The Capacity of Three-Receiver AWGN Broadcast Channels with Receiver Message Side Information

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***Abstract*—This paper investigates the capacity region of three-receiver AWGN broadcast channels where the receivers (i) have private-message requests and (ii) know the messages requested by some other receivers as side information. We classify these channels based on their side information into eight groups, and construct different transmission schemes for the groups. For six groups, we characterize the capacity region, and show that it improves both the best known inner and outer bounds. For the remaining two groups, we improve the best known inner bound by using side information during channel decoding at the receivers.**

I. Introduction

We study the capacity region of three-receiver additive white Gaussian noise broadcast channels (AWGN BCs) where the receivers have private-message requests and know some of the transmitted messages, aimed for other receivers, a priori.

A. Background

Broadcast channels [1] are considered as one of the main components of multi-sender multi-receiver wireless networks. The capacity region of broadcast channels is not known in general, except for a few special classes, e.g., degraded broadcast channels, which include AWGN BCs [2].

A variant of broadcast channels is where the receivers have some information about the source messages a priori (referred to as receiver message side information). This models several practical applications, e.g., sensor networks where the receivers know noisy versions of the source messages [3]. For some applications, e.g., multimedia broadcasting with packet loss or the downlink phase of multi-way relay channels, the receivers know some noise-free parts of the source messages.

The capacity region of broadcast channels with receiver message side information where each receiver must decode all the source messages (or equivalently, all the messages not known a priori) has been established by Tuncel [3] and Oechtering et al. [4].

However, the case where the receivers need not decode all the messages remains unsolved to date. Wu characterized the capacity region of *two-receiver* AWGN BCs with general message request and receiver message side information [5]. Extending the results to three or more receivers is "highly nontrivial" [5]. Oechtering et al. characterized the capacity region of some classes of *three-receiver* less-noisy and more-capable broadcast channels, where (i) only two receivers possess side information and (ii) the request of the third receiver is only restricted to a common message [6].

This work is supported by the Australian Research Council under grants FT110100195 and DE120100246.

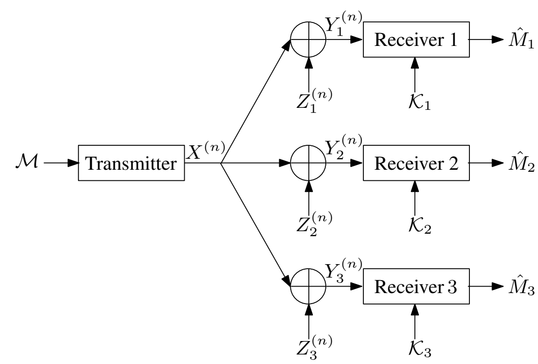

Fig. 1. The AWGN broadcast channel with receiver message side information, where $\mathcal{M} = \{M_1, M_2, M_3\}$ is the set of independent messages, each demanded by one receiver, and $\mathcal{K}_i \subseteq \mathcal{M} \setminus \{M_i\}$ is the set of messages known to receiver i a priori.

B. Existing Results and Contributions

In this paper, we consider *private-message* broadcasting over three-receiver AWGN BCs where the receivers know the messages requested by some other receivers as side information. The best known inner and outer bounds are within a constant gap of the capacity region [7]; the inner bound (achievability) uses a separate index and channel coding scheme, developed based on the deterministic approach [8].

One of the difficulties in deriving the capacity region is to find a unified scheme for all side information configurations. To make the problem more tractable, we first classify the channels into eight groups based on their side information, and construct different transmission schemes for different groups.

For six groups, we establish the capacity region. Our classification proves to be useful in grouping the channels with the same capacity-achieving transmission scheme. This result also shows the looseness of the best known inner and outer bounds [7]. For the remaining two groups, we improve the capacity inner bound by using side information during channel decoding at the receivers.

II. AWGN BC with Side Information

In the channel model under consideration, as depicted in Fig. 1, the signals received by receiver i, $Y_i^{(n)} = (Y_{i1}, Y_{i2}, \dots, Y_{in})$ $i = 1, 2, 3$, is the sum of the transmitted codeword, $X^{(n)}$, and an i.i.d. noise sequence, $Z_i^{(n)}$ $i = 1, 2, 3$, with normal distribution, $Z_i \sim \mathcal{N}(0, N_i)$. This channel is stochastically degraded, and without loss of generality, we can

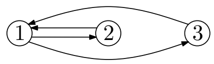

Fig. 2. Sample side information graph where receiver 1 knows M_2 and M_3, receiver 2 knows M_1, and receiver 3 knows M_1.

