance input to the output.

Conventional $\mathcal{L}_2$-gain, where the gain is characterized by a finite number, actually represents a linear upper bound function for the energy amplification from input to output. The fact that the output energy is usually related to the input energy for a general nonlinear system through a nonlinear gain function makes the conventional linear $\mathcal{L}_2$-gain bound function restrictive in many cases. The fact that many nonlinear systems do not possess finite $\mathcal{L}_2$-gain, yet retain asymptotic stability, motivates a generalization to functional nonlinear $\mathcal{L}_2$-gain bounds in order to make gain-based analysis and design methods applicable to more general nonlinear systems [3].

The bounded real lemma plays a central role in the conventional linear $\mathcal{L}_2$-gain methodology which characterizes the $\mathcal{L}_2$-gain properties with the solutions of an algebraic Riccati equation for linear systems [5] or a Hamilton-Jacobi-Bellman equation [1], [7], [9] in the nonlinear case. In this paper, we develop a bounded real lemma to characterize the nonlinear $\mathcal{L}_2$-gain properties. This generalization is achieved through a dissipative systems interpretation of an augmented system to the nonlinear $\mathcal{L}_2$-gain inequalities. It turns out the storage functions of the new dissipative systems are the solutions we seek to the partial differential inequalities in the bounded real lemma.

In this paper, we denote $\mathbf{R}_{>0} = \{x \in \mathbf{R} | x > 0\}$ and $\mathbf{R}_{\geq 0} = \{x \in \mathbf{R} | x \geq 0\}$, $B(x,r) = \{y \in \mathbf{R}^n : |y-x| < r\}$ and $\bar{B}(x,r) = \{y \in \mathbf{R}^n : |y-x| \leq r\}$. A continuous function $\gamma : \mathbf{R}_{\geq 0} \to \mathbf{R}_{\geq 0}$ is of class $\bar{\mathcal{K}}$ if it is nondecreasing and $\gamma(0) = 0$.

II. FINITE $\mathcal{L}_2$-GAINS AND BOUNDED REAL LEMMA

This section presents the problem we study in this paper. We try to establish the Bounded Real Lemma characterization of nonlinear $\mathcal{L}_2$-gain for a nonlinear system.

Consider a nonlinear system of the form

$$\Sigma : \begin{cases} \dot{x}(t) &= f(x(t)) + g(x(t))w(t), \quad x(0) = x \\ z(t) &= h(x(t)), \end{cases} \tag{1}$$

where $x(t) \in \mathbf{R}^n, w(t) \in \mathbf{R}^s, z(t) \in \mathbf{R}^l$ are the state, input and output respectively. The input space is taken as

$$\mathcal{W} = \mathcal{L}_2^{loc}[0,\infty) = \{w : [0,\infty) \to \mathbf{R}^s | w_{[0,T]} \in \mathcal{L}_2[0,T], \ \forall \ T \geq 0\}.$$

Conditions such as A3.1I, A3.2I, A3.3I in page 39-40 of [12] on functions f, g, h are assumed for the system although the framework applies to much more general settings.

For comparison purposes, the notion of linear finite $\mathcal{L}_2$ gain [14], [7] is given first.

Definition 2.1: The nonlinear system Σ has finite $\mathcal{L}_2$-gain less than $\bar{\gamma} \in \mathbf{R}_{>0}$ if there exists a bias function $\beta \in \bar{\mathcal{K}}$ such that

$$\begin{cases} \int_0^T |z(t)|^2 dt \leq \bar{\gamma}^2 \int_0^T |w(t)|^2 dt + \beta(|x|) \\ \text{for all } x \in \mathbf{R}^n, w \in \mathcal{L}_2[0,T], T \geq 0. \end{cases} \tag{2}$$

For linear systems, the well-known bounded real lemma provides an algebraic test for the $\mathcal{L}_2$-gain inequality (i.e. $||\Sigma||_{\mathbf{H}^\infty} < \bar{\gamma}$) in terms of solutions of Riccati inequalities (equations) [5]. In the nonlinear context, there is a similar characterization of the (linear) $\mathcal{L}_2$-gain property in terms of appropriate solutions of the following Partial Differential Inequalities (PDI)

$$\nabla_x V \cdot f(x) + \frac{1}{4\bar{\gamma}^2} \nabla_x V g(x) g^T(x) \nabla_x V^T + h^T(x)h(x) \leq 0. \tag{3}$$

The following Bounded Real Lemma for nonlinear systems can be found in [1], [7].

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Theorem 2.2: System Σ has finite $\mathcal{L}_2$-gain less than $\bar{\gamma}$ if and only if there exists a viscosity solution $\mathbf{V} \geq 0$, $\mathbf{V}(0) = 0$ of the PDI (3).

