# A Generalization of Input-to-State Stability

Christopher M. Kellett<sup> $\dagger$ </sup> and Peter M. Dower<sup> $\ddagger$ </sup>

Abstract—Input-to-State Stability (ISS) and its many derivatives have proved to be extremely useful in the analysis and design of robustly stable nonlinear systems. In this paper, we present a generalization of ISS that subsumes several ISS-type properties and discuss cases where this generalization may fail.

Index Terms—Input-to-State Stability, Input-to-Output Stability, Lyapunov methods, Stability with respect to Two Measures

#### I. INTRODUCTION

The Input-to-State Stability (ISS) framework introduced by Sontag in [18] has proven to be among the most successful paradigms for simultaneously analyzing both input-output as well as internal stability for nonlinear systems. Recently, Sontag [21] presented a comprehensive overview of the stateof-the-art of ISS including sample applications. Of particular interest is the unification of several ISS-type results as suggested in [8] and [21, Section 10]. In this paper, we will present initial results towards such a unification.

Consider the system

$$\frac{d}{dt}x(t) = f(x(t), u(t)), \qquad x(0) = x \tag{1}$$

where  $x \in \mathbb{G} \subset \mathbb{R}^n$  and  $u \in \mathbb{R}^m$ . In what follows, we denote by  $\mathcal{U}$  the set of admissible (measurable and essentially bounded) input functions. Note that, by a slight abuse of notation, we will generally use  $u \in \mathbb{R}^m$  and  $u \in \mathcal{U}$  where u being a vector or function, respectively, will be clear from context. We denote the essential supremum of the function  $u \in \mathcal{U}$  by  $||u||_{\infty}$ . We denote solutions to (1) by  $\phi : \mathbb{R}_{\geq 0} \times \mathbb{G} \times \mathcal{U} \to \mathbb{R}^n$ . In what follows, we make the standing assumptions that  $f(\cdot, \cdot)$  is locally Lipschitz and that system (1) is forward complete on  $\mathbb{G}$ ; i.e., for every  $x \in \mathbb{G}$  and  $u \in \mathcal{U}$ , solutions  $\phi(t, x, u)$  exist and remain in  $\mathbb{G}$  for all  $t \geq 0$ .

Following on from the original notion of Input-to-State Stability (ISS) [18], many similar notions have been proposed providing different relationships between inputs, outputs, and states such as Input-to-Output Stability [25], Output-to-State Stability [24], and Input-Output-to-State Stability [12]. These various notions frequently have similar properties, such as Lyapunov characterizations, that are tailored to the specific relationships posited by the differing properties.

With the aim of unifying and generalizing many of these notions, we propose the following notion of multiplemeasure input stability:

Definition 1: Let  $\omega_i : \mathbb{G} \to \mathbb{R}_{\geq 0}$ , i = 1, 2 be continuous, positive semi-definite functions and let  $\omega_3 : \mathbb{G} \to \mathbb{R}^p$  be continuous. System (1) is said to be *multiple-measure input stable (MMIS)* if it is forward complete on  $\mathbb{G}$  and there exist functions  $\beta \in \mathcal{KL}$  and  $\rho, \gamma \in \mathcal{K}$  such that

$$\omega_1(\phi(t, x, u)) \le \max \left\{ \beta(\omega_2(x), t), \\ \rho(\|\omega_3(\phi(t, x, u))\|_{\infty}), \gamma(\|u\|_{\infty}) \right\}, \quad \forall t \ge 0.$$
 (2)

The classical definition of ISS is covered by Definition 1 by taking  $\omega_1(\cdot) = \omega_2(\cdot) = |\cdot|$  and  $\omega_3(\cdot) \equiv 0$ . Table I describes how several ISS-type properties can be seen to be special cases of Definition 1 where for certain properties the system (1) is augmented by an output

$$y(t) = h(\phi(t, x, u)); \quad h : \mathbb{R}^n \to \mathbb{R}^p.$$
(3)

*Remark 1:* Definition 1 generalizes the notion of *Input-Output-to-State Stability (IOSS)* as proposed in [24] and characterized in [12]. In [26], Sontag and Wang proposed a generalization of IOSS similar to the above definition of MMIS where  $\omega_1(\cdot) = |h(\cdot)|, \omega_2(\cdot) = |\cdot|_{\mathcal{A}}$  (distance to a closed set  $\mathcal{A}$ ), and  $\omega_3(\cdot) = k(\cdot)$  and the functions  $h(\cdot)$  and  $k(\cdot)$  define regulated and measured outputs, respectively. We note that, in contrast to [26], the MMIS measurement functions need not satisfy the properties of a norm and may only be defined on a subset  $\mathbb{G} \subset \mathbb{R}^n$ .