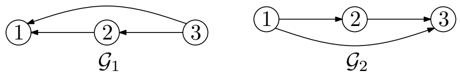

Fig. 3. Defined graphs in order to classify the problem.

assume that receiver 1 is the strongest, and receiver 3 is the weakest, in the sense that $N_1 \leq N_2 \leq N_3$.

The transmitted codeword has a power constraint of $\sum_{l=1}^{n} E\left(X_l^2\right) \leq nP$ and is a function of source messages $\mathcal{M} = \{M_1, M_2, M_3\}$. The messages $\{M_i\}_{i=1}^3$ are independent, and M_i is intended for receiver i at rate R_i bits per channel use i.e., $m_i \in \{1, 2, \ldots, 2^{nR_i}\}$. To model the side information of each receiver, we define the *knows* set $\mathcal{K}_i$ as the set of messages known to receiver i.

The side information configuration of each channel is modeled by a side information graph, $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{A}_\mathcal{G})$, where $\mathcal{V}_\mathcal{G} = \{1, 2, 3\}$ is the set of *vertices* and $\mathcal{A}_\mathcal{G}$ is the set of *arcs*. As we have only private messages, vertex i represents both M_i and receiver i requesting it. An arc from vertex i to vertex j, denoted by (i, j), exists if and only if receiver i knows M_j. The set of out-neighbors of vertex i is then $\mathcal{O}_i \triangleq \{j | (i, j) \in \mathcal{A}_\mathcal{G}\} = \{j | M_j \in \mathcal{K}_i\}$. A sample side information graph is shown in Fig. 2.

III. Problem Classification

We classify the channels of interest into eight groups based on their side information graphs. To this end, we define two graphs, $\mathcal{G}_1 = (\mathcal{V}_{\mathcal{G}_1}, \mathcal{A}_{\mathcal{G}_1})$ and $\mathcal{G}_2 = (\mathcal{V}_{\mathcal{G}_2}, \mathcal{A}_{\mathcal{G}_2})$ where $\mathcal{V}_{\mathcal{G}_1} = \mathcal{V}_{\mathcal{G}_2} = \mathcal{V}_\mathcal{G}$, as shown in Fig. 3. Any side information graph is the union of an arc subgraph* of $\mathcal{G}_1$ (denoted by $\mathcal{G}_{1j}$) and an arc subgraph of $\mathcal{G}_2$ (denoted by $\mathcal{G}_{2j}$). The arc subgraphs of $\mathcal{G}_1$ are considered as group leaders; Fig. 4 depicts all the group leaders. For instance, $\mathcal{G}_{13}$ in this figure is the leader of group 3. Group j is the set of side information graphs constructed by the union of $\mathcal{G}_{1j}$ with each of $\{\mathcal{G}_{2k}\}_{k=1}^8$. For instance, Fig. 5 depicts the elements of group 6.

IV. Transmission Schemes

In this section, we first establish the capacity region of six groups, stated as Theorem 1. We then enlarge the best existing inner bound for the two remaining groups using a joint decoding approach. Lastly, we demonstrate the looseness of the best existing outer bound.

A. Deriving the Capacity for Groups 1, 2, 3, 5, 6, and 8

Before presenting Theorem 1, we explain our proposed capacity-achieving transmission schemes. Table I shows these schemes for six groups. All the members of each group use the same scheme; there is one exception in group 5 and one in group 8 that use slightly modified schemes.

If the codebook of the transmission scheme is composed of multiple subcodebooks, the transmitted codeword, $x^{(n)}$, is constructed from the linear superposition of multiple codewords, $\sum_k x_k^{(n)}$. Each subcodebook consists of i.i.d. codewords, $x_k^{(n)}$, generated according to an independent normal distribution $X_k \sim \mathcal{N}(0, \alpha_k P)$, where $\alpha_k \geq 0$ and $\sum_k \alpha_k = 1$ to satisfy the transmission power constraint. Multiplexing coding [9], index coding [10] and dirty paper coding [11] are employed to construct the subcodebooks.

In multiplexing coding, two or more messages are bijectively mapped to a single message, and then, the codewords are generated for this message. For instance, the first subcodebook of group 3 is constructed using multiplexing coding. In this scheme, the single message $M_\text{m} = [M_1, M_2]$, where $[\cdot]$ denotes a bijective map, is first formed from M_1 and M_2. Then, the codewords of the first subcodebook are generated for this single message, M_m, where $m_\text{m} \in \{1, 2, \ldots, 2^{n(R_1+R_2)}\}$.

