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A weak $\mathcal{L}_2$-gain property for nonlinear systems

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***Abstract*— Nonlinear $\mathcal{L}_2$-gain is a generalization of the conventional (linear) notion of $\mathcal{L}_2$-gain in which the linear scaling of input energy is replaced by a nonlinear comparison function scaling. In this paper, this nonlinear $\mathcal{L}_2$-gain property is formalized as being strictly weaker than the conventional linear property. This is achieved by appealing to existing results in the literature that demonstrate qualitative equivalences between linear $\mathcal{L}_2$-gain and input-to-state stability (ISS), and between nonlinear $\mathcal{L}_2$-gain and the strictly weaker integral ISS (iISS) property.**

***Index Terms*— $\mathcal{L}_2$-gain analysis, input-to-state stability, integral input-to-state stability, nonlinear systems.**

I. Introduction

The conventional notion of $\mathcal{L}_2$-gain is a physically inspired generalization of input / output energy gain for dynamical systems. Systems with finite $\mathcal{L}_2$-gain have been widely studied in the context of linear and nonlinear systems, yielding connections between the finite $\mathcal{L}_2$-gain property and notions of stability and dissipation [8], [9], [19], [20]. The role of $\mathcal{L}_2$-gain as a performance measure in feedback control design is well documented for both linear and nonlinear systems, see for example [8], [19].

Conventionally, the finite $\mathcal{L}_2$-gain property has been formulated for dynamical systems with inputs and outputs as an inequality in which the output energy is bounded above by the sum of a linear scaling of the input energy and a term summarizing the effect of the initial state. However, it seems reasonable that the output energy may in fact scale in a nonlinear way with the input energy when nonlinear dynamical systems are considered. Consequently, a generalized notion of nonlinear $\mathcal{L}_2$-gain [3], [4], [5], [6], [14] is considered here, where the output energy is bounded above using a nonlinear scaling of the input energy. This concept of nonlinear scaling or gain is well known in the ISS community, e.g. [13], [16], [17].

By generalizing the notion of $\mathcal{L}_2$-gain to incorporate a nonlinear gain, it is contended that (i) a broader class of nonlinear systems can admit the nonlinear $\mathcal{L}_2$-gain property, and (ii) tighter gain bounds can be achieved, the latter of which is of importance in small-gain based design techniques [1], [10], [11], [13], [19]. Naturally the utility of this generalization is predicated on the nonlinear $\mathcal{L}_2$-gain property proposed [3], [5] being strictly weaker than the conventional (linear) $\mathcal{L}_2$-gain property [19]. In this paper, this assertion is formalized by appealing to qualitative equivalences between linear $\mathcal{L}_2$-gain and input-to-state stability (ISS) [15], [7], and between nonlinear $\mathcal{L}_2$-gain and integral ISS (iISS) [2]. As iISS is strictly weaker than ISS, it is inferred that the nonlinear $\mathcal{L}_2$-gain property is strictly weaker than the linear $\mathcal{L}_2$-gain property. Indeed, the fact that iISS is a natural generalization of ISS suggests that the nonlinear $\mathcal{L}_2$-gain property is a similarly natural generalization of the linear $\mathcal{L}_2$-gain property.

In view of the aforementioned qualitative equivalences, it is noted that iISS tools remain applicable in dealing with nonlinear systems which satisfy the nonlinear $\mathcal{L}_2$-gain property. In particular, small-gain results [1], [10], [11] continue to apply for feedback interconnections of the form shown in Figure 1. However, it is noted that the *qualitative* equivalence between nonlinear $\mathcal{L}_2$-gain and iISS does not imply a *quantitative* equivalence. Hence, tight comparison function bounds characterized for the iISS property do not necessarily translate into tight comparison function bounds for the nonlinear $\mathcal{L}_2$-gain property. So, if the nonlinear $\mathcal{L}_2$-gain property is the desired closed-loop robust stability property to be implemented via a small-gain based design, the characterization of tight comparison function bounds for this property remains important, as do the analogous small-gain results. As results concerning the characterization of tight bounds for the nonlinear $\mathcal{L}_2$-gain exist [3], [4], [21], further attention here is restricted to the development of appropriate small-gain results for the nonlinear $\mathcal{L}_2$-gain properties presented.

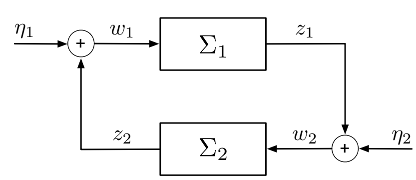

Fig. 1. Feedback interconnection.

Throughout, $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the reals and non-negative reals respectively. $\mathbb{R}^m$ denotes m-dimensional Euclidean space. $\mathcal{L}_2([0,t];\mathbb{R}^m)$, $t \in \mathbb{R}_{\geq 0}$, denotes the space of square integrable functions mapping interval $[0,t]$ to $\mathbb{R}^m$. The associated norm is denoted by $\|\cdot\|_{\mathcal{L}_2[0,t]}$. A function $\gamma : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}$ if it is zero at zero, continuous and non-decreasing. Additionally, $\gamma \in \mathcal{K}_\infty \subset \mathcal{K}$ if it is also strictly increasing and unbounded. Id $\in \mathcal{K}_\infty$ is used to represent the identity operator. $\beta \in \mathcal{KL}$ if $\beta(\cdot,t) \in \mathcal{K}_\infty$ and $\beta(s,\cdot) \in \mathcal{L}$, i.e. continuous and non-increasing, for each $s,t \in \mathbb{R}_{\geq 0}$.