An alternative to Definition 1 would be to replace the essential supremum norm on the trajectory with a pointwise, rather than functional, norm; for example, let  $\omega_3 : \mathbb{G} \to \mathbb{R}_{\geq 0}$  be a continuous, positive semi-definite function, similar to  $\omega_1(\cdot)$  and  $\omega_2(\cdot)$ , then an alternate definition of MMIS is

$$\omega_1(\phi(t, x, u)) \le \max\left\{\beta(\omega_2(x), t), \\ \rho(\omega_3(\phi(t, x, u))), \gamma(\|u\|_{\infty})\right\}, \quad \forall t \ge 0.$$
(4)

A concept similar to MMIS was studied in [8] for systems without inputs. In particular, [8] considered a notion of measurement-to-error stability for the differential inclusion

 $\dot{x} \in F(x)$ 

augmented by a measurement function of the state,  $w(t) = g(\phi(t, x))$ , and an error function of the state,  $y(t) = h(\phi(t, x))$ , where  $g, h : \mathbb{R}^n \to \mathbb{R}^p$  are continuous functions. This is a special case of Definition 1 where  $\gamma(s) \equiv 0$ ,  $\omega_1(x) = |h(x)|$ , and  $\omega_3(x) = |g(x)|$ . Note that, similar to Definition 1, [8] allows an arbitrary measurement function

<sup>&</sup>lt;sup>†</sup>C.M. Kellett is with the School of Electrical Engineering and Computer Science, University of Newcastle, Callaghan, New South Wales, Australia Chris.Kellett@newcastle.edu.au. Kellett is supported by ARC Future Fellowship FT1101000746. <sup>‡</sup> P.M. Dower is with the Department of Electrical & Electronic Engineering, University of Melbourne, Melbourne, Victoria, Australia pdower@unimelb.edu.au. Dower is supported by AFOSR grant FA2386-09-1-4111.

TABLE I Special cases of MMIS recovering ISS-type estimates.

Property	Measurement / Gain Functions
Input-to-State Stability (ISS)	$\omega_1(\cdot) = \omega_2(\cdot) =  \cdot ,  \omega_3(\cdot) \equiv 0$
Input-to-Output Stability (IOS)	$\omega_1(\cdot) =  h(\cdot) ,  \omega_2(\cdot) =  \cdot ,  \omega_3 \equiv 0$
Output-to-State Stability (OSS)	$\omega_1(\cdot) = \omega_2(\cdot) =  \cdot , \ \omega_3(\cdot) = h(\cdot), \ \gamma(\cdot) \equiv 0$
Input-Output-to-State Stability (IOSS)	$\omega_1(\cdot) = \omega_2(\cdot) =  \cdot ,  \omega_3(\cdot) = h(\cdot)$
State-Independent Input-Output Stability	$\omega_1(\cdot) = \omega_2(\cdot) =  h(\cdot) ,  \omega_3(\cdot) \equiv 0$

of the initial condition, with the slightly more restrictive condition that [8] requires a continuous and proper measure of the initial condition, while Definition 1 only requires continuity and positive semidefiniteness.

In [8], the above stability notion was termed *Measurement*to-Error Stability or Stability in Three Measures. The nomenclature Input Measurement to Error Stability (IMES) was proposed for the relationship (2) with measurement and error signals as defined in the previous paragraph. We prefer the terminology of Multiple-Measure Input Stability as it avoids a priori assigning particular meanings, such as measurement or error signals, to the functions  $\omega_1$  and  $\omega_3$ .

In Table I we note that two important properties – Input-to-State Stability (ISS), and State-Independent Input-to-Output Stability (SI-IOS) [25] – both arise via the consideration of  $\omega_1 = \omega_2$  and  $\omega_3(\cdot) \equiv 0$ . We will refer to this property as *Input-to-State Stability with respect to*  $\omega$ , or for brevity,  $\omega$ ISS.

Definition 2: Let  $\omega : \mathbb{G} \to \mathbb{R}_{\geq 0}$  be a continuous, positive semidefinite function. System (1) is said to be *Input-to-State Stable with respect to*  $\omega$  ( $\omega ISS$ ) if it is forward complete on  $\mathbb{G}$  and there exist functions  $\beta \in \mathcal{KL}$  and  $\gamma \in \mathcal{K}$  such that

$$\omega(\phi(t, x, u)) \le \max\left\{\beta(\omega(x), t), \gamma(\|u\|_{\infty})\right\}, \ \forall t \ge 0.$$
(5)

We observe that, as with the standard notion of ISS, the above definition is qualitatively equivalent to an additive-formulation of  $\omega$ ISS

$$\omega(\phi(t, x, u)) \le \hat{\beta}(\omega(x), t) + \hat{\gamma}(\|u\|_{\infty}), \quad \forall t \ge 0.$$
 (6)

ISS with respect to  $\omega$  clearly subsumes ISS and SI-IOS. Furthermore, we note that this also covers local or partial-state versions of these properties as captured by the arbitrary domain  $\mathbb{G} \subseteq \mathbb{R}^n$ . In addition, set-stability versions are also covered by allowing the measurement function  $\omega$ to be defined as the distance to a (closed) set as well as stability of a prescribed motion. This includes a weak form of incremental ISS [1] where the input-dependent bound is the essential supremum of both inputs rather than the difference between the two inputs.