In index coding (which is also called network coding [12] in some of the works on broadcast channels), the transmitter XORs the messages to accomplish compression prior to channel coding. The same function can also be achieved using modulo addition [13]. The transmission schemes of the exceptions in groups 5 and 8 use index coding. In these schemes, $M_2 \oplus M_3$ is first formed, where $\oplus$ denotes the bitwise XOR with zero padding for messages of unequal length i.e. $m_2 \oplus m_3 \in \{1, 2, \ldots, 2^{n \max\{R_2, R_3\}}\}$. Then, the messages M_1 and $M_2 \oplus M_3$ are fed to the channel encoder (who performs multiplexing coding and superposition coding).

Dirty paper coding [11] is employed to construct the transmission scheme of group 2. In this scheme, first, $[M_2, M_3]$ is encoded using $2^{n(R_2+R_3)}$ i.i.d. codewords, $x_2^{(n)}([m_2, m_3])$, generated according to $X_2 \sim \mathcal{N}(0, \alpha_2 P)$. Then, considering

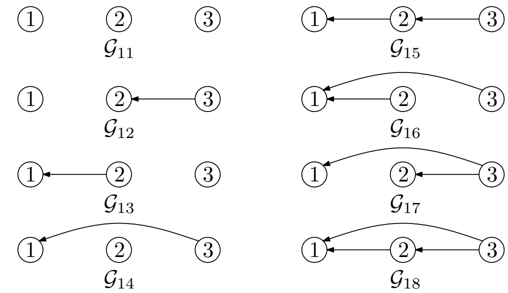

Fig. 4. Group leaders, capturing if each receiver knows the message(s) requested by stronger receiver(s).

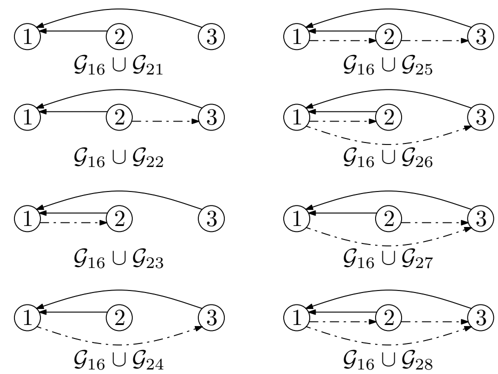

Fig. 5. The elements of group 6, where the arcs of $\mathcal{G}_{16}$ (the group leader) are drawn with solid lines, and those of $\mathcal{G}_{2j}$ dotted lines. As it can be seen, the group leader is actually the first element of each group.

*$\mathcal{G}' = (\mathcal{V}_{\mathcal{G}'}, \mathcal{A}_{\mathcal{G}'})$ is an arc subgraph of $\mathcal{G}'' = (\mathcal{V}_{\mathcal{G}''}, \mathcal{A}_{\mathcal{G}''})$ if $\mathcal{A}_{\mathcal{G}'} \subseteq \mathcal{A}_{\mathcal{G}''}$ and $\mathcal{V}_{\mathcal{G}'} = \mathcal{V}_{\mathcal{G}''}$. The union of $\mathcal{G}'$ and $\mathcal{G}''$ is equal to $\mathcal{G}' \cup \mathcal{G}'' = (\mathcal{V}_{\mathcal{G}'} \cup \mathcal{V}_{\mathcal{G}''}, \mathcal{A}_{\mathcal{G}'} \cup \mathcal{A}_{\mathcal{G}''})$.