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In terms of organization, the linear and nonlinear $\mathcal{L}_2$-gain properties (e.g. [19], [3], [5]) are recalled in Section II, along with the input-to-state stability (ISS) and integral ISS (iISS) properties, and key qualitative equivalences between pairs of these properties. The relative strength of the two $\mathcal{L}_2$-gain properties is inferred in Section III, followed by a discussion of the utility of this equivalence from a computational and small-gain point of view. Finally, some concluding remarks are provided in Section IV.

II. Robust stability properties

Attention is restricted to finite dimensional nonlinear dynamical systems of the form

$$\Sigma \,:\, \begin{cases} \dot{x}(t) = f(x(t),\, w(t))\,, & x(0) = x\,, \\ z(t) = h(x(t))\,, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^m$, and $z(t) \in \mathbb{R}^n$ denote the state, input, and output at time $t \in \mathbb{R}_{\geq 0}$, and $x \in \mathbb{R}^n$ denotes the initial state. In (1), sufficient regularity of f is assumed so as to guarantee existence and uniqueness of solutions. It is also assumed that $f(0,0) = 0$ and $h(0) = 0$, where h is proper in the sense that there exists $\underline{\alpha}_h$, $\overline{\alpha}_h \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_h(|x|) \leq |h(x)| \leq \overline{\alpha}_h(|x|) \tag{2}$$

for all $x \in \mathbb{R}^n$. (Note that this properness assumption is used to render an equivalence between input-to-output and input-to-state properties, see Lemma 2.12.) The input spaces

$$\mathscr{W}_2 \doteq \bigcap_{t\geq 0} \mathcal{L}_2\left([0,t];\, \mathbb{R}^m\right), \tag{3}$$

$$\mathscr{W}_\infty \doteq \bigcap_{t\geq 0} \mathcal{L}_\infty\left([0,t];\, \mathbb{R}^m\right), \tag{4}$$

$$\mathscr{W}_\gamma \doteq \bigcap_{t\geq 0} \left\{ w : [0,t] \mapsto \mathbb{R}^m \;\middle|\; \int_0^t \gamma(|w(s)|)\, ds < \infty \right\}, \tag{5}$$

are used in proposing a variety of robust stability properties to follow. In (5), the subscript γ is used to denote the dependence on a comparison function $\gamma \in \mathcal{K}_\infty$.

A. Linear and nonlinear $\mathcal{L}_2$-gain properties

Linear and nonlinear $\mathcal{L}_2$-gain properties, e.g. [5], are examples of robust stability properties for systems of the form (1). These properties are expressed in terms of input / output gain inequalities, where each inequality provides an upper bound on the $\mathcal{L}_2$-norm of the system output in terms of the $\mathcal{L}_2$-norm of the system input.

Definition 2.1: System (1) has finite (linear) $\mathcal{L}_2$-gain with transient and gain bound $(\beta,\, \bar{\gamma}) \in \mathcal{K}_\infty \times \mathbb{R}_{\geq 0}$ if

$$\|z\|^2_{\mathcal{L}_2[0,t]} \leq \beta(|x|) + \bar{\gamma}^2\, \|w\|^2_{\mathcal{L}_2[0,t]} \tag{6}$$

holds for all initial conditions $x \in \mathbb{R}^n$, all inputs $w \in \mathscr{W}_2$, and time horizons $t \in \mathbb{R}_{\geq 0}$.

In interpreting the (conventional) linear $\mathcal{L}_2$-gain inequality (6), the gain bound $\bar{\gamma}$ may be regarded as parameterizing a comparison function $\gamma \in \mathcal{K}_\infty$ given by $\gamma(s) \doteq \bar{\gamma}^2\, s$, $s \in \mathbb{R}_{\geq 0}$. Consequently, the linear $\mathcal{L}_2$-gain property may be regarded as a special case of a more general nonlinear $\mathcal{L}_2$-gain property, where the gain parameter $\bar{\gamma} \in \mathbb{R}_{\geq 0}$ is generalized to a function $\gamma \in \mathcal{K}_\infty$.

Definition 2.2 ([4]): System (1) has finite nonlinear $\mathcal{L}_2$-gain with transient and gain bound $(\beta,\, \gamma) \in \mathcal{K}_\infty \times \mathcal{K}_\infty$ if

$$\|z\|^2_{\mathcal{L}_2[0,t]} \leq \beta(|x|) + \gamma\left(\|w\|^2_{\mathcal{L}_2[0,t]} \right) \tag{7}$$

holds for all initial conditions $x \in \mathbb{R}^n$, all inputs $w \in \mathscr{W}_2$, and time horizons $t \in \mathbb{R}_{\geq 0}$.

Remark 2.3: Note that in [3], [4], etc, the transient and gain bounds are selected from a different class of comparison functions, denoted there by $\bar{\mathcal{K}}$. This class of comparison functions is a relaxation of $\mathcal{K}$, with continuity replaced by right continuity at 0. The purpose of this relaxation is to allow the characterization of tight comparison function bounds, as these tight bounds need not be continuous away from 0 or strictly increasing. However, it may be shown that the nonlinear $\mathcal{L}_2$-gain property defined in those references (with respect to $\bar{\mathcal{K}}$) is *qualitatively equivalent* to the definition (7) used here. A discussion of qualitative equivalence appears in Section II-C.

By inspection, the nonlinear $\mathcal{L}_2$-gain property (7) necessarily implies that the input / output operator defined by (1) is well-defined in the sense that

$$\Sigma \,:\, \mathscr{W}_2 \mapsto \mathscr{W}_2$$

for all initial states $x(0) = x \in \mathbb{R}^n$. Property (7) also implies that Σ is $\mathcal{L}_2$-stable [19] in the sense that

$$w \in \mathcal{L}_{2,\infty}\left(\mathbb{R}^m\right) \Longrightarrow z \in \mathcal{L}_{2,\infty}\left(\mathbb{R}^n\right)\,, \tag{8}$$

where $\mathcal{L}_{2,\infty}(\mathbb{R}^m) \doteq \mathcal{L}_2([0,\infty];\, \mathbb{R}^m)$.