Note a finite $\mathcal{L}_2$-gain $\bar{\gamma} > 0$ defines a linear $\mathcal{L}_2$-gain bound function $\gamma(\xi) = \bar{\gamma}^2\xi$. For many nonlinear systems, the energy gain from input to output may not be bounded by a linear function with respect to the input energy. This may be even possible for globally asymptotically stable systems [3]. This motivates the following generalization of $\mathcal{L}_2$-gain concepts.

Definition 2.3: A function $\gamma \in \bar{\mathcal{K}}$ is a nonlinear $\mathcal{L}_2$-gain bound function of system Σ if there exists a bias (transient bound) function $\beta \in \bar{\mathcal{K}}$ such that

$$\begin{cases} \int_0^T |z(t)|^2 dt \leq \gamma\left(\int_0^T |w(t)|^2 dt\right) + \beta(|x|) \\ \text{for all } x \in \mathbf{R}^n, w \in \mathcal{L}_2[0,T], T \geq 0. \end{cases} \quad (4)$$

For simplification of the following development, we restrict attention to the following space of gain functions

$$\widehat{\mathcal{K}}^\infty \doteq \{\gamma \in \bar{\mathcal{K}} \cap C^1(\mathbf{R}_{>0}) | \exists \kappa > 0 \text{ s.t. } \gamma'(\xi) > \kappa, \forall \xi > 0\}.$$

Here $C^1(\mathbf{R}_{>0})$ denotes the space of differentiable functions on $\mathbf{R}_{>0}$.

The aim of this paper is to derive a Bounded Real Lemma for nonlinear $\mathcal{L}_2$-gain similar to Theorem 2.2, that is, to obtain a characterization of the nonlinear $\mathcal{L}_2$-gain property in terms of the solutions of a PDI.

III. Dissipative Systems Interpretation

We shall follow a similar line to the derivation of the linear $\mathcal{L}_2$-gain bounded real lemma in establishing the nonlinear one. That is, we shall first characterize the nonlinear $\mathcal{L}_2$-gain property in terms of a dissipativity property of some system. Then the infinitesimal version of the dissipation inequalities turns out to be the PDIs we seek and the storage functions are the solutions of these PDIs.

For linear gain, the dissipative systems characterization is immediate by choosing a supply rate [7]

$$s(w,z) = \bar{\gamma}^2|w|^2 - |z|^2.$$

Then system Σ has finite $\mathcal{L}_2$-gain less then $\bar{\gamma}$ if and only if Σ is dissipative with respect to the supply rate. That is, there is a storage function $V \geq 0, V(0) = 0$ such that

$$\begin{cases} V(x) + \int_0^T s(w(t), z(t))dt \geq V(x(T)) \\ \text{for all } x = x(0) \in \mathbf{R}^n, w \in \mathcal{L}_2[0,T], T \geq 0. \end{cases} \quad (5)$$

In particular, the function

$$V_a(x) = \sup_{T\geq 0} \sup_{w \in \mathcal{L}_2[0,T]} \int_0^T \left(|z(t)|^2 - \bar{\gamma}^2|w(t)|^2\right) dt \quad (6)$$

is the smallest storage function (available storage).

For nonlinear gain, in general, there does not exist a supply rate $l(w,z)$ such that

$$\begin{aligned} W_a(x) = & \\ & \sup_{T\geq 0} \sup_{w\in\mathcal{L}_2[0,T]} \int_0^T |z(t)|^2 dt - \gamma\left(\int_0^T |w(t)|^2 dt\right) \\ = & \sup_{T\geq 0} \sup_{w\in\mathcal{L}_2[0,T]} \int_0^T l(w(t), z(t))dt \end{aligned}$$

due to the lack of ordering for the gain bound function, namely,

$$\gamma\left(\int_0^T |w(t)|^2 dt\right) \neq \gamma\left(\int_0^\tau |w(t)|^2 dt\right) + \gamma\left(\int_\tau^T |w(t)|^2 dt\right)$$

for $0 \leq \tau \leq T$. However, it is possible to interchange the nonlinear gain with the integration in a very specific way. This is detailed in the following Lemma.

Lemma 3.1: Assume $\gamma \in \widehat{\mathcal{K}}^\infty$, then for all $0 \leq t \leq T$

$$\begin{aligned} &\gamma\left(\textstyle\int_0^t |w(s)|^2 ds + \xi\right) - \gamma(\xi) \\ &\quad = \int_0^t \gamma'\left(\int_0^s |w(\tau)|^2 d\tau + \xi\right)|w(s)|^2 ds \end{aligned} \quad (7)$$

for all $\xi \in \mathbf{R}_{\geq 0}, x \in \mathbf{R}^n, T \geq 0, w \in \mathcal{L}_2[0,T]$.