TABLE I

THE CAPACITY AND OUR PROPOSED CAPACITY-ACHIEVING TRANSMISSION SCHEME FOR DIFFERENT GROUPS

Group	Transmitted Codeword	Capacity Region
Group 1	$x_1^{(n)}(m_1) + x_2^{(n)}(m_2) + x_3^{(n)}(m_3)$	$R_1 < C\left(\frac{\alpha_1 P}{N_1}\right)$, $R_2 < C\left(\frac{\alpha_2 P}{\alpha_1 P + N_2}\right)$, $R_3 < C\left(\frac{\alpha_3 P}{(\alpha_1+\alpha_2)P + N_3}\right)$
Group 2	$x_1^{(n)}\left(x_2^{(n)}([m_2, m_3]), m_1\right) + x_2^{(n)}([m_2, m_3])$	$R_1 < C\left(\frac{\alpha_1 P}{N_1}\right)$, $\sum_{i\in\{2,3\}\setminus\mathcal{O}_2} R_i < C\left(\frac{\alpha_2 P}{\alpha_1 P + N_2}\right)$, $R_3 < C\left(\frac{\alpha_2 P}{\alpha_1 P + N_3}\right)$
Group 3	$x_1^{(n)}([m_1, m_2]) + x_2^{(n)}(m_3)$	$\sum_{i\in\{1,2\}\setminus\mathcal{O}_1} R_i < C\left(\frac{\alpha_1 P}{N_1}\right)$, $R_2 < C\left(\frac{\alpha_1 P}{N_2}\right)$, $R_3 < C\left(\frac{\alpha_2 P}{\alpha_1 P + N_3}\right)$
Group 5	$x_1^{(n)}([m_1, m_2]) + x_2^{(n)}([m_2, m_3])$	$\sum_{i\notin\mathcal{O}_1} R_i < C\left(\frac{P}{N_1}\right)$, $R_1 < C\left(\frac{\alpha_1 P}{N_1}\right)$, $\sum_{i\notin\mathcal{O}_2} R_i < C\left(\frac{P}{N_2}\right)$, $R_3 < C\left(\frac{\alpha_2 P}{\alpha_1 P + N_3}\right)$
	$\mathcal{G}_{15} \cup \mathcal{G}_{22}$:	$\mathcal{G}_{15} \cup \mathcal{G}_{22}$:
	$x_1^{(n)}([m_1, m_2 \oplus m_3]) + x_2^{(n)}(m_2 \oplus m_3)$	$R_1 + \max\{R_2, R_3\} < C\left(\frac{P}{N_1}\right)$, $R_1 < C\left(\frac{\alpha_1 P}{N_1}\right)$, $R_2 < C\left(\frac{P}{N_2}\right)$, $R_3 < C\left(\frac{\alpha_2 P}{\alpha_1 P + N_3}\right)$
Group 6	$x_1^{(n)}([m_1, m_2]) + x_2^{(n)}([m_1, m_3])$	$\sum_{i\notin\mathcal{O}_1} R_i < C\left(\frac{P}{N_1}\right)$, $R_2 < C\left(\frac{\alpha_1 P}{N_2}\right)$, $R_3 < C\left(\frac{\alpha_2 P}{\alpha_1 P + N_3}\right)$
Group 8	$x^{(n)}([m_1, m_2, m_3])$	$\sum_{i\notin\mathcal{O}_1} R_i < C\left(\frac{P}{N_1}\right)$, $\sum_{i\notin\mathcal{O}_2} R_i < C\left(\frac{P}{N_2}\right)$, $R_3 < C\left(\frac{P}{N_3}\right)$
	$\mathcal{G}_{18} \cup \mathcal{G}_{22}$: $x^{(n)}([m_1, m_2 \oplus m_3])$	$\mathcal{G}_{18} \cup \mathcal{G}_{22}$: $R_1 + \max\{R_2, R_3\} < C\left(\frac{P}{N_1}\right)$, $R_2 < C\left(\frac{P}{N_2}\right)$, $R_3 < C\left(\frac{P}{N_3}\right)$

$x_2^{(n)}$ as interference for receiver 1, which is known non-causally at the transmitter, M_1 is encoded using dirty paper coding. The auxiliary random variable in dirty paper coding is defined as $U = X_1 + \beta X_2$ where $X_1 \sim \mathcal{N}(0, \alpha_1 P)$ is independent of X_2, and $\beta = \alpha_1 P / (\alpha_1 P + N_1)$.

We now state the results for the six groups in Table I.

Theorem 1: The capacity region and the optimal scheme for three-receiver AWGN BCs with private messages and side information graphs not in groups 4 and 7 are shown in Table I. The capacity region for each channel is the closure of the set of all rate triplets (R_1, R_2, R_3), each satisfying the conditions in the respective row for some $\alpha_k \geq 0$ such that $\sum_k \alpha_k = 1$.