In view of the linear and nonlinear $\mathcal{L}_2$-gain properties (6) and (7), the following result summarizes the obvious generalization at work.

Theorem 2.4 ([5]): Suppose that a nonlinear system (1) has finite linear $\mathcal{L}_2$-gain with transient and gain bound $(\beta, \bar{\gamma}) \in \mathcal{K}_\infty \times \mathbb{R}_{\geq 0}$. Then, that system has finite nonlinear $\mathcal{L}_2$-gain with transient and gain bound $(\beta,\, \gamma) \in \mathcal{K}_\infty \times \mathcal{K}_\infty$, where $\gamma(s) = \bar{\gamma}^2\, s$ for all $s \in \mathbb{R}_{\geq 0}$. That is,

$$\left(\begin{array}{c} \text{Linear } \mathcal{L}_2\text{-gain} \\ \text{property (6)} \end{array} \right) \Longrightarrow \left(\begin{array}{c} \text{Nonlinear } \mathcal{L}_2\text{-gain} \\ \text{property (7)} \end{array} \right)$$

Theorem 2.4 indicates that the nonlinear $\mathcal{L}_2$-gain property of (7) cannot be a stronger robust stability property than the linear $\mathcal{L}_2$-gain property of (6). However, it is important to establish that the nonlinear $\mathcal{L}_2$-gain property (7) is strictly weaker than the linear $\mathcal{L}_2$-gain property. This can be demonstrated using existing results in the literature concerning various characterizations of input-to-state stability (ISS) and integral ISS (iISS). These properties are recalled in the following section.

B. Input-to-state stability (ISS) and integral ISS (iISS)

ISS and iISS, e.g. [15], [16], [17], [18], are well-known robust stability properties that are similarly expressed as gain inequalities. In the case of ISS, this inequality provides an upper bound on the $\mathcal{L}_\infty$-norm of the state in terms of an $\mathcal{L}_\infty$-norm of the input. For iISS, the upper bound is an integrated nonlinear scaling of the instantaneous norm of the input.

Definition 2.5 ([15]): A nonlinear system (1) is input-to-state stable (ISS) with transient and gain bound $(\beta, \gamma) \in \mathcal{KL} \times \mathcal{K}_\infty$ if

$$|x(t)| \leq \beta(|x|, t) + \gamma\left(\|w\|_{\mathcal{L}_\infty[0,t]}\right) \tag{9}$$

holds for all initial conditions $x \in \mathbb{R}^n$, inputs $w \in \mathscr{W}_\infty$, and time horizons $t \in \mathbb{R}_{\geq 0}$.

Inequality (9) is a trajectory based characterization of ISS, with the instantaneous norm of the state bounded by a decaying term in the initial condition and a gain term. A number of qualitatively equivalent characterizations of ISS exist [18], [15]. One such qualitatively equivalent definition of ISS is integral-input-to-integral-state stability (iIiSS, [15]).

Definition 2.6 ([15]): Given $\alpha \in \mathcal{K}_\infty$, a nonlinear system (1) is integral-input-to-integral-state stable (iIiSS) with transient and gain bound $(\beta, \gamma) \in \mathcal{K}_\infty \times \mathcal{K}_\infty$ if

$$\int_0^t \alpha(|x(s)|)\, ds \leq \beta(|x|) + \int_0^t \gamma(|w(s)|)\, ds \tag{10}$$

holds for all initial conditions $x \in \mathbb{R}^n$, inputs $w \in \mathscr{W}_\gamma$, and time horizons $t \in \mathbb{R}_{\geq 0}$.

Theorem 2.7 ([15], Theorem 1): A nonlinear system (1) is input-to-state stable (ISS) if and only if it is integral-input-to-integral-state stable (iIiSS).

The iISS property takes a similar form to (9), except that the instantaneous input enters via an integrated nonlinear scaling of its norm.

Definition 2.8 ([15]): Given $\alpha \in \mathcal{K}_\infty$, a nonlinear system (1) is integral input-to-state stable (iISS) with transient and gain bound $(\beta, \gamma) \in \mathcal{KL} \times \mathcal{K}_\infty$ if

$$\alpha(|x(t)|) \leq \beta(|x|, t) + \int_0^t \gamma(|w(s)|)\, ds \tag{11}$$

holds for all initial conditions $x \in \mathbb{R}^n$, inputs $w \in \mathscr{W}_\gamma$, and time horizons $t \in \mathbb{R}_{\geq 0}$.

Property (11) is a trajectory based characterization of iISS, with an analogous form to (9). An equivalent definition [2] that is of a similar form to the iIiSS property (10) also exists.

Definition 2.9 ([2]): Given $\alpha, \sigma \in \mathcal{K}_\infty$, a nonlinear system (1) is iISS [2] with transient and gain bound $(\beta, \gamma) \in \mathcal{K}_\infty \times \mathcal{K}_\infty$ if

$$\int_0^t \alpha(|x(s)|)\, ds \leq \beta(|x|) + \gamma\left(\int_0^t \sigma(|w(s)|)\, ds\right) \tag{12}$$

holds for all initial conditions $x \in \mathbb{R}^n$, inputs $w \in \mathscr{W}_\sigma$, and time horizons $t \in \mathbb{R}_{\geq 0}$.