Our ultimate aim is similar to that in [8], namely an attempt to unify IOS and IOSS notions, but from a different starting point. The authors of [8] considered systems without *inputs* and obtained an appropriate lower-semicontinuous Lyapunov-like function. Herein, we consider systems without *measurements* (as defined in [8]) and obtain a smooth Lyapunov-like function. Deriving a smooth Lyapunov function for MMIS remains an open problem.

This paper is organized as follows: in Section II we present the Lyapunov characterization of  $\omega$ ISS, including a condition providing for the equivalence of implication and dissipation forms of the decrease condition. In Section III-A we review the notion of stability with respect to two measures and in Section III-B describe how a converse Lyapunov theorem for two measure stability does not directly lead to an IOS-Lyapunov function. Section IV describes how  $\omega$ ISS subsumes incremental ISS in the case of linear systems. We conclude in Section V by pointing to directions for future work.

#### II. $\omega$ ISS-Lyapunov Functions

One of the most useful tools in the ISS-based nonlinear systems literature has been the equivalent Lyapunov characterization of ISS.

Definition 3: An ISS-Lyapunov function for (1) is a (smooth) function  $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$  such that, for  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$  and  $\chi \in \mathcal{K}$  the following hold:

$$\alpha_1(|x|) \le V(x) \le \alpha_2(|x|) \tag{7}$$

$$|x| \ge \chi(|u|) \quad \Rightarrow \quad \langle \nabla V(x), f(x, u) \rangle \le -V(x). \tag{8}$$

In [23], Sontag and Wang demonstrated that ISS is equivalent to the existence of an ISS-Lyapunov function. Furthermore, in [22] they demonstrated that this result holds equally for compact invariant sets  $\mathcal{A}$ . We note that this is a special case of  $\omega$ ISS when  $\omega_1(\cdot) = \omega_2(\cdot) = |\cdot|_{\mathcal{A}}$ . Furthermore, in [14] it was shown that for parametrized systems and consideration of a closed (not necessarily compact) set  $\mathcal{A}$ , an appropriate ISS-Lyapunov function implies ISS. The ISS-Lyapunov function defined in [14] for robust stability of a closed set is a special case of the  $\omega$ ISS-Lyapunov function we now define.

Definition 4: Let  $\omega : \mathbb{G} \to \mathbb{R}_{\geq 0}$  be a continuous, positive semidefinite function. An  $\omega ISS$ -Lyapunov function for  $\omega ISS$ of (1) is a (smooth) function  $V : \mathbb{G} \to \mathbb{R}_{\geq 0}$  such that, for  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$  and  $\chi \in \mathcal{K}$  the following hold:

$$\alpha_1(\omega(x)) \le V(x) \le \alpha_2(\omega(x)) \tag{9}$$

$$\omega(x) \ge \chi(|u|) \quad \Rightarrow \quad \langle \nabla V(x), f(x, u) \rangle \le -V(x).$$
 (10)

This definition matches that of an SI-IOS Lyapunov function when  $\omega(\cdot) = |h(\cdot)|$  (see [26]).

Theorem 1: System (1) is  $\omega$ ISS if and only if it admits an  $\omega$ ISS-Lyapunov function.

The proof of Theorem 1 is entirely straightforward from the proof of the similar result for the equivalence of ISS and an ISS-Lyapunov function as presented in [23]. A significant technical result required in [23] to demonstrate that ISS implies the existence of an ISS-Lyapunov function is a converse Lyapunov theorem for differential inclusions developed in [15]. A two-measure generalization of this converse Lyapunov theorem was presented in [27] and a special case of this result serves the same purpose in proving Theorem 1. A sketch of the proof of Theorem 1 is included in the Appendix.

We denote a sequence of points  $x \in \mathbb{G}$  converging to a point on the boundary of  $\mathbb{G}$  by  $x \to \partial \mathbb{G}^{\infty}$ . If  $\mathbb{G}$  is unbounded, the notation implies  $|x| \to \infty$ . Our next result requires the following definition:

Definition 5: Let  $\mathbb{A} \subset \mathbb{G}$  be compact. A continuous function  $\omega : \mathbb{G} \to \mathbb{R}_{\geq 0}$  is a proper indicator for  $\mathbb{A}$  on  $\mathbb{G}$  if  $\omega(x) = 0$  if and only if  $x \in \mathbb{A}$  and  $\lim_{x \to \partial \mathbb{G}^{\infty}} \omega(x) = \infty$ .