Proof: Due to space limitations, the complete proof is omitted and is available in the extended version of this paper [14]. The proof for group 6 is given in the appendix to illustrate the employed decoding and bounding techniques. ■

B. Improving the Existing Inner Bound

We are unable to establish the capacity region for groups 4 and 7. However, in this subsection, we improve the best prior known inner bound for these groups. The best prior known inner bound, which is achieved by a separate index and channel coding scheme (developed based on the deterministic approach), is the set of all rate triples (R_1, R_2, R_3), each satisfying [7]

$$\sum_{i\in\mathcal{V}_\mathcal{S}} R_i < \max_{i\in\mathcal{V}_\mathcal{S}} A_i, \tag{1}$$

for all induced acyclic subgraphs, $\mathcal{S}$, of the side information graph. In (1), $A_i = \sum_{k=i}^{3} B_k$ where $B_1 = C(\alpha_1 P/N_1)$, $B_2 = C(\alpha_2 P/(\alpha_1 P + N_2))$, and $B_3 = C(\alpha_3 P/((\alpha_1 + \alpha_2)P + N_3))$ for some $\alpha_k \geq 0$ $k = 1, 2, 3$ such that $\sum_{k=1}^{3} \alpha_k = 1$. Here, $C(q) \triangleq \frac{1}{2}\log(1+q)$.

For instance, the achievable rate region for $\mathcal{G}_{17} \cup \mathcal{G}_{24}$, a member of group 7, is the set of all rate triples (R_1, R_2, R_3), each satisfying

$$\begin{aligned} R_1 + R_2 &< B_1 + B_2 + B_3, \\ R_2 + R_3 &< B_2 + B_3, \\ R_3 &< B_3, \end{aligned} \tag{2}$$

for some $\alpha_k \geq 0$ $k = 1, 2, 3$ such that $\sum_{k=1}^{3} \alpha_k = 1$. The region in (2) can be achieved using the encoding scheme (which utilizes rate splitting, index coding, multiplexing coding and superposition coding)

$$x_1^{(n)}(m_{10}) + x_2^{(n)}([m_{11}, m_{20}]) + x_3^{(n)}([m_{21}, m_{12} \oplus m_3]),$$

and a separate index and channel decoding scheme (where side information is not utilized during channel decoding). Using rate splitting, the message M_1 is divided into independent messages M_{10} at rate R_{10}, M_{11} at rate R_{11}, and M_{12} at rate R_{12} such that $R_1 = \sum_{k=0}^{2} R_{1k}$; the message M_2 is also divided into independent messages M_{20} at rate R_{20}, and M_{21} at rate R_{21} such that $R_2 = R_{20} + R_{21}$. We can verify the achievability of the region in (2) using Fourier-Motzkin elimination subsequent to successive decoding.

We now show that using the same encoding scheme, but utilizing the side information during successive decoding (i.e., joint decoding), the achievable rate region can be improved. For the given example ($\mathcal{G}_{17} \cup \mathcal{G}_{24}$), consider the decoding of $x_3^{(n)}$ by the receivers while treating $x_1^{(n)} + x_2^{(n)}$ as noise. Using separate decoding, we get the condition $R_{21} + \max\{R_{12}, R_3\} < B_3$ on achievability. Using joint decoding, we can relax this condition to $R_3 < C(\alpha_3 P/N_3)$ and $R_{21} + \max\{R_{12}, R_3\} < B'_3$ where $B'_3 = C(\alpha_3 P/((\alpha_1 + \alpha_2)P + N_2)) \geq B_3$ for any choice of $\{\alpha_k\}_{k=1}^{3}$. This gives an improved achievable rate region for $\mathcal{G}_{17} \cup \mathcal{G}_{24}$ as the set of all rate triples (R_1, R_2, R_3), each satisfying

$$\begin{aligned} R_1 + R_2 &< B_1 + B_2 + B'_3, \\ R_2 + R_3 &< B_2 + B'_3, \\ R_3 &< \min\{C(\alpha_3 P/N_3), B'_3\}, \end{aligned} \tag{3}$$

for some $\alpha_k \geq 0$ $k = 1, 2, 3$ such that $\sum_{k=1}^{3} \alpha_k = 1$.

This joint decoding approach can be used for all the channels in groups 2 to 8[†] in order to enlarge the rate region in (1); however, the expression for the enlarged region depends on the particular side information configuration. Therefore, the capacity region of the six groups in Secion IV-A except group 1 must also be strictly larger than the best prior known inner

[†]For group 1, the capacity region is the same as the three-receiver AWGN BC without receiver message side information.

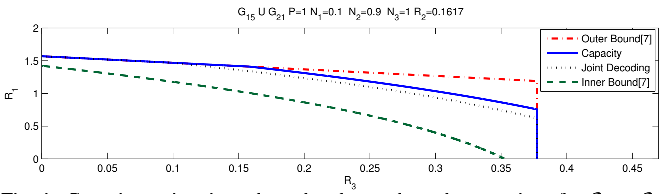

Fig. 6. Capacity region, inner bound and outer bound comparison for $\mathcal{G}_{15} \cup \mathcal{G}_{21}$

bound in (1) when $N_1 < N_2 < N_3$. For example, Fig. 6 depicts the looseness of the best prior known inner bound for $\mathcal{G}_{15} \cup \mathcal{G}_{21}$. This figure also shows that the encoding scheme which is developed based on the deterministic approach cannot achieve the capacity region by using the proposed joint decoding approach.