Proof. Fix any $\xi \in \mathbf{R}_{\geq 0}, x \in \mathbf{R}^n, T \geq 0, w \in \mathcal{L}_2[0,T]$, let

$$\phi_1(t) = \gamma\left(\int_0^t |w(s)|^2 ds + \xi\right) - \gamma(\xi)$$

and

$$\phi_2(t) = \int_0^t \gamma'\left(\int_0^s |w(\tau)|^2 d\tau + \xi\right)|w(s)|^2 ds$$

for $t \in [0,T]$. Obviously $\phi_1(0) = \phi_2(0) = 0$. Also, for $t \in (0,T)$

$$\phi_1'(t) = \gamma'\left(\int_0^t |w(s)|^2 ds + \xi\right)|w(t)|^2 = \phi_2'(t).$$

Hence $\phi_1(t) = \phi_2(t),\ 0 \leq t < T$ and by continuity of ϕ_1 and ϕ_2, we also have $\phi_1(T) = \phi_2(T)$. □

Lemma 3.1 implies that additional dynamics can be introduced to system Σ in order to characterize the nonlinear $\mathcal{L}_2$-gain property of Σ via a dissipation property of the new augmented system Σ^a. To this end, define Σ^a by

$$\Sigma^a : \begin{cases} \begin{bmatrix} \dot{x}(t) \\ \dot{\xi}(t) \end{bmatrix} = \begin{bmatrix} f(x(t)) + g(x(t))w(t) \\ |w(t)|^2 \end{bmatrix}, \\ \begin{bmatrix} z(t) \\ \eta(t) \end{bmatrix} = \begin{bmatrix} h(x(t)) \\ \xi(t) \end{bmatrix} \end{cases} \quad (8)$$

with initial state $\begin{bmatrix} x(0) \\ \xi(0) \end{bmatrix} = \begin{bmatrix} x \\ \xi \end{bmatrix}$.

To make connections with the nonlinear $\mathcal{L}_2$-gain bound property, we define a supply rate for Σ^a to be

$$S(w,z,\eta) = \gamma'(\eta)|w|^2 - |z|^2 \quad (9)$$

for all $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}$. The available storage is now defined as

$$W_a(x,\xi) = \sup_{T\geq 0} \sup_{w\in\mathcal{L}_2[0,T]} \int_0^T -S(w(t),z(t),\eta(t))dt \quad (10)$$

where the integral is evaluated along the state trajectory $(x(s),\xi(s)), 0 \leq s \leq T$ with initial state $(x(0),\xi(0)) = (x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}$.

Remark 3.2: If we have a linear gain bound function $\gamma(\xi) = \bar{\gamma}^2\xi$, then it is easy to see that the available storage W_a in (10) reduces to V_a (6) in the sense that

$$W_a(x,\xi) = V_a(x),\ (x,\xi) = \mathbf{R}^n \times \mathbf{R}_{\geq 0}.$$

In this paper, we take it as an assumption that W_a is continuous on $\mathbf{R}^n \times \mathbf{R}_{\geq 0}$. Provided with general results on the continuity of value functions of general nonlinear optimal control problems, this is a rather mild assumption. In special cases, the continuity could be proved similarly as in [4] under an "incremental gain property".

As in the linear $\mathcal{L}_2$-gain (nonlinear H_∞) case [1], [7], the available storage function W_a plays a key role in the development of the bounded real lemma. To this end, various properties of W_a are investigated first.

W_a is closely related to the function W which is investigated in [3], [4]

$$W(x,\rho) = \sup_{T\geq 0} \sup_{||w||^2_{\mathcal{L}_2[0,T]}\leq \rho} ||z||^2_{\mathcal{L}_2[0,T]}.$$

It can be shown that the worst input is always with energy ρ, that is

$$W(x,\rho) = \sup_{T\geq 0} \sup_{||w||^2_{\mathcal{L}_2[0,T]}= \rho} ||z||^2_{\mathcal{L}_2[0,T]}. \quad (11)$$

The continuity of W was proved in [4] under some conditions, here we assume the continuity of W directly in the following development.

Theorem 3.3: If the system Σ has nonlinear $\mathcal{L}_2$-gain bound function $\gamma \in \widehat{\mathcal{K}}^\infty$ and transient bound $\beta \in \bar{\mathcal{K}}$, then the available storage W_a in (10) satisfies the following properties:

1) For any $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}$, it holds

$$0 \leq W_a(x,\xi) \leq \beta(|x|) + \gamma(\xi); \quad (12)$$

and furthermore, there exist $\delta > 0, \kappa > 0$ such that for all $0 \leq \triangle\xi \leq \delta$

$$(W_a(x,\xi) - \gamma(\xi)) - (W_a(x,\xi+\triangle\xi) - \gamma(\xi+\triangle\xi)) \geq \kappa\triangle\xi \quad (13)$$

for all $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}$.