The "implication" form of an  $\omega$ ISS-Lyapunov function is equivalent to a dissipation-type  $\omega$ ISS-Lyapunov function only when the measurement function is, in fact, a proper indicator for a compact set.

**Proposition 1:** Let  $\mathbb{A} \subset \mathbb{G}$  be a compact set and let  $\omega : \mathbb{G} \to \mathbb{R}_{\geq 0}$  be a proper indicator for  $\mathbb{A}$  on  $\mathbb{G}$ . The  $\omega$ ISS-Lyapunov function in Definition 4 is equivalent to the existence of a (smooth) function  $W : \mathbb{G} \to \mathbb{R}_{\geq 0}$  and functions  $\hat{\alpha}_1, \hat{\alpha}_2 \in \mathcal{K}_{\infty}$  and  $\sigma \in \mathcal{K}$  such that

$$\hat{\alpha}_1(\omega(x)) \le W(x) \le \hat{\alpha}_2(\omega(x)) \tag{11}$$

$$\langle \nabla W(x), f(x, u) \rangle \le -W(x) + \sigma(|u|).$$
 (12)

*Proof:* That (9)-(10) imply (11)-(12) follows as in [23]. In particular, let  $\sigma \in \mathcal{K}_{\infty}$  satisfy  $\sigma(r) \ge \max\{0, \hat{\sigma}(r)\}$  where

$$\hat{\sigma}(r) \doteq \max\left\{ \langle \nabla V(x), f(x, u) \rangle + \alpha_2 \left( \chi(|u|) \right) : \\ |u| \le r, \ \omega(x) \le \chi(r) \right\}.$$

Then, if  $\omega(x) \ge \chi(|u|)$ , (10) immediately yields (12) with  $W(x) \doteq V(x)$  for all  $x \in \mathbb{G}$ . On the other hand, if  $\omega(x) \le \chi(|u|)$ , then by the definition of  $\sigma \in \mathcal{K}_{\infty}$ , we see that

$$\sigma(r) \ge \sup_{\substack{|u|=r}} \langle \nabla V(x), f(x, u) \rangle + \alpha_2 \left( \chi(|u|) \right)$$
  
$$\ge \sup_{|u|=r} \langle \nabla V(x), f(x, u) \rangle + \alpha_2(\omega(x))$$
  
$$\ge \langle \nabla V(x), f(x, u) \rangle + V(x).$$

To see that (11)-(12) imply (9)-(10), we simply take  $V \doteq W^2$ ,  $\alpha_1 \doteq \hat{\alpha}_1^2$ ,  $\alpha_2 \doteq \hat{\alpha}_2^2$ , and  $\chi \doteq \alpha_2^{-1} \circ 4\sigma^2$ . Then the bounds (9) follow directly from (11) and the decrease condition follows from a simple application of the chain rule and completion of squares.

*Remark 2:* We note that the assumption that  $\omega$  is a proper indicator is needed only to demonstrate that (9)-(10) imply (11)-(12). In particular, it is needed to guarantee that the function  $\hat{\sigma}$  is well-defined. The converse statement only requires that  $\omega : \mathbb{G} \to \mathbb{R}_{>0}$  be continuous.

# III. STABILITY WITH RESPECT TO TWO MEASURES AND IOS-LYAPUNOV FUNCTIONS

# A. Stability with respect to two measures

The MMIS generalization of IOSS proposed in Definition 1 is inspired by the notion of stability in two measures initially proposed by Movchan [16] and is a general case of partial stability [28]. A general treatment of stability with respect to two measures can be found in [13].

More recently, Teel and Praly [27] made use of the modern application of comparison functions in stability analysis to introduce the notion of  $\mathcal{KL}$ -stability with respect to two measures. We briefly review the notion of  $\mathcal{KL}$ -stability with respect to two measures and present its Lyapunov characterization prior to discussing the generalization of IOS and, consequently, IOSS.

Consider the differential inclusion

$$\dot{x} \in F(x). \tag{13}$$

Denote the set of solutions of (13) from an initial condition  $x \in \mathbb{G}$  by  $\mathcal{S}(x)$  and a particular solution by  $\phi : \mathbb{R}_{\geq 0} \times \mathbb{G} \to \mathbb{G}$ . Let  $\omega_i : \mathbb{G} \to \mathbb{R}_{\geq 0}, i = 1, 2$ , be continuous functions. The differential inclusion (13) is  $\mathcal{KL}$ -stable with respect to  $(\omega_1, \omega_2)$  if it is forward complete on  $\mathbb{G}$  and all solutions  $\phi \in \mathcal{S}(x)$  satisfy

$$\omega_1(\phi(t,x)) \le \beta(\omega_2(x),t),\tag{14}$$

for all  $t \ge 0$  and  $x \in \mathbb{G}$ . It was shown in [27, Proposition 1] that  $\mathcal{KL}$ -stability with respect to  $(\omega_1, \omega_2)$  for (13) is equivalent to uniform stability, global boundedness, and attractivity (all defined in an appropriate two-measure sense) and forward completeness.