It is well known that iISS is a strictly weaker property than ISS, see Corollary 4 in [15] and the discussion thereafter.

Theorem 2.10 ([15]): Any nonlinear system (1) that is ISS is also iISS. That is,

$$\left(\begin{array}{c}\text{ISS}\\ \text{property (9)}\end{array}\right) \overset{\not\Leftarrow}{\Longrightarrow} \left(\begin{array}{c}\text{iISS}\\ \text{property (11)}\end{array}\right)$$

However, systems (1) exist that are iISS but not ISS, for example $f(x, w) \doteq -x + x\, w$, $h(x) \doteq x$.

C. Invariance and qualitative equivalence

Statements concerning the robust stability of system (1) are fundamental in nature and should be independent of the realization employed. Indeed, [17] states that "*notions of stability should be invariant under (nonlinear) changes of variables*" of the form

$$w = S(\omega)\,, \quad x = T(\xi)\,, \tag{13}$$

where $S : \mathbb{R}^m \mapsto \mathbb{R}^m$ and $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ are homeomorphisms satisfying $S(0) = 0$, $T(0) = 0$, and play the role of input and state transformations respectively. By inspection of (13), it may be shown [17] that S, T are proper in the sense of (2), i.e. there exist $\underline{\alpha}_S, \underline{\alpha}_T, \overline{\alpha}_S, \overline{\alpha}_T \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_S(|\omega|) \leq |S(\omega)| \leq \overline{\alpha}_S(|\omega|)\,, \tag{14}$$

$$\underline{\alpha}_T(|\xi|) \leq |T(\xi)| \leq \overline{\alpha}_T(|\xi|)\,, \tag{15}$$

for all $\omega \in \mathbb{R}^m$, $\xi \in \mathbb{R}^n$. In the case of the iISS property (12) for example, application of these state and input changes of variables yields

$$\begin{aligned}
&\int_0^t \alpha \circ \underline{\alpha}_T(|\xi(s)|)\, ds \\
&\quad \leq \int_0^t \alpha(|T(\xi(s))|)\, ds = \int_0^t \alpha(|x(s)|)\, ds \\
&\quad \leq \beta(|x|) + \gamma\left(\int_0^t \sigma(|w(s)|)\, ds\right) \\
&\quad \leq \beta \circ \overline{\alpha}_T(|\xi|) + \gamma\left(\int_0^t \sigma \circ \overline{\alpha}_S(|\omega(s)|)\, ds\right).
\end{aligned} \tag{16}$$

While the comparison functions differ between properties (12) and (16), the form of the inequality described remains the same. That is, the form of (12) is retained under the state and input transformation (13). Consequently, properties (12) and (16) are regarded as being *qualitatively equivalent*. (Where the comparison functions are the same, this equivalence becomes *quantitative*.)

This notion of equivalence can be extended to pairs of properties of different form. In particular, suppose one property implies a second of different form, which in turn implies a third of the same form as the first. If the first and third properties are qualitatively equivalent, then all three may be regarded as qualitatively equivalent. For example, a qualitatively equivalent formulation of the nonlinear $\mathcal{L}_2$-gain property (7) may be obtained by replacing "+" with "max". This takes a different form to (7), with

$$\|z\|^2_{\mathcal{L}_2[0,T]} \leq \max\left\{\beta\left(|x_\circ|\right),\, \gamma\left(\|w\|^2_{\mathcal{L}_2[0,T]}\right)\right\}. \tag{17}$$

However, by inspection, (17) implies (7) with (in this case, the same) transient / gain bound pair $(\beta, \gamma) \in \mathcal{K}_\infty \times \mathcal{K}_\infty$. Furthermore, note that for any $a, b \in \mathbb{R}_{\geq 0}$, $a + b \leq 2\max(a,\, b) = \max(2a,\, 2b)$. Hence, with $a \doteq \beta(|x|)$, $b \doteq \gamma\left(\|w\|^2_{\mathcal{L}_2[0,T]}\right)$, (7) implies (17) with transient / gain bound pair $(\hat{\beta}, \hat{\gamma}) \doteq (2\beta,\, 2\gamma) \in \mathcal{K}_\infty \times \mathcal{K}_\infty$. That is, (7) and (17) are qualitatively equivalent robust stability properties.

The following lemma concerns changes of variables used in demonstrating a number of qualitative equivalences.

Lemma 2.11: Given any $\alpha \in \mathcal{K}_\infty$, there exist changes of variable $\underline{U}$, $\overline{U} : \mathbb{R}^p \mapsto \mathbb{R}^p$ such that for all $u \in \mathbb{R}^p$,

$$|\underline{U}(u)| \leq \alpha(|u|) \leq |\overline{U}(u)| \,. \tag{18}$$

Proof: Construct the change of variables $\overline{U} : \mathbb{R}^p \mapsto \mathbb{R}^p$ with i^{th} coordinate

$$\overline{U}_i(u) \doteq \operatorname{sgn}(u_i)\, \alpha(|u_i|\sqrt{p}) \,, \tag{19}$$

which is invertible by inspection. (Following [17, Section 2.8], differentiability of this change of variables is not required.) Then,

$$\begin{aligned}|\overline{U}(u)| &= \sqrt{\sum_i |\overline{U}_i(u)|^2} \geq \max_i |\overline{U}_i(u)| \\ &= \max_i \alpha\left(|u_i|\sqrt{p}\right) = \alpha\left(\max_i |u_i|\sqrt{p}\right) \geq \alpha(|u|) \,.\end{aligned}$$