2) For any $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}$

$$W_a(x,\xi) = \sup_{\rho\geq 0}\{W(x,\rho) - \gamma(\xi+\rho)\} + \gamma(\xi). \quad (14)$$

3) For any $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}$, there is a sequence of sub-optimal inputs (T_n, w_n) of W_a for $\epsilon_n \downarrow 0, n \to \infty$ such that the energy is bounded by a $\bar{\rho}(x,\xi) < \infty$, i.e.

$$||w_n||^2_{\mathcal{L}_2[0,T_n]} \leq \bar{\rho}(x,\xi),\ \forall\ n \quad (15)$$

if and only if there is a finite maximum $\rho^*(x,\xi) < \infty$, i.e.

$$\rho^*(x,\xi) \in \operatorname*{argmax}_{\rho\geq 0}\{W(x,\rho) - \gamma(\xi+\rho)\}. \quad (16)$$

4) For any $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}$, there exists $\epsilon_0 > 0$ such that the energy of all ϵ-optimal inputs (T_ϵ, w_ϵ) is bounded by a finite number $\bar{\rho}(x,\xi) > 0$, i.e.

$$||w_\epsilon||^2_{\mathcal{L}_2[0,T_\epsilon]} \leq \bar{\rho}(x,\xi),\ \forall\ 0 \leq \epsilon \leq \epsilon_0 \quad (17)$$

if and only if there is a $0 < \rho_0(x,\xi) < \infty$ such that

$$\max_{\rho\leq\rho_0}\{W(x,\rho) - \gamma(\xi+\rho)\} > \sup_{\rho>\rho_0}\{W(x,\rho) - \gamma(\xi+\rho)\}.$$

Proof.

1) The nonnegativity of W_a is immediate from the definition (10) by taking null input $w(t) \equiv 0,\ t \geq 0$. By Lemma 3.1 and (8),

$$\begin{aligned}
&W_a(x,\xi) = \\
&\sup_{T\geq 0} \sup_{w\in\mathcal{L}_2[0,T]} \int_0^T \left(|z(t)|^2 - \gamma'(\eta(t))|w(t)|^2\right) dt \\
&= \sup_{T\geq 0} \sup_{w\in\mathcal{L}_2[0,T]} \left\{ \int_0^T |z(t)|^2 dt - \gamma\left(\int_0^T |w(t)|^2 dt + \xi\right) \right\} + \gamma(\xi) \\
&\leq \sup_{T\geq 0} \sup_{w\in\mathcal{L}_2[0,T]} \left\{ \int_0^T |z(t)|^2 dt - \gamma\left(\int_0^T |w(t)|^2 dt\right) \right\} + \gamma(\xi) \\
&\leq \beta(|x|) + \gamma(\xi).
\end{aligned} \quad (18)$$

where the last inequality follows from (4).

Let

$$\tilde{W}(x,\xi) = W_a(x,\xi) - \gamma(\xi) = \sup_{T\geq 0} \sup_{w\in\mathcal{L}_2[0,T]} \int_0^T |z(t)|^2 dt - \gamma\left(\int_0^T |w(t)|^2 dt + \xi\right).$$

For $\gamma \in \widehat{\mathcal{K}}^\infty$, there exist $\delta > 0, \kappa > 0$ such that

$$\gamma(\xi+\triangle\xi) - \gamma(\xi) \geq \kappa\triangle\xi,\ \forall\ \xi \geq 0, 0 \leq \triangle\xi \leq \delta.$$

Fix any $(x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{\geq 0}, 0 \leq \triangle\xi \leq \delta$, and $\epsilon > 0$, let (T^ϵ, w^ϵ) be ϵ-optimal input for $\tilde{W}(x,\xi+\triangle\xi)$, i.e.

$$\tilde{W}(x,\xi+\triangle\xi) < ||z^\epsilon||^2_{\mathcal{L}_2[0,T^\epsilon]} - \gamma\left(||w^\epsilon||^2_{\mathcal{L}_2[0,T^\epsilon]} + \xi + \triangle\xi\right) + \epsilon.$$

Then

$$\begin{aligned}
&\tilde{W}(x,\xi) - \tilde{W}(x,\xi+\triangle\xi) \\
&> \gamma\left(||w^\epsilon||^2_{\mathcal{L}_2[0,T^\epsilon]} + \xi + \triangle\xi\right) - \gamma\left(||w^\epsilon||^2_{\mathcal{L}_2[0,T^\epsilon]} + \xi\right) - \epsilon \\
&\geq \kappa\triangle\xi - \epsilon
\end{aligned}$$

which proves (13) by taking $\epsilon \downarrow 0$.

2) From (18), it holds

$$W_a(x,\xi) = \sup_{T\ge 0}\sup_{w\in\mathcal{L}_2[0,T]}\left\{\int_0^T |z(t)|^2dt - \gamma\left(\int_0^T |w(t)|^2dt+\xi\right)\right\}+\gamma(\xi)$$
$$= \sup_{\rho\ge 0}\sup_{T\ge 0}\sup_{||w||^2_{\mathcal{L}_2[0,T]=\rho}}\{||z||^2_{\mathcal{L}_2[0,T]}-\gamma(\xi+\rho)\} + \gamma(\xi)$$
$$= \sup_{\rho\ge 0}\{W(x,\rho)-\gamma(\xi+\rho)\}+\gamma(\xi).$$