C. Demonstrating the Looseness of the Existing Outer Bound

In this subsection, we demonstrate the looseness of the best known outer bound prior to this work [7] compared to the capacity region of the six groups in Secion IV-A. The best known outer bound states that if the rate triple (R_1, R_2, R_3) is achievable, it must satisfy [7], $\sum_{i \in \mathcal{V}_\mathcal{S}} R_i \leq \max_{i \in \mathcal{V}_\mathcal{S}} C(\frac{P}{N_i})$, for all induced acyclic subgraphs, $\mathcal{S}$, of the side information graph. This outer bound is a polyhedron, and is loose for the six groups with known capacity except group 8, since the capacity-achieving transmission schemes of these groups are functions of α_k, and therefore the capacity region has some curved surfaces. As an example, for $\mathcal{G}_{15} \cup \mathcal{G}_{21}$ the outer bound is characterized by the inequalities $R_1 + R_2 + R_3 \leq C\left(P/N_1\right)$, $R_2 + R_3 \leq C\left(P/N_2\right)$ and $R_3 \leq C\left(P/N_3\right)$, and Fig. 6 depicts the looseness of it.

V. Remarks on the Transmission Schemes

In this section, we make some observations about our capacity-achieving schemes which may be useful for extending the results to AWGN BCs with more than three receivers.

Considering private-message broadcasting, multiplexing the requested message of a receiver, M_i, with a set of messages, $\{M_j\}$, which are known to this receiver, is performed when (i) there is at least one message in $M_i \cup \{M_j\}$ that is not known a priori to any weaker receiver, (ii) each M_j is requested by a stronger receiver, and (iii) for each M_j, all in-between receivers (the receivers that are stronger than the receiver requesting M_i, and weaker than the receiver requesting M_j) also know M_j as side information.

An exception occurs when two receivers know each other's requested messages as side information and a stronger receiver knows neither message (e.g., the exceptions in groups 5 and 8, and $\mathcal{G}_{12} \cup \mathcal{G}_{22}$). Then, index coding is employed prior to multiplexing coding and superposition coding. Asadi et al. [15] have recently shown why this can lead to a larger achievable region. However, dirty paper coding provides a unified scheme for group 2 to avoid having the exception, $\mathcal{G}_{12} \cup \mathcal{G}_{22}$.

VI. Conclusion

In this work, we have classified three-receiver AWGN BCs where the receivers have private-message requests and know the messages demanded by some other receivers as side information. The classification generates eight groups. For six groups, we have established the capacity region by proposing their capacity-achieving transmission schemes. This result (i) demonstrates the effectiveness of the classification method in building the groups with the same capacity-achieving transmission scheme and (ii) shows the looseness of the best known inner and outer bounds prior to this work. For the remaining two groups, we have improved the achievable rate region prior to this work by joint decoding, which utilizes side information during channel decoding.

Appendix

In this section, we prove, for group 6, that the proposed transmission scheme achieves the capacity region. The proof is based on those for AWGN BCs without side information [2], [16]. In the converse, we use Fano's inequality and the entropy power inequality (EPI). Based on Fano's inequality,

$$H(M_i \mid Y_i^{(n)}, \mathcal{K}_i) \leq n\epsilon_{n,i}, \quad i = 1, 2, 3, \tag{4}$$

where $\epsilon_{n,i} \to 0$ as $n \to \infty$. For the sake of simplicity we use ϵ_n instead of $\epsilon_{n,i}$ for the remainder. In the converse, we also use the fact that the capacity region of a stochastically degraded broadcast channel without feedback is the same as its equivalent physically degraded broadcast channel [16, p. 444] in which the channel input and outputs form a Markov chain, $X \to Y_1 \to Y_2 \to Y_3$, i.e., $Y_1 = X + Z_1$ and $Y_i = Y_{i-1} + \tilde{Z}_i \;\; i = 2, 3$ where $\tilde{Z}_i \sim \mathcal{N}\left(0, N_i - N_{i-1}\right) \;\; i = 2, 3$.

We first prove a lemma that will be used in the converse.