Similarly, construct the change of variables $\underline{U} : \mathbb{R}^p \mapsto \mathbb{R}^p$ with i^{th} coordinate

$$\underline{U}_i(u) \doteq \tfrac{1}{\sqrt{p}} \operatorname{sgn}(u_i)\, \alpha(|u_i|) \,, \tag{20}$$

which is again invertible by inspection. Then,

$$\begin{aligned}|\underline{U}(u)| &= \sqrt{\sum_i |\underline{U}_i(u)|^2} \leq \sqrt{p}\, \max_i |\overline{U}_i(u)| \\ &= \sqrt{p}\, \max_i \tfrac{1}{\sqrt{p}}\, \alpha(|u_i|) \leq \alpha\left(\max_i |u_i|\right) \leq \alpha(|u|) \,.\end{aligned}$$

So, applying the changes of variables (19) and (20) completes the proof. ∎

Lemma 2.12: The nonlinear $\mathcal{L}_2$-gain property (7) for system (1) with output map h satisfying (2) is qualitatively equivalent to the same property (using the same nonlinear gain) for system (1) employing the identity output map.

Proof: For convenience, note that the nonlinear $\mathcal{L}_2$-gain property (6) holds for system (1) with identity output map $h(x) = x$ and transient and gain bound $(\beta, \gamma) \in \mathcal{K}_\infty \times \mathcal{K}_\infty$ if

$$\|x\|^2_{\mathcal{L}_2[0,t]} \leq \beta(|x_\circ|) + \gamma\left(\|w\|^2_{\mathcal{L}_2[0,t]}\right) \tag{21}$$

holds for all initial states $x_\circ \doteq x(0) \in \mathbb{R}^n$, inputs $w \in \mathscr{W}_2$, and time horizons $t \in \mathbb{R}_{\geq 0}$. The objective is to demonstrate that the nonlinear $\mathcal{L}_2$-gain properties (7) and (21) are qualitatively equivalent subject to (2).

Applying Lemma 2.11 with $\underline{\alpha}_h^{-1} \in \mathcal{K}_\infty$, there exists a change of state variables $\overline{T} : \mathbb{R}^n \mapsto \mathbb{R}^n$, $x = \overline{T}(\xi)$, such that $\underline{\alpha}_h^{-1}(|\xi|) \leq |\overline{T}(\xi)| = |x|$. Hence, (2) implies that $|\xi| \leq \underline{\alpha}_h(|x|) \leq |h(x)|$. So, applying change of state $\overline{T}$ in the nonlinear $\mathcal{L}_2$-gain property (7) yields that

$$\|\xi\|^2_{\mathcal{L}_2[0,t]} \leq \beta \circ \overline{\alpha}_{\overline{T}}(|\xi_\circ|) + \gamma\left(\|w\|^2_{\mathcal{L}_2[0,t]}\right) \tag{22}$$

for all transformed initial states $\xi_\circ \doteq \xi(0) \in \mathbb{R}^n$, inputs $w \in \mathscr{W}_2$, and time horizons $t \in \mathbb{R}_{\geq 0}$, where $\overline{\alpha}_{\overline{T}} \in \mathcal{K}_\infty$ is as per (15) for change of variables $\overline{T}$. That is, property (21) holds with transient and gain bound $(\beta \circ \overline{\alpha}_T, \gamma) \in \mathcal{K}_\infty \times \mathcal{K}_\infty$.

Once again applying Lemma 2.11, this time with $\overline{\alpha}_h^{-1} \in \mathcal{K}_\infty$, there exists a change of state variables $\underline{T} : \mathbb{R}^n \mapsto \mathbb{R}^n$, $\tilde{x} = \underline{T}(\xi)$, such that $|\tilde{x}| = |\underline{T}(\xi)| \leq \overline{\alpha}_h^{-1}(|\xi|)$. Hence, (2) implies that $|h(\tilde{x})| \leq \overline{\alpha}_h(|\tilde{x}|) \leq |\xi|$. So, applying change of state $\underline{T}$ in the nonlinear $\mathcal{L}_2$-gain property (22) yields that

$$\|\tilde{x}\|^2_{\mathcal{L}_2[0,t]} \leq \beta \circ \overline{\alpha}_{\overline{T}} \circ \underline{\alpha}_{\underline{T}}^{-1}(|\xi_\circ|) + \gamma\left(\|w\|^2_{\mathcal{L}_2[0,t]}\right) \tag{23}$$

for all transformed initial states $\xi_\circ \doteq \xi(0) \in \mathbb{R}^n$, inputs $w \in \mathscr{W}_2$, and time horizons $t \in \mathbb{R}_{\geq 0}$, where $\tilde{z} \doteq h(\tilde{x})$ and $\underline{\alpha}_{\underline{T}} \in \mathcal{K}_\infty$ is as per (15) for change of variables $\overline{T}$. That is, property (7) holds with transient and gain bound $(\beta \circ \overline{\alpha}_{\overline{T}} \circ \underline{\alpha}_{\underline{T}}^{-1}, \gamma) \in \mathcal{K}_\infty \times \mathcal{K}_\infty$.

By inspection of (6), (21), (22), (23), property (21) is qualitatively equivalent to the nonlinear $\mathcal{L}_2$-gain property (7). ∎

III. Nonlinear $\mathcal{L}_2$-gain is strictly weaker than linear $\mathcal{L}_2$-gain

A. Key qualitative equivalences

Using the notion of qualitative equivalence for robust stability properties discussed in Section II-C, it may be shown using results taken directly from the literature that the nonlinear $\mathcal{L}_2$-gain property (7) is strictly weaker than the linear $\mathcal{L}_2$-gain property (6).