3) **Only if:** For the sequence ϵ_n, we have

$$\begin{aligned}&W_a(x,\xi)-\gamma(\xi)\\ &\le ||z_n||^2_{\mathcal{L}_2[0,T_n]}-\gamma(\xi+||w_n||^2_{\mathcal{L}_2[0,T_n]})+\epsilon_n\\ &\le W(x,||w_n||^2_{\mathcal{L}_2[0,T_n]})-\gamma(\xi+||w_n||^2_{\mathcal{L}_2[0,T_n]})+\epsilon_n\\ &= W(x,\rho_n)-\gamma(\xi+\rho_n)+\epsilon_n.\end{aligned}$$

From the assumption (15), the sequence $\rho_n = ||w_n||^2_{\mathcal{L}_2[0,T_n]}$ is bounded by $\bar{\rho}(x,\xi)$. Then there is a subsequence (without loss of generality, we assume it is ρ_n itself) such that $\rho_n\to\rho^*\le\bar{\rho}$. Let $n\to\infty$, we have by the continuity of W and γ

$$W_a(x,\xi)\le W(x,\rho^*)-\gamma(\xi+\rho^*).$$

Hence we know $W_a(x,\xi)=W(x,\rho^*)-\gamma(\xi+\rho^*)+\gamma(\xi)$, that is, ρ^* is a finite maximum point.
If: Assume $\rho^*<\infty$ is a maximum point, then

$$W_a(x,\xi)-\gamma(\xi)=W(x,\rho^*)-\gamma(\xi+\rho^*).$$

By the continuity of W and γ, there exists a sequence $\rho_n\to\rho^*$ such that

$$W(x,\rho_n)-\gamma(\xi+\rho_n)\to W(x,\rho^*)-\gamma(\xi+\rho^*).$$

The claim will follow if it can be shown there is a suboptimal sequence (T_n,w_n) of some $\hat{\epsilon}_n\downarrow 0$ for $W_a(x,\xi)$ such that

$$||w_n||^2_{\mathcal{L}_2[0,T_n]}=\rho_n\to\rho^*.$$

If it holds

$$W(x,\rho_n)-\gamma(\xi+\rho_n)=W(x,\rho^*)-\gamma(\xi+\rho^*),\ \forall\ n,$$

then take any suboptimal inputs (T_n,w_n) corresponding to $\hat{\epsilon}_n\downarrow 0$ for $W(x,\rho_n)$, it holds

$$\begin{aligned}W_a(x,\xi)-\gamma(\xi) &= W(x,\rho^*)-\gamma(\xi+\rho^*)\\ &= W(x,\rho_n)-\gamma(\xi+\rho_n)\\ &< ||z_n||^2_{\mathcal{L}_2[0,T_n]}-\\ &\quad\gamma\left(\xi+||w_n||^2_{\mathcal{L}_2[0,T_n]}\right)+\hat{\epsilon}_n.\end{aligned}$$

So (T_n,w_n) is the $\hat{\epsilon}_n$-optimal input for $W_a(x,\xi)$, it holds

$$||w_n||^2_{\mathcal{L}_2[0,T_n]}=\rho_n\to\rho^*.$$

It is left to show that there is a subsequence (without loss of generality, assume it is ϵ_n itself) such that

$$W(x,\rho_n)-\gamma(\xi+\rho_n)<W(x,\rho^*)-\gamma(\xi+\rho^*)$$

for all n. Take $\hat{\epsilon}_n$ such that

$$\frac{1}{2}\hat{\epsilon}_n=(W(x,\rho^*)-\gamma(\xi+\rho^*))-(W(x,\rho_n)-\gamma(\xi+\rho_n)),$$

then $\hat{\epsilon_n}>0$ and $\hat{\epsilon}_n\downarrow 0$.

$$\begin{aligned}W_a(x,\xi)-\gamma(\xi) &= W(x,\rho^*)-\gamma(\xi+\rho^*)\\ &= W(x,\rho_n)-\gamma(\xi+\rho_n)+\frac{1}{2}\hat{\epsilon}_n.\end{aligned}$$

Then take suboptimal input (T_n,w_n) for $W(x,\rho_n)$ such that $||w_n||^2_{\mathcal{L}_2[0,T_n]}=\rho_n$ and

$$W(x,\rho_n)<||z_n||^2_{\mathcal{L}_2[0,T_n]}+\frac{1}{2}\hat{\epsilon}_n.$$

It holds

$$\begin{aligned}W_a(x,\xi)-\gamma(\xi) &< ||z_n||^2_{\mathcal{L}_2[0,T_n]}\\ &\quad -\gamma(\xi+||w_n||^2_{\mathcal{L}_2[0,T_n]})+\hat{\epsilon}_n.\end{aligned}$$

So subsequently (T_n,w_n) is a near-optimal sequence corresponding to $\hat{\epsilon}_n\to 0$ for $W_a(x,\xi)$. Again it holds $||w_n||^2_{\mathcal{L}_2[0,T_n]}=\rho_n\to\rho^*$.