As with classical stability analysis, Lyapunov functions provide a useful method of demonstrating stability with respect to two measures without the need to directly solve the differential inclusion. In the two measure case, a Lyapunov function is a (smooth) function  $V : \mathbb{G} \to \mathbb{R}_{>0}$  satisfying

$$\alpha_1(\omega_1(x)) \le V(x) \le \alpha_2(\omega_2(x)) \tag{15}$$

$$\max_{w \in F(x)} \langle \nabla V(x), w \rangle \le -V(x) \tag{16}$$

for some  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$ .

*Remark 3:* Note that the decrease condition (16) provides for an exponential decrease of the Lyapunov function. The function V(x) on the right-hand side of (16) can, in fact, be replaced by any class- $\mathcal{K}_{\infty}$  function of V(x) and  $\mathcal{KL}$ -stability with respect to two measures still follows using standard comparison lemmas.

The main result of [27] is the equivalence between *robust*  $\mathcal{KL}$ -stability with respect to two measures and the existence of a smooth Lyapunov function. Without using a Lyapunov function, several conditions for robustness are also provided, yielding several converse Lyapunov theorems.

#### B. IOS-Lyapunov functions

The equivalence of robust  $\mathcal{KL}$ -stability with respect to two measures and the existence of a Lyapunov function (15)-(16) naturally leads to the question of whether or not Theorem 1 holds in the two-measure case and, hence, for the property of Input-to-Output Stability. In particular, we are tempted to conjecture that two-measure ISS is equivalent to a Lyapunov function of the form

$$\alpha_1(\omega_1(x)) \le V(x) \le \alpha_2(\omega_2(x)) \tag{17}$$

$$V(x) \ge \chi(|u|) \quad \Rightarrow \quad \langle \nabla V(x), f(x, u) \rangle \le -V(x).$$
 (18)

While the upper and lower bounds (17) are consistent with the definition of an IOS-Lyapunov function, Sontag and Wang [26] demonstrated that, in general, an IOS system does not admit an IOS-Lyapunov function with the decrease (18). Rather, the required decrease condition is

$$V(x) \geq \chi(|u|) \ \Rightarrow \ \langle \nabla V(x), f(x,u) \rangle \leq -\kappa(V(x),|x|),$$

where  $\kappa \in \mathcal{KL}$ .

This can be seen through the example (presented in [26])

$$\begin{array}{rcl} \dot{x}_1 & = & 0 \\ \dot{x}_2 & = & \frac{-2x_2 + u}{1 + x_1^2} \end{array}$$

where the output is taken to be the state  $x_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ . Intuitively, finding a Lyapunov function with a decrease condition of the form (18) is impossible since the decrease rate would need to be independent of  $x_1$ . However, choosing, for example, an initial condition  $x_1(0)$  very large results in a decrease rate of  $x_2$  that can be very small.

On the basis of this discussion, as well as the form of an "Input/Measurement to Output Stability" Lyapunov function conjectured in [26], we anticipate the appropriate definition of an MMIS Lyapunov function to be a function  $V : \mathbb{G} \to \mathbb{R}_{>0}$  satisfying

$$\alpha_1(\omega_1(x)) \le V(x) \le \alpha_2(\omega_2(x)) \tag{19}$$

and

$$\langle \nabla V(x), f(x, u) \rangle \leq -\kappa(V(x), \omega_2(x)) + \sigma_1(|u|) + \sigma_2(\omega_3(x))$$
 (20)

where  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}, \sigma_1, \sigma_2 \in \mathcal{K}$ , and  $\kappa \in \mathcal{KL}$ . The equivalence of MMIS (2) and an MMIS-Lyapunov function (19)-(20) remains to be demonstrated.