Lemma 1: If $\mathcal{L} \subseteq \{M_1, M_2, M_3\}$, then

$$H(M_l \mid Y_i^{(n)}, \mathcal{L}) \leq H(M_l \mid Y_j^{(n)}, \mathcal{L}),$$
$$\forall l, i, j \in \{1, 2, 3\} \text{ such that } i < j.$$

Proof: The proof is similar to the proof for the data processing inequality [16, p. 25]. We just need to expand $I\left(M_l; Y_i^{(n)}, Y_j^{(n)} | \mathcal{L}\right)$ in two ways by the mutual information chain rule and use the Markov chain, resulted from the physically degradedness. ■

We now present the achievability and the converse proofs for group 6.

Proof: The achievability for $\mathcal{G}_{16} \cup \mathcal{G}_{21}$ is proved using successive decoding at receivers 2 and 3 and simultaneous decoding at receiver 1. Since receivers 2 and 3 know M_1, the second and the third inequalities, given in Table I for this group, are required for achievability. Receiver 1, using simultaneous decoding, decodes $\hat{m}_1$ if there exists a unique $\hat{m}_1$ such that $(X_1^{(n)}([\hat{m}_1, m_2]), X_2^{(n)}([\hat{m}_1, m_3]), Y_1^{(n)}) \in \mathcal{T}_\delta^{(n)}$ for some m_2, m_3, where $\mathcal{T}_\delta^{(n)}$ is the set of jointly δ-typical n-sequences [17, p. 521]; otherwise the error is declared. Assuming the transmitted messages are equal to one by the symmetry of the code generation, the error events at receiver 1 for $\mathcal{G}_{16} \cup \mathcal{G}_{21}$ are

$$\mathcal{E}_{11} : \left(X_1^{(n)}([1, m_2]), X_2^{(n)}([1, m_3]), Y_1^{(n)}\right) \notin \mathcal{T}_\delta^{(n)} \quad \text{for all } m_2, m_3,$$

$$\mathcal{E}_{12} : \left(X_1^{(n)}([m_1, m_2]), X_2^{(n)}([m_1, m_3]), Y_1^{(n)}\right) \in \mathcal{T}_\delta^{(n)} \quad \text{for some } m_1 \neq 1, m_2, m_3.$$

From the properties of joint typicality [17, Theorems 15.2.1 and 15.2.3], it can be seen for $\mathcal{G}_{16} \cup \mathcal{G}_{21}$, the first inequality, $R_1 + R_2 + R_3 < C(P/N_1)$, guarantees that the probability of error at receiver 1 tends to zero as n increases.

For all other elements in group 6, we use the same encoding and decoding schemes, but each receiver makes its decoding decision based on its extra side information.

Here, we prove the converse for $\mathcal{G}_{16} \cup \mathcal{G}_{21}$. The rate R_3 in this channel is upper bounded as

$$\begin{aligned} nR_3 &= H(M_3) = H(M_3 \mid Y_3^{(n)}, M_1) + I(M_3; Y_3^{(n)}, M_1) \\ &\overset{(a)}{=} H(M_3 \mid Y_3^{(n)}, M_1) + I(M_3; Y_3^{(n)} \mid M_1) \\ &= H(M_3 \mid Y_3^{(n)}, M_1) + h(Y_3^{(n)} \mid M_1) - h(Y_3^{(n)} \mid M_1, M_3) \\ &\overset{(b)}{\leq} n\epsilon_n + h(Y_3^{(n)} \mid M_1) - h(Y_3^{(n)} \mid M_1, M_3) \\ &\overset{(c)}{\leq} n\epsilon_n + \frac{n}{2}\log 2\pi e(P + N_3) - h(Y_3^{(n)} \mid M_1, M_3) \\ &\overset{(d)}{=} n\epsilon_n + \frac{n}{2}\log 2\pi e(P + N_3) - \frac{n}{2}\log 2\pi e(\alpha P + N_3), \end{aligned} \quad (5)$$

where (a) follows from the independence of M_1 and M_3, (b) from (4), and (c) from $h(Y_3^{(n)} \mid M_1) \leq h(Y_3^{(n)}) \leq \frac{n}{2}\log 2\pi e(P + N_3)$. In (5), (d) is due to

$$\begin{aligned} &\frac{n}{2}\log 2\pi e N_3 = h(Z_3^{(n)}) = h(Y_3^{(n)} \mid X^{(n)}) \\ &\overset{(e)}{\leq} h(Y_3^{(n)} \mid M_1, M_3) \leq h(Y_3^{(n)}) \leq \frac{n}{2}\log 2\pi e(P + N_3), \end{aligned}$$

where (e) is because $(M_1, M_3) \to X^{(n)} \to Y_3^{(n)}$ form a Markov chain; then since $\frac{n}{2}\log 2\pi e N_3 \leq h(Y_3^{(n)} \mid M_1, M_3) \leq \frac{n}{2}\log 2\pi e(P + N_3)$ there must exist an $0 \leq \alpha \leq 1$ such that $h(Y_3^{(n)} \mid M_1, M_3) = \frac{n}{2}\log 2\pi e(\alpha P + N_3)$.