4) **Only if:** The assumption implies for any $\rho>\bar{\rho}(x,\xi)$, we have

$$W(x,\rho)-\gamma(\xi+\rho)\le W_a(x,\xi)-\gamma(\xi)-\frac{1}{2}\epsilon_0.$$

Otherwise, choosing a suboptimal input for $W(x,\rho)$ such that $||w_\epsilon||^2_{\mathcal{L}_2[0,T_\epsilon]}=\rho$ and

$$W(x,\rho)<||z_\epsilon||^2_{\mathcal{L}_2[0,T_\epsilon]}+\frac{1}{2}\epsilon_0,$$

then

$$\begin{aligned}W_a(x,\xi)-\gamma(\xi) &< ||z_\epsilon||^2_{\mathcal{L}_2[0,T_\epsilon]}\\ &\quad -\gamma(\xi+||w_\epsilon||^2_{\mathcal{L}_2[0,T_\epsilon]})+\epsilon_0.\end{aligned}$$

Hence (T_ϵ,w_ϵ) is an ϵ_0-optimal input of $W_a(x,\xi)$ with $||w_\epsilon||^2_{\mathcal{L}_2[0,T_\epsilon]}=\rho>\bar{\rho}(x,\xi)$ which contradicts the assumption. Consequently, we know that

$$\sup_{\rho\ge\bar{\rho}}\{W(x,\rho)-\gamma(\xi+\rho)\}\le W_a(x,\xi)-\gamma(\xi)-\frac{1}{2}\epsilon_0.$$

We know it must hold

$$\begin{aligned}W_a(x,\xi)-\gamma(\xi) &= \sup_{\rho\le\bar{\rho}}\{W(x,\rho)-\gamma(\xi+\rho)\}\\ &\ge \sup_{\rho>\bar{\rho}}\{W(x,\rho)-\gamma(\xi+\rho)\}+\frac{1}{2}\epsilon_0.\end{aligned}$$

If: Take $\epsilon_0=\sup_{\rho\le\rho_0}\{W(x,\rho)-\gamma(\xi+\rho)\}-\sup_{\rho>\rho_0}\{W(x,\rho)-\gamma(\xi+\rho)\}$, then for any $0\le\epsilon\le\epsilon_0$, the ϵ-optimal input (T_ϵ,w_ϵ) of $W_a(x,\xi)$

$$\begin{aligned}&W_a(x,\xi)-\gamma(\xi)\\ &< ||z_\epsilon||^2_{\mathcal{L}_2[0,T_\epsilon]}-\gamma\left(\xi+||w_\epsilon||^2_{\mathcal{L}_2[0,T_\epsilon]}\right)+\epsilon\\ &\le W(x,\rho_\epsilon)-\gamma(\xi+\rho_\epsilon)+\epsilon_0\\ &= W(x,\rho_\epsilon)-\gamma(\xi+\rho_\epsilon)+\sup_{\rho\le\rho_0}\{W(x,\rho)\\ &\quad -\gamma(\xi+\rho)\}-\sup_{\rho>\rho_0}\{W(x,\rho)-\gamma(\xi+\rho)\}\\ &= W(x,\rho_\epsilon)-\gamma(\xi+\rho_\epsilon)+W_a(x,\xi)\\ &\quad -\gamma(\xi)-\sup_{\rho>\rho_0}\{W(x,\rho)-\gamma(\xi+\rho)\}\end{aligned}$$

where $\rho_\epsilon = ||w_\epsilon||^2_{\mathcal{L}_2[0,T_\epsilon]}$. Then

$$W(x,\rho_\epsilon) - \gamma(\xi+\rho_\epsilon) > \sup_{\rho>\rho_0}\{W(x,\rho) - \gamma(\xi+\rho)\}$$

which implies $\rho_\epsilon \le \rho_0$.

□

Remark 3.4: Item 3) and item 4) provide necessary and sufficient conditions to guarantee that the energy of the worst input signals are bounded, which is to say that the augmented state $\xi(t)$ converges to an equilibrium point under the optimal (worst case) input. More concrete and testable sufficient conditions on system data f, g, h and γ may be obtained to guarantee the boundedness of the worst case inputs such as those for the linear $\mathcal{L}_2$-gain case in Theorem 3.10 of [12].

Now we have the following theorem:

Theorem 3.5: A function $\gamma \in \widehat{\mathcal{K}}^\infty$ is a nonlinear $\mathcal{L}_2$-gain bound function of system Σ if and only if the system Σ^a is dissipative with respect to the supply rate (9).

Proof. Assume $\gamma \in \widehat{\mathcal{K}}^\infty$ is a nonlinear $\mathcal{L}_2$-gain bound function. We know from Theorem 3.3 that W_a is finite and well defined. By a standard dynamic programming argument, it holds

$$W_a(x,\xi) = \sup_{w\in\mathcal{L}_2[0,T]} \left\{\int_0^T - S(w(t),z(t),\eta(t))dt + W_a(x(T),\xi(T))\right\}. \tag{19}$$

Then we have

$$W_a(x,\xi) + \int_0^T S(w(t),z(t),\eta(t))dt \ge W_a(x(T),\xi(T)) \tag{20}$$

for all $T \ge 0$, $w \in \mathcal{L}_2[0,T]$ which is the dissipation inequality required.