Theorem 3.1 ([7], [15]): The linear $\mathcal{L}_2$-gain property (6) is qualitatively equivalent to the ISS property (9), or

$$\begin{pmatrix} \text{Linear } \mathcal{L}_2\text{-gain} \\ \text{property (6)} \end{pmatrix} \iff \begin{pmatrix} \text{ISS} \\ \text{property (9)} \end{pmatrix}$$

Proof: For convenience, note that the linear version of the nonlinear $\mathcal{L}_2$-gain property (21) holds with transient and gain bound $(\beta, \bar{\gamma}) \in \mathcal{K} \times \mathbb{R}_{\geq 0}$ if

$$\|x\|^2_{\mathcal{L}_2[0,t]} \leq \beta(|x_\circ|) + \bar{\gamma}^2\, \|w\|^2_{\mathcal{L}_2[0,t]} \tag{24}$$

holds for all initial states $x_\circ = x(0) \in \mathbb{R}^n$, inputs $w \in \mathscr{W}_2$, and time horizons $t \in \mathbb{R}_{\geq 0}$. So, Lemma 2.12 implies that the linear $\mathcal{L}_2$-gain properties (6) and (24) are qualitatively equivalent, with the same linear gain $\bar{\gamma} \in \mathbb{R}_{\geq 0}$.

Next, suppose system (1) satisfies the ISS property (9) with transient and gain bound $(\beta, \gamma) \in \mathcal{KL} \times \mathcal{K}_\infty$. Then, Theorems 3 and 4 of [7] imply that there exists change of state and input variables such that the linear $\mathcal{L}_2$-gain property (24) holds for the transformed system, with transient and gain bound $(|\cdot|^2, 1) \in \mathcal{K}_\infty \times \mathbb{R}_{\geq 0}$. However, this is precisely the iIiISS property (10) with $\alpha(s) = \beta(s) = \gamma(s) = s^2$. So, applying Theorem 1 of [15], the transformed system must satisfy the ISS property (9) for some transient and gain bound $(\hat{\beta}, \hat{\gamma}) \in \mathcal{KL} \times \mathcal{K}_\infty$. That is, the ISS property (9) is qualitatively equivalent to the linear $\mathcal{L}_2$-gain property (24), as required.

Hence, combining these equivalences, it follows that the linear $\mathcal{L}_2$-gain property (6) is qualitatively equivalent to the ISS property (9). ∎

Theorem 3.2 ([2]): The nonlinear $\mathcal{L}_2$-gain property (7) is qualitatively equivalent to the iISS property (11), or

$$\begin{pmatrix} \text{Nonlinear } \mathcal{L}_2\text{-gain} \\ \text{property (7)} \end{pmatrix} \iff \begin{pmatrix} \text{iISS} \\ \text{property (11)} \end{pmatrix}$$

Proof: By Lemma 2.12, the nonlinear $\mathcal{L}_2$-gain property (7) is qualitatively equivalent to the same property for system

(1) modified to employ an identity output map. In particular, property (21) holds. Fix any $\alpha,\,\sigma\in\mathcal{K}_\infty$, define $\tilde\alpha(s)\doteq\sqrt{\alpha(s)}$, $\tilde\sigma(s)\doteq\sqrt{\sigma(s)}$, and note that $\tilde\alpha,\,\tilde\sigma\in\mathcal{K}_\infty$. Applying Lemma 2.11 given $\tilde\alpha\in\mathcal{K}_\infty$, there exists a change of state variables $\tilde T:\mathbb{R}^n\mapsto\mathbb{R}^n$, $x=\tilde T(\xi)$, such that $\tilde\alpha(|\xi|)\le|\tilde T(\xi)|$ for all $\xi\in\mathbb{R}^n$. Similarly, applying Lemma 2.11 given $\tilde\sigma\in\mathcal{K}_\infty$, there exists a change of input variables $\tilde S:\mathbb{R}^m\mapsto\mathbb{R}^m$, $w=\tilde S(\omega)$, such that $|\tilde S(\omega)|\le\tilde\sigma(|\omega|)$ for all $\omega\in\mathbb{R}^m$. Hence, applying $\tilde S$ and $\tilde T$ in property (21),

$$\begin{aligned}\int_0^t \alpha(|\xi(s)|)\,ds &= \int_0^t |\tilde\alpha(|\xi(s)|)|^2\,ds\\ &\le \int_0^t |x(s)|^2\,ds \le \beta(|x_\circ|)+\gamma\left(\int_0^t |w(s)|^2\,ds\right)\\ &\le \beta\circ\overline{\alpha}_{\tilde T}(|\xi_\circ|)+\gamma\left(\int_0^t |\tilde\sigma(|\omega(s)|)|^2\,ds\right)\\ &= \beta\circ\overline{\alpha}_{\tilde T}(|\xi_\circ|)+\gamma\left(\int_0^t \sigma(|\omega(s)|)\,ds\right), \end{aligned}\tag{25}$$

where $\overline{\alpha}_{\tilde T}\in\mathcal{K}_\infty$ is an upper bound for $\tilde T$ as per (15). That is, property (12) holds. Again applying Lemma 2.11 twice, given $\tilde\alpha,\,\tilde\sigma\in\mathcal{K}_\infty$ as defined above, there exist changes of input and state variables $\hat S:\mathbb{R}^p\mapsto\mathbb{R}^p$, $\hat T:\mathbb{R}^n\mapsto\mathbb{R}^n$, $\hat w=\hat S(\omega)$, $\hat x=\hat T(\xi)$ such that $\tilde\sigma(|\omega|)\le|\hat S(\omega)|$, $|\hat T(\xi)|\le\tilde\alpha(|\xi|)$ for all $\omega\in\mathbb{R}^p$, $\xi\in\mathbb{R}^n$. Applying $\hat S$ and $\hat T$ in property (25), a similar argument yields that

$$\int_0^t |\hat x(s)|^2\,ds \le \beta\circ\overline{\alpha}_{\tilde T}\circ\underline{\alpha}_{\hat T}^{-1}(|\hat x_\circ|)+\gamma\left(\int_0^t |\hat w(s)|^2\,ds\right),$$

where $\underline{\alpha}_{\hat T}\in\mathcal{K}_\infty$ is a lower bound for $\hat T$ as per (15). That is, property (21) holds, and is qualitatively equivalent to itself via the iISS property (12), which is itself qualitatively equivalent to the iISS property (11), see [2]. Hence, the nonlinear $\mathcal{L}_2$-gain property (7) and the iISS property (11) are qualitatively equivalent as required. ■