In this channel, R_2 is also upper bounded as

$$\begin{aligned} nR_2 &= H(M_2) = H(M_2 | Y_2^{(n)}, M_1, M_3) + I(M_2; Y_2^{(n)} | M_1, M_3) \\ &= H(M_2 \mid Y_2^{(n)}, M_1, M_3) \\ &\qquad + h(Y_2^{(n)} \mid M_1, M_3) - h(Y_2^{(n)} \mid M_1, M_2, M_3) \\ &\overset{(a)}{\leq} n\epsilon_n + h(Y_2^{(n)} \mid M_1, M_3) - h(Y_2^{(n)} \mid M_1, M_2, M_3) \\ &\overset{(b)}{\leq} n\epsilon_n + \frac{n}{2}\log 2\pi e(\alpha P + N_2) - h(Y_2^{(n)} \mid M_1, M_2, M_3) \\ &\overset{(c)}{=} n\epsilon_n + \frac{n}{2}\log 2\pi e(\alpha P + N_2) - \frac{n}{2}\log 2\pi e N_2, \end{aligned} \quad (6)$$

where (a) follows from (4) and $H(M_2 \mid Y_2^{(n)}, M_1, M_3) \leq H(M_2 \mid Y_2^{(n)}, M_1)$, and (b) from using the conditional EPI [16, p. 22] for $Y_3^{(n)} = Y_2^{(n)} + \tilde{Z}_3^{(n)}$, and substituting $h(\tilde{Z}_3^{(n)} \mid M_1, M_3) = \frac{n}{2}\log 2\pi e(N_3 - N_2)$ and $h(Y_3^{(n)} \mid M_1, M_3) = \frac{n}{2}\log 2\pi e(\alpha P + N_3)$. In (6), (c) is due to

$$h(Y_2^{(n)} | M_1, M_2, M_3) = h(Y_2^{(n)} | X^{(n)}) = h(Z_2^{(n)}) = \frac{n}{2}\log 2\pi e N_2.$$

We also have

$$\begin{aligned} n(R_1 + R_2 + R_3) &= H(M_1, M_2, M_3) \\ &= H(M_1, M_2, M_3 \mid Y_1^{(n)}) + I(M_1, M_2, M_3; Y_1^{(n)}) \\ &= H(M_1, M_2, M_3 \mid Y_1^{(n)}) + h(Y_1^{(n)}) - h(Y_1^{(n)} \mid M_1, M_2, M_3) \\ &\overset{(a)}{\leq} 3n\epsilon_n + h(Y_1^{(n)}) - h(Y_1^{(n)} \mid M_1, M_2, M_3) \\ &\overset{(b)}{\leq} 3n\epsilon_n + \frac{n}{2}\log 2\pi e(P + N_1) - \frac{n}{2}\log 2\pi e N_1, \end{aligned} \quad (7)$$

where (a) follows from adding the following inequalities which are the results of using Lemma 1 and (4) as

$$\begin{aligned} &H(M_1 \mid Y_1^{(n)}) \leq n\epsilon_n, \\ &H(M_2 \mid Y_1^{(n)}, M_1) \leq H(M_2 \mid Y_2^{(n)}, M_1) \leq n\epsilon_n, \\ &H(M_3 \mid Y_1^{(n)}, M_1, M_2) \leq H(M_3 \mid Y_3^{(n)}, M_1) \leq n\epsilon_n. \end{aligned}$$

In (7), (b) follows from $h(Y_1^{(n)}) \leq \frac{n}{2}\log 2\pi e(P + N_1)$ and

$$h(Y_1^{(n)} | M_1, M_2, M_3) = h(Y_1^{(n)} | X^{(n)}) = h(Z_1^{(n)}) = \frac{n}{2}\log 2\pi e N_1.$$

From (5)–(7) and since $\epsilon_n \to 0$ as $n \to \infty$, the converse for $\mathcal{G}_{16} \cup \mathcal{G}_{21}$ is proven. The converse for the other channels in this group is straightforward; we only need to modify (7), if receiver 1 knows M_2 or M_3. ■

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