Conversely, if (20) is true, then we know for all $x \in \mathbf{R}^n$, $T \ge 0$ and $w \in \mathcal{L}_2[0,T]$

$$\begin{aligned}
&\int_0^T |z(t)|^2 dt - \gamma\left(\int_0^T |w(t)|^2 dt\right) = \\
&\quad \int_0^T |z(t)|^2 dt - \int_0^T \gamma'\left(\int_0^t |w(\tau)|^2 d\tau\right)|w(t)|^2 dt \\
&\quad = \int_0^T \left(|z(t)|^2 - \gamma'(\eta(t))|w(t)|^2\right) dt \\
&\quad = -\int_0^T S(w(t),z(t),\eta(t))dt \\
&\quad \le W_a(x,0) - W_a(x(T),\xi(T)) \\
&\quad \le W_a(x,0) \\
&\quad \le \beta(|x|).
\end{aligned}$$

Where $\beta \in \bar{\mathcal{K}}$ is defined as

$$\beta(r) = \sup_{x\in\bar{B}(0,r)}\{W_a(x,0)\}.$$

This completes the proof. □

IV. Bounded Real Lemma

The PDI which characterizes the nonlinear $\mathcal{L}_2$-gain property is

$$\begin{cases}
\nabla_x W(x,\xi)\cdot f(x) + \\
\quad \dfrac{\nabla_x W(x,\xi) g(x) g^T(x) \nabla_x W^T(x,\xi)}{4(\gamma'(\xi) - \nabla_\xi W(x,\xi))} + h^T(x)h(x) \le 0 \\
\nabla_\xi W(x,\xi) - \gamma'(\xi) < 0, \qquad (x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{>0}.
\end{cases} \tag{21}$$

Remark 4.1: When the $\mathcal{L}_2$-gain bound function is linear $\gamma(\xi) = \bar{\gamma}^2\xi$, we know from Remark 3.2 that W_a does not depend on ξ and $\gamma'(\xi) = \bar{\gamma}^2$, so the set of PDI (21) reduces to PDI (3) for the linear $\mathcal{L}_2$-gain case.

Theorem 4.2: (Bounded Real Lemma) A function $\gamma \in \widehat{\mathcal{K}}^\infty$ is a nonlinear $\mathcal{L}_2$-gain bound function of system Σ if and only if there exists a viscosity solution $W \ge 0$, $W(0,0) = 0$ to the PDI (21).

Proof. Assume $\gamma \in \widehat{\mathcal{K}}^\infty$ is a nonlinear $\mathcal{L}_2$-gain bound function of system Σ, then we know W_a in (10) is well defined. We show it is a viscosity solution.

Let $(x,\xi) \in \mathbf{R}\times\mathbf{R}_{>0}$ be a local minimum of $W_a - \varphi$ for a $\varphi \in \mathcal{C}^1(\mathbf{R}^n \times \mathbf{R}_{>0})$. That is, there is $r > 0$ such that

$$W_a(x,\xi) - W_a(y,\zeta) \le \varphi(x,\xi) - \varphi(y,\zeta)$$

for all $(y,\zeta) \in B((x,\xi),r)$. Fix a constant input $w(t) \equiv w$ for $w \in \mathbf{R}^l$ and $|w|^2 = 1$, we know there is $0 < t_0 \le T$ such that $(x,\xi(s)) \in B((x,\xi),r)$. Hence

$$\varphi(x,\xi(s)) - \varphi(x,\xi) \le W_a(x,\xi(s)) - W_a(x,\xi).$$

From Item 1 of Theorem 3.3, we know there exist $\kappa > 0$, $t_0 > 0$ such that

$$(W_a(x,\xi(s)) - \gamma(\xi(s))) - (W_a(x,\xi) - \gamma(\xi)) < -\kappa(\xi(s) - \xi)$$

for all $0 \le s \le t_0$. Hence one obtains

$$(\phi(x,\xi(s)) - \phi(x,\xi)) - (\gamma(\xi(s)) - \gamma(\xi)) < -\kappa(\xi(s) - \xi).$$

Dividing by s and sending $s \downarrow 0$, one gets

$$\nabla_\xi \phi(x,\xi)|w|^2 - \gamma'(\xi)|w|^2 \le -\kappa|w|^2$$

which is

$$\nabla_\xi \phi(x,\xi) - \gamma'(\xi) \le -\kappa < 0.$$

The following input for some $T > 0$ is well defined

$$w(t) = \begin{cases} \frac{g^T(x)\nabla_x \varphi^T(x,\xi)}{2(\gamma'(\xi) - \nabla_\xi \varphi(x,\xi))}, & 0 \le t \le T \\ 0, & t \ge 0. \end{cases} \tag{22}$$