B. Nonlinear $\mathcal{L}_2$-gain is strictly weaker

In view of Theorems 2.10, 3.1 and 3.2, the fact that the iISS property (11) is strictly weaker than the ISS property (9) may be used to conclude the relative weakness of the nonlinear $\mathcal{L}_2$-gain property (7) relative to the linear $\mathcal{L}_2$-gain property (6). The proof of the following result is thus immediate.

Theorem 3.3: The nonlinear $\mathcal{L}_2$-gain property (7) is strictly weaker than the linear $\mathcal{L}_2$-gain property (6), or

$$\begin{array}{ccc}\left(\begin{array}{c}\text{Linear } \mathcal{L}_2\text{-gain}\\ \text{property (6)}\end{array}\right) & \overset{\not\Leftarrow}{\Longrightarrow} & \left(\begin{array}{c}\text{Nonlinear } \mathcal{L}_2\text{-gain}\\ \text{property (7)}\end{array}\right)\\ \Updownarrow & & \Updownarrow\\ \left(\begin{array}{c}\text{ISS}\\ \text{property (9)}\end{array}\right) & \overset{\not\Leftarrow}{\Longrightarrow} & \left(\begin{array}{c}\text{iISS}\\ \text{property (11)}\end{array}\right)\end{array}$$

C. Verification and tight comparison function bounds

Verification of the linear $\mathcal{L}_2$-gain property (6) exploits well-known connections between finite gain and dissipation [19]. In particular, by inspection of (6), it is evident that system (1) satisfies the linear $\mathcal{L}_2$-gain property (6) with transient and gain bound $(\beta,\,\bar\gamma)\in\mathcal{K}_\infty\times\mathbb{R}_{\ge0}$ if a solution $V:\mathbb{R}^n\mapsto\mathbb{R}_{\ge0}$ of the dynamic programming equation (DPP)

$$V(x)=\sup_{w\in\mathscr{W}_2}\left\{\begin{array}{c|c}\|z\|^2_{\mathcal{L}_2[0,t]} & \text{(1) holds}\\ -\bar\gamma^2\,\|w\|^2_{\mathcal{L}_2[0,t]} & \text{with}\\ +V(x(t)) & x(0)=x\end{array}\right\}\tag{26}$$

exists and is dominated by β in the sense that $V(x)\le\beta(|x|)$ for all $x\in\mathbb{R}^n$. In the limit $t\to0^+$ of vanishingly small time horizons, DPP (26) yields the Hamilton-Jacobi-Bellman (HJB) partial differential equation (PDE)

$$0=H(x,\,\nabla_x V(x))\tag{27}$$

where $H(x,p)\doteq-|h(x)|^2-\sup_{w\in\mathbb{R}^m}\{\langle p,\,f(x,w)\rangle-\bar\gamma^2\,|w|^2\}$. HJB PDE (27) is instrumental in the verification of the linear $\mathcal{L}_2$-gain property [8], [12], [19].

In providing an analogous verification result for the nonlinear $\mathcal{L}_2$-gain property (7), a key difficulty in the corresponding argument is encountered. In the linear gain case, in passing to the small time horizon limit in (26), commutation of the linear gain term $\bar\gamma^2$ and the integration defining $\|w\|^2_{\mathcal{L}_2[0,t]}$ is exploited. However, in the nonlinear gain case, this commutation is not possible. One way around this problem [5] is to augment the state of system (1) with additional dynamics to keep track of the accumulation of the input norm $\|w\|^2_{\mathcal{L}_2[0,t]}$. This leads to an analogous dissipation argument in which the gain function (assumed to be differentiable) and integration are commuted via the fundamental theorem of calculus. The result is a more complicated verification result [5]. In particular, the nonlinear $\mathcal{L}_2$-gain property (7) holds with transient and gain bound $(\beta,\gamma)\in\mathcal{K}_\infty\times\mathcal{K}_\infty$, γ differentiable, if there exists a continuous viscosity supersolution $V:\mathbb{R}^n\times\mathbb{R}_{\ge0}\mapsto\mathbb{R}_{\ge0}$ of the HJB PDE

$$0=H(x,\xi,\nabla_x V(x,\xi),\,\nabla_\xi V(x,\xi))\tag{28}$$

for all $(x,\xi)\in\mathbb{R}^n\times\mathbb{R}_{>0}$, with V dominated by β in the sense that $V(x,0)\le\beta(|x|)$ for all $x\in\mathbb{R}^n$, and

$$\begin{aligned}&H(x,\xi,p,q)\\ &\quad\doteq-|h(x)|^2-\sup_{w\in\mathbb{R}^m}\left\{\langle p,\,f(x,w)\rangle+(q-\gamma'(\xi))\,|w|^2\right\}.\end{aligned}$$

That is, an equation on a higher dimensional space must be solved in the nonlinear gain case. Where this nonlinear gain is selected to be linear, i.e. $\gamma(s)=\bar\gamma^2\,s$, HJB PDE (28) simplifies to that of (27). Consequently, it is apparent that verification is a more difficult computational problem for the nonlinear $\mathcal{L}_2$-gain property (7) than for the linear $\mathcal{L}_2$-gain property (6).