#### IV. INCREMENTAL INPUT-TO-STATE STABILITY

The notion of *incremental Input-to-State Stability* ( $\delta ISS$ ) was introduced by Angeli in [1] with the aim of providing tools for incremental stability notions, particularly aimed at observer synthesis problems. In the case of a linear system, we see that  $\omega ISS$  captures incremental ISS of the error system. Specifically, for differing initial conditions and inputs to the system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

we consider two copies of the system with different initial conditions and different inputs:

$$\dot{x}_1(t) = Ax_1(t) + Bu_1(t), \quad x_1(0) = x_1 \dot{x}_2(t) = Ax_2(t) + Bu_2(t), \quad x_2(0) = x_2.$$
(21)

Define the error dynamics as  $e \doteq x_1 - x_2$  and the input difference  $v \doteq u_1 - u_2$ . Then the error dynamics satisfy

$$\dot{e}(t) = Ae(t) + Bv(t).$$

We now see that ISS of the error dynamics is, in fact,  $\omega$ ISS of the system (21) where we are not necessarily interested in convergence of  $x_1(t)$  or  $x_2(t)$  to a particular value, but rather we want a robust stability estimate on the distance between trajectories. Let  $x \doteq [x_1 \ x_2]^T$  and  $\omega(x) \doteq |x_1 - x_2|$ . Then ISS of the error dynamics implies

$$\begin{aligned} \omega(x(t)) &= |e(t)| \\ &\leq \max \left\{ \beta(|e(0)|, t), \gamma(||v||_{\infty}) \right\} \\ &= \max \left\{ \beta(\omega(x), t), \gamma(||u_1 - u_2||_{\infty}) \right\}. \end{aligned}$$

We observe that the above argument does not extend to nonlinear systems, which indicates that, in general,  $\omega$ ISS does not subsume incremental ISS other than in the linear case. This is perhaps surprising, but is consistent with a recent result on the relationship of  $\delta$ ISS and incremental *integral* ISS ( $\delta$ IISS) [2]. Specifically, Angeli demonstrated that, in contrast to the standard ISS result that iISS is *strictly weaker* than ISS,  $\delta$ iISS in fact implies  $\delta$ ISS. However, the one class of systems where ISS and iISS coincide is the class of linear systems. More generally, the precise relationship between  $\omega$ ISS and integral Input-to-State Stability with respect to  $\omega$  ( $\omega$ iISS) remains to be investigated.

#### V. CONCLUSION

We have presented an "arbitrary measure" generalization of Input-to-State Stability, termed Input-to-State Stability with respect to  $\omega$  ( $\omega$ ISS) and demonstrated its equivalence to an appropriate Lyapunov function. The use of this arbitrary measurement function subsumes ISS and State-Independent Input-to-Output Stability, as well as local or partial-state ISS, ISS for systems with multiple equilibrium points, ISS for a prescribed motion, and ISS for closed (but not necessarily compact) sets. We note in the latter case, this may be of particular interest in the area of consensus algorithms (see, for example, [17]) where the interest is in all variables or outputs agreeing on a value without being concerned about the specific numerical value.

The key technical results to enable this equivalence rely on results for  $\mathcal{KL}$ -stability with respect to two measures and are drawn from [27]. Similar discrete-time results for  $\mathcal{KL}$ stability with respect to two measures were developed in [10], [11], and hence we expect that a similar version of Theorem 1 will hold in discrete-time, providing a generalization of the results found in discrete-time ISS [9].

With results for both continuous and discrete time systems, the natural next step is to exploit similar results for  $\mathcal{KLL}$ -stability with respect to two measures of hybrid systems

developed in [6]. Hence, a similar process of generalizing ISS notions may be possible for hybrid systems, keeping in mind that even standard ISS properties do not always translate easily to the hybrid domain [5].

Finally, we conjecture that generalization of integral ISS [4], [20] variants also give rise to a generalized iISS-Lyapunov function equivalence as many of the key technical results from [15] used to derive the equivalence of integral ISS and an integral ISS Lyapunov function [3] have been appropriately generalized in [27].

## VI. APPENDIX

## A. $\omega$ ISS-Lyapunov implies $\omega$ ISS

The proof directly follows that for the ISS case presented in [19] and [23, Lemma 2.14].

Fix  $x \in \mathbb{G}$  and  $u \in \mathcal{U}$ . Define

$$S \doteq \{\eta \in \mathbb{G} : V(\eta) \le \alpha_2 \circ \chi(||u||_{\infty})\}.$$

As in [19], we can demonstrate that S is forward invariant. We omit the details due to space constraints.

Let  $t_1 \doteq \inf\{t \ge 0 : \phi(t, x, u) \in S\}$ . Then, for all  $t \ge t_1$ , using (9) we have

$$\alpha_1(\omega(\phi(t, x, u))) \le V(\phi(t, x, u)) \le \alpha_2 \circ \chi(||u||_{\infty})$$

which implies

$$\omega(\phi(t, x, u)) \le \alpha_1^{-1} \circ \alpha_2 \circ \chi(||u||_{\infty}) =: \gamma(||u||_{\infty}).$$