By continuity of the dynamics data, we know there exists $t_0 > 0$ such that $(x(s),\xi(s)) \in B((x,\xi),r)$, for all $s \in [0,t_0]$. From DPP (19), we have

$$\begin{aligned}
&\varphi(x(s),\xi(s)) - \varphi(x,\xi) + \\
&\quad \int_0^s - S(w(\tau),z(\tau),\eta(\tau))d\tau \\
&\le W_a(x(s),\xi(s)) - W_a(x,\xi) + \\
&\quad \int_0^s - S(w(\tau),z(\tau),\eta(\tau))d\tau \le 0
\end{aligned}$$

Dividing by s and sending $s \downarrow 0$, we have

$$-(\gamma'(\xi) - \nabla_\xi \varphi(x,\xi))|w|^2 + \nabla_x \varphi(x,\xi) \cdot g(x)w + \nabla_x \varphi(x,\xi) \cdot f(x) + |h(x)|^2 \le 0.$$

Substituting the w (22) into the above yields (21).

Conversely, assuming $\widehat{W} \in \mathcal{C}(\mathbf{R}^n \times \mathbf{R}_{\ge} 0)$ is a viscosity solution of PDI (21), we show the the nonlinear inequality (4). We adopt a technique from [8].

Fix any $T > 0$ and let $\mathcal{W}_R$ to be the set of inputs taking values in $\bar{B}(0,R) = \{w \in \mathbf{R}^l | |w| \le R\}$. Now define

$$\mathbf{W}(x,\xi,s) = \sup_{w \in \mathcal{W}_R} \left\{ \int_s^T -S(w(t), z(t), \eta(t))dt + \widehat{W}(x(T), \xi(T)) \right\}$$

where $(x(t), \xi(t)), s \le t \le T$ are the state trajectories of system Σ^a with initial state $(x(s), \xi(s)) = (x, \xi)$. Now $\mathbf{W} \in \mathcal{C}(\mathbf{R}^n \times \mathbf{R}_{\ge 0} \times [0,T])$ is the unique viscosity solution of the PDE

$$\begin{cases} \nabla_t W + \nabla_x W \cdot f(x) + h^T(x)h(x) + \\ \quad \sup_{w \in \bar{B}(0,R)} \left\{ (\nabla_\xi W - \gamma'(\xi))|w|^2 + \nabla_x W \cdot g(x)w \right\} = 0 \\ \nabla_\xi W - \gamma'(\xi) < 0 \\ W(w, \xi, T) = \widehat{W}(x, \xi). \end{cases}$$

for $(x,\xi,s) \in \mathbf{R}^n \times \mathbf{R}_{>0} \times (0,T)$. Since $\mathbf{W}$ is a supersolution of this PDE, the comparison theorem implies that

$$\widehat{W}(x,\xi) \ge \mathbf{W}(x,\xi,s), \ \forall \ (x,\xi) \in \mathbf{R}^n \times \mathbf{R}_{>0} \times (0,T).$$

Thus, seting $s = 0$, for any $R \ge 0$ it follows that

$$\widehat{W}(x,\xi) \ge \sup_{w \in \mathcal{W}_R} \left\{ \int_0^T - S(w(t), z(t), \eta(t))dt + \widehat{W}(x(T), \xi(T)) \right\}.$$

Consequently, we have

$$\begin{aligned} \widehat{W}(x,\xi) &\ge \sup_{r \ge 0} \sup_{w \in \mathcal{W}_R} \left\{ \int_0^T -S(w(t), z(t), \eta(t))dt + \widehat{W}(x(T), \xi(T)) \right\} \\ &\ge \sup_{w \in \mathcal{L}_2[0,T]} \left\{ \int_0^T -S(w(t), z(t), \eta(t))dt + \widehat{W}(x(T), \xi(T)) \right\} \end{aligned}$$

Since T is arbitrary, we conclude $\widehat{W}$ is a storage function of system Σ^a with supply rate (9). Hence γ is a nonlinear $\mathcal{L}_2$-gain bound function of system Σ according to Theorem 3.5.

□

Remark 4.3: In general, to actually solve PDI (21) we have to use numerical schemes such as grid-based methods [11]. Recently a numerical approximation based on the max-plus algebra method [12] has been developed to approximate the solutions of PDI (21) [16].

V. Conclusions

A bounded real lemma for a generalized nonlinear $\mathcal{L}_2$-gain concept was developed. The bounded real lemma characterizes the nonlinear $\mathcal{L}_2$-gain property in terms of the solutions of a set of partial differential inequalities. Key to the derivation of the bounded real lemma is a dissipative systems interpretation for an augmented system. The developed bounded real lemma can be used to design controllers achieving a nonlinear $\mathcal{L}_2$-gain bound as in the case for the conventional linear $\mathcal{L}_2$-gain ($\mathcal{H}_\infty$ control) case.

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