With regard to the computation of tight comparison function bounds, the linear gain case corresponds to the computation of the $\mathcal{H}_\infty$-norm of system (1). This can be achieved via a bisection approach [12], where the HJB PDE (27) is solved repeatedly while searching for the minimum gain bound $\bar\gamma\in\mathbb{R}_{\ge0}$. In the nonlinear gain case [3], [4], [21], a HJB PDE similar to (28) may be derived, also defined on $\mathbb{R}^{n+1}$, whose solution characterizes the tight transient and gain bounds. The higher state-space dimension in that case means that the linear and nonlinear tight bound computations appear to be similarly computationally expensive.

D. Small-gain results

Recent results [1], [10], [11] have demonstrated cases where small-gain conditions associated with iISS are both sufficient and necessary. In particular, [11] demonstrated that stability of a feedback interconnection of two iISS systems implies a small-gain condition if one of the systems is strictly ISS. Conversely, condition (H3) of [11, Theorem 2] provides a sufficient small-gain condition for the feedback interconnection of two iISS systems to be iISS. Here, attention is restricted to sufficient conditions (proofs omitted). The search for tight gain bounds for which the nonlinear $\mathcal{L}_2$-gain property (7) holds is easily motivated via such sufficient small-gain arguments.

In the remainder of this section, a small-gain theorem is stated for the interconnection shown in Figure 1, for the "+" formulation of the nonlinear $\mathcal{L}_2$-gain property (7). An analogous small-gain result for the nonlinear $\mathcal{L}_2$-gain property (17) also holds, but is omitted for brevity.

Theorem 3.4: Suppose that two systems $\Sigma_{1,2}$ of the form (1), with compatible input / output dimensions ($m_{1,2} = p_{2,1}$), satisfy the nonlinear $\mathcal{L}_2$-gain property (7) with gain bounds $\gamma_{1,2} \in \mathcal{K}_\infty$. Using these systems, construct a feedback interconnection via

$$w_1 = z_2 + \eta_1 \,, \quad w_2 = z_1 + \eta_2 \,, \tag{29}$$

where $\eta_{1,2} \in \mathcal{L}_{2,\infty}(\mathbb{R}^{m_{1,2}})$ denote external inputs. Then, this feedback interconnection is $\mathcal{L}_2$-stable (8) if the small-gain condition

$$\begin{aligned} \mathrm{Id} - (\gamma_1 \circ \rho_1 \circ \hat{\rho}_1 \circ \pi_1 \circ \gamma_2 \circ \tilde{\rho}_1 \circ \hat{\pi}_1) \in \mathcal{K}_\infty \\ \mathrm{Id} - (\gamma_2 \circ \rho_2 \circ \hat{\rho}_2 \circ \pi_2 \circ \gamma_1 \circ \tilde{\rho}_2 \circ \hat{\pi}_2) \in \mathcal{K}_\infty \end{aligned} \tag{30}$$

is satisfied, where the functions $\rho_i, \hat{\rho}_i, \tilde{\rho}_i \in \mathcal{K}_\infty$, $i = 1, 2$, are such that $\rho_i - \mathrm{Id}, \hat{\rho}_i - \mathrm{Id}, \tilde{\rho}_i - \mathrm{Id} \in \mathcal{K}_\infty$, and $\pi_i, \hat{\pi}_i \in \mathcal{K}_\infty$ are of the form

$$\pi_i(s) \doteq (1 + c_i)\, s \,, \quad \hat{\pi}_i \doteq (1 + \hat{c}_i)\, s \,, \tag{31}$$

where $c_i, \hat{c}_i \in \mathbb{R}_{>0}$.

Remark 3.5: By choosing ρ_i, $\hat{\rho}_i$, $\tilde{\rho}_i$, π_i, $\hat{\pi}_i$ of the form of (31) with constant $c_i \ll 1$, the small gain condition (30) approaches that of the classical linear small-gain results, namely, $\mathrm{Id} - \gamma_1 \circ \gamma_2,\ \mathrm{Id} - \gamma_2 \circ \gamma_1 \in \mathcal{K}_\infty$. Where gains $\gamma_{1,2}$ are linear, both of these conditions correspond to $\gamma_1 \gamma_2 < 1$. However, with $c_i \ll 1$, it may be shown that large bounds on external inputs and transients result.

Theorem 3.4 provides a sufficient condition for closed-loop stability. For sufficiently large gain bounds $\gamma_{1,2}$, (30) cannot hold, and so neither stability nor instability can be concluded. Consequently, knowledge of tight gain bounds for which the nonlinear $\mathcal{L}_2$-gain property (7) (or (17)) holds is important from the point of view of application of this sufficient condition. As indicated in Section III-C, this motivates the development of methods for the computation of these tight gain bounds, see for example [3], [4], [21].

IV. Conclusion

The conventional notion of linear $\mathcal{L}_2$-gain can be generalized to incorporate a nonlinear gain by replacing the linear gain parameter with a nonlinear comparison function. The resulting nonlinear $\mathcal{L}_2$-gain property can be shown to be strictly weaker than the conventional linear $\mathcal{L}_2$-gain property, by appealing to qualitative equivalences between ISS and the linear $\mathcal{L}_2$-gain property, and between iISS and the nonlinear $\mathcal{L}_2$-gain property. Small-gain results for the more general property are readily established, following standard arguments.

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