Now, for  $t < t_1$ ,  $\phi(t, x, u) \notin S$  which implies  $\omega(\phi(t, x, u)) > \chi(||u||_{\infty})$ . Consequently,

$$\frac{d}{dt} V(\phi(t, x, u)) = \langle \nabla V(\eta), f(\eta, u) \rangle |_{\eta = \phi(t, x, u)}$$
  
 
$$\leq -V(\phi(t, x, u)).$$

A comparison result [15, Lemma 4.4] yields the existence of  $\hat{\beta} \in \mathcal{KL}$  such that

$$\alpha_1(\omega(\phi(t, x, u))) \le V(\phi(t, x, u))$$
$$\le \hat{\beta}(V(x), t) \le \hat{\beta}(\alpha_2(\omega(x)), t).$$

Defining  $\beta \in \mathcal{KL}$  by  $\beta(r,s) \doteq \alpha_1^{-1} \left( \hat{\beta}(\alpha_2(r),s) \right)$  then yields

$$\omega(\phi(t, x, u)) \le \max\{\beta(\omega(x), t), \gamma(||u||_{\infty})\}.$$

#### B. *wISS implies wISS-Lyapunov Function*

The proof in this direction closely follows that of the ISS result presented in [23, Lemmas 2.12, 2.13]. Hence, we consider two tasks: (1) we demonstrate that  $\omega$ ISS implies a form of robust feedback stability; and then (2) demonstrate that this robust feedback stability implies the existence of an  $\omega$ ISS-Lyapunov function. By robust feedback stability, as in [23] we mean the existence of a stabilizing state feedback  $\varphi : \mathbb{G} \to \mathbb{R}_{\geq 0}$  such that, for any small perturbation of the feedback,  $\mathcal{KL}$ -stability with respect to  $(\omega, \omega)$  is maintained.

1)  $\omega$ *ISS implies Robust Feedback Stability:* For the differential inclusion  $\dot{x} \in F(x)$ , stability with respect to  $(\omega, \omega)$  was shown to be equivalent to (see [27, Proposition 1]):

- The differential inclusion  $\dot{x} \in F(x)$  is forward complete;
- (Uniform stability and global boundedness): There exists γ ∈ K<sub>∞</sub> such that, for each x ∈ G, all solutions φ ∈ S(x) satisfy

$$\omega(\phi(t,x)) \le \gamma(\omega(x)), \quad \forall t \ge 0;$$

 (Uniform global attractivity): For each r > 0 and ε > 0, there exists T(r, ε) > 0 such that, for each x ∈ G, all solutions φ ∈ S(x) satisfy

$$\omega(x) \le r, \quad t \ge T \quad \Rightarrow \quad \omega(\phi(t,x)) \le \varepsilon.$$

Let  $d: \mathbb{R}_{\geq 0} \to [-1, 1]^m$ . By an abuse of notation, we will also use  $d \in [-1, 1]^m$ . We will construct a state feedback  $\varphi: \mathbb{G} \to \mathbb{R}_{\geq 0}$  and demonstrate that

$$\dot{x} \in F(x) \doteq \bigcup_{d \in [-1,1]^m} f(x, d\varphi(x))$$

is  $\mathcal{KL}$ -stable with respect to  $(\omega, \omega)$ .

From the functions in the  $\omega$ ISS -estimate (5) define functions  $\alpha, \sigma \in \mathcal{K}_{\infty}$  by

$$\alpha(r) \doteq \max\left\{\beta(r,0), \frac{3}{2}r\right\}, \quad \text{and} \tag{22}$$

$$\sigma(r) \doteq \frac{1}{2}\gamma^{-1}\left(\frac{1}{4}\alpha^{-1}(r)\right) < \gamma^{-1}\left(\frac{1}{4}\alpha^{-1}(r)\right).$$
(23)

Let  $\psi \in \mathcal{K}_{\infty}$  and  $\varphi : \mathbb{G} \to \mathbb{R}_{\geq 0}$  be any smooth function satisfying

$$\psi(\omega(x)) \le \varphi(x) \le \sigma(\omega(x)).$$
 (24)

That this is always possible is demonstrated in [23] as follows: Let  $\tilde{\sigma}(r) = \sigma(\sqrt{r})$  and choose any smooth  $\chi \in \mathcal{K}_{\infty}$  with  $\chi(r) \leq \tilde{\sigma}(r)$ . Set  $\varphi(x) = \chi(\omega(x)^2)$  and  $\psi(r) = \chi(r^2)$ .

We now demonstrate that  $\dot{x} = f(x, d\varphi(x))$  is  $\mathcal{KL}$ -stable with respect to  $(\omega, \omega)$ . We denote solutions to this differential equation by  $\phi_{\varphi}(t, x, d)$ .

*Claim:* For any  $x \in \mathbb{G}$  and  $d : \mathbb{R}_{>0} \to [-1, 1]^m$  we have

$$\gamma(\varphi(\phi_{\varphi}(t, x, d))) \le \frac{1}{2}\omega(x) \quad \forall t \ge 0.$$
(25)

Proof: We first note that the ISS estimate implies

$$\omega(x) \le \beta(\omega(x), 0) \le \alpha(\omega(x)).$$
(26)

Consequently, from (24), (22), and (26) we have

$$\gamma(\varphi(x)) \le \gamma \circ \sigma(\omega(x)) < \frac{1}{4}\alpha^{-1}(\omega(x)) \le \frac{1}{4}\omega(x).$$

From the strict inequality above we have that for sufficiently small t > 0

$$\gamma(\varphi(\phi_{\varphi}(t,x,d))) \le \frac{1}{4}\omega(x).$$
(27)

Let

$$t_1 \doteq \inf \left\{ \gamma(\varphi(\phi_{\varphi}(t, x, d))) > \frac{1}{2}\omega(x) \right\}$$

Assume that  $t_1 < \infty$ , then (25) holds for all  $t \in [0, t_1)$ . As a consequence,  $\gamma(|d(t)\varphi(\phi_{\varphi}(t, x, d))|) \leq \frac{1}{2}\alpha(\omega(x))$  for almost all  $t \in [0, t_1)$ . The  $\omega$ ISS estimate (5) yields

$$\omega(\phi_{\varphi}(t,x,d)) \leq \max\left\{\beta(\omega(x),0),\frac{1}{2}\alpha(\omega(x))\right\} \\
 \leq \alpha(\omega(x)), \quad \forall t \in [0,t_1].$$
(28)

Hence, using (23), (24), and (28) we have

$$\gamma(\varphi(\phi_{\varphi}(t_1, x, d))) \le \gamma(\sigma(\omega(\phi_{\varphi}(t_1, x, d)))) \le \frac{1}{4}\omega(x)$$

which contradicts the definition of  $t_1$  above and proves the claim.

Uniform stability and global boundedness: Combining (5) and (25) we see that, for all  $t \ge 0$ ,

$$\omega(\phi_{\varphi}(t,x,d)) \le \max\left\{\beta(\omega(x),0), \frac{1}{2}\omega(x)\right\} \le \alpha(\omega(x)).$$

Uniform global attractivity: We note that (5) and (25) imply

$$\omega(\phi_{\varphi}(t,x,d)) \leq \max\left\{\beta(\omega(x),t),\frac{1}{2}\omega(x)\right\}, \quad \forall x \in \mathbb{G}.$$

Also, since  $\beta \in \mathcal{KL}$ , for each r > 0 there exists T(r) > 0such that  $\beta(r,t) < \frac{1}{2}r$  for all  $t \ge T(r)$ . Consequently, for  $\omega(x) \le r$ ,

$$\omega(\phi_{\varphi}(t,x,d)) \leq \frac{1}{2}r, \quad \forall t \geq T(r).$$

Fix any  $\varepsilon > 0$  and let k be a positive integer such that  $2^{-k}r < \varepsilon$ . Let  $r_1 = r$  and  $r_i = \frac{1}{2}r_{i-1}$  for  $i \ge 2$ . Let  $T(r,\varepsilon) = T(r_1) + T(r_2) + \cdots + T(r_k)$ . Therefore we have

$$\omega(\phi_{\varphi}(t, x, d)) \leq \frac{r}{2^{k}} < \varepsilon, \quad \forall \omega(x) \leq r, \quad t \geq T(r, \varepsilon).$$

2) Robust Feedback Stability implies  $\omega$ ISS-Lyapunov: Having demonstrated strong  $\mathcal{KL}$ -stability of the difference inclusion  $\dot{x} \in F(x)$ , we can appeal to a converse theorem [27, Theorems 1 and 2] to obtain a Lyapunov function  $V : \mathbb{G} \to \mathbb{R}_{>0}$  and functions  $\alpha_1, \alpha_2 \in \mathcal{K}_{\infty}$  such that

$$\alpha_1(\omega(x)) \le V(x) \le \alpha_2(\omega(x)) \tag{29}$$

$$\max_{w \in F(x)} \langle \nabla V(x), w \rangle \le -V(x). \tag{30}$$

We can rewrite the decrease condition as

$$\langle \nabla V(x), f(x, u) \rangle \le -V(x)$$

whenever  $|u| \leq \varphi(x)$ . Since  $\psi(\omega(x)) \leq \varphi(x)$  for all x, the above will also hold for  $|u| \leq \psi(\omega(x))$ . Hence, taking  $\chi(r) = \psi^{-1}(r)$  for all  $r \geq 0$  yields

$$\chi(|u|) \leq \omega(x) \quad \Longrightarrow \quad \langle \nabla V(x), f(x,u) \rangle \leq -V(x)$$

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