QUANTIFYING THE ACCURACY OF HAMMERSTEIN MODEL ESTIMATION

Brett Ninness [*,1] **Stuart Gibson** [*]

[*] *Dept. of Elec. & Comp. Eng, Uni. Newcastle, Australia.*
*email:*brett@ee.newcastle.edu.au, *FAX: +61 49 21 69 93*

Abstract: This paper investigates the accuracy of the linear model estimate that forms a part of an overall Hammerstein model structure. A key finding here is that the process of estimating a non-linear element can have a dramatic effect on the associated estimate of the linear component. This effect is not explained simply by way of considering how the input spectrum is changed by the non-linearity. Instead, it arises that the estimate variability may be dominated by a term that depends on the frequency response of the linear system itself. This has implications for experiment design for Hammerstein system estimation.

Keywords: Hammerstein Models, System Identification, Parameter Estimation, Non-linear systems.

1. INTRODUCTION

Over the last decade there has been an intense activity devoted to quantifying the estimation error that arises due to undermodelling; see, for example, (Koust *et al.* 1992, Milanese and Vicino 1991, Ninness and Goodwin 1995, Wahlberg and Ljung 1992, Smith and (editors) 1994) and the references therein.

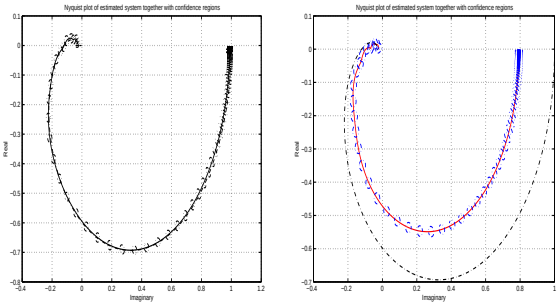
However, as recently exposed by Guo and Ljung (Ljung and Guo 1997), when the undermodelling is due only to linear time-invariant (LTI) model mismatch, then this indicates that model validation has not been adequately performed and, vice versa, in the presence of proper model validation, if undermodelling exists, then the errors involved are small enough to be accommodated by merely doubling the variance error quantification.

In recognition of this dispatch of the error quantification issue in the linear system case, the attention here is on non-linear systems of the Hammerstein class (Tao and Kokotović 1996). In this instance, if the non-linearity is ignored and only a linear model is fitted to the data, then the estimation bias is likely to be so large that any error bounds based on variance error quantifications will be strongly misleading.

For example, consider the case of fitting an $n = 15^{th}$ order finite impulse response (FIR) model to input-output measurements of the system

$$G(q) = \frac{0.1548q + 0.0939}{(q - 0.6065)(q - 0.3679)} \qquad (1)$$

when the input $\{u_t\}$ is unit variance Gaussian distributed white noise and the output $\{y_t\}$ is corrupted by Gaussian white noise, with variance $\sigma^2 = 0.001$. Then the true process and estimated systems together with confidence bounds for a typical realisation are shown in Figure 1(a). Clearly, these confidence regions are an informative indication of overall model accuracy.



(a) FIR model fitted to linear system

(b) FIR model fitted to Hammerstein system

Fig. 1. *Nyquist Plots of the true system (dot-dashed line) and the estimated FIR model (solid line). The ellipses are the 95% confidence bounds associated with the FIR model.*

However, now suppose that the input is saturated symmetrically when its magnitude exceeds $\alpha = 1.0$ so that the true system is of Hammerstein type. In this case, if the same 15^{th} order linear FIR model is fitted to this data then, the estimation results and confidence bounds are shown in Figure 1(b). As can be seen, there is a dramatic difference between Figures 1(a) and 1(b) in the reliability of the confidence region error quantifications - essentially the error bound validity has been destroyed in Figure 1(b) by the estimation bias induced by the model structure being linear and the true system being non-linear.

In reaction to these sorts of difficulties, many investigations have been pursued that seek to derive alternative

[1] This work was supported by the Australian Research Council and the Centre for Integrated Dynamics and Control (CIDAC). Part of this work was completed while visiting S3-Automatic Control, The Royal Institute of Technology, Stockholm, Sweden.

error quantification methods that account for both the bias and variance error (Smith and (editors) 1994, *Special Issue on System Identification for Robust Cont Design* 1992, Ninness and Goodwin 1995).

The focus of this paper is to examine a more direct strategy of actually estimating the non-linearity in a prediction error framework and then, since in this instance the true system is in the model class, again using confidence region error bounds as model uncertainty quantification (as in 1(a)).

A key concern in pursuing this strategy involves the following question. What price is paid, in terms of the quality of the linear system estimate, by including a parameterisation of a non-linear component in the model structure? Certainly any bias error should be eliminated via the model class being enriched to encompass the true system, but what of the variance error component which is included in the total error?

More specifically, for the case of an FIR model estimate $\widehat{G}(q)$, a widely known and used approximate expression for its variability is (L.Ljung and Z.D.Yuan 1985)

$$\mathrm{Var}\left\{\widehat{G}(e^{j\omega})\right\} \approx \frac{n}{N}\frac{\sigma^2}{\Phi_u(\omega)} \tag{2}$$

where N is the length of the data records $\{y_t\}$ and $\{u_t\}$, $\Phi_u(\omega)$ is the spectral density of $\{u_t\}$, σ^2 is the variance of any white measurement noise, and n is the order of the FIR model. In the Hammerstein case the actual system input is a signal $x_t = X(u_t, \alpha)$ (where $X(\cdot, \alpha)$ is a memoryless and possibly non-linear function of $\{u_t\}$ that is parameterised by a vector α) so it would be natural to suppose that the effect of $X(\cdot, \alpha)$ would be to modify the variance error for the linear system estimate as

$$\mathrm{Var}\left\{\widehat{G}(e^{j\omega})\right\} \approx \frac{n}{N}\frac{\sigma^2}{\Phi_x(\omega)},$$

where (of course) $\Phi_x(\omega)$ is the spectral density of $\{x_t\}$.

The main result of this paper is to show, both empirically and analytically, that in fact

$$\mathrm{Var}\left\{\widehat{G}(e^{j\omega})\right\} \approx \frac{1}{N}\left[\frac{\sigma^2 n}{\Phi_x} + \left\|\frac{\mathrm{d}}{\mathrm{d}\alpha}\log\Phi_x\right\|_\alpha^2 |G(\beta_\circ)|^2\right] \tag{3}$$

is a more accurate estimate of the variability of the linear FIR system estimate. Here

$$\left\|\frac{\mathrm{d}}{\mathrm{d}\alpha}\log\Phi_x\right\|_\alpha^2 \triangleq \frac{1}{4}\left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\log\Phi_x\right)^T P_\alpha \left(\frac{\mathrm{d}}{\mathrm{d}\alpha}\log\Phi_x\right) \tag{4}$$

(P_α is the covariance matrix of the estimate of α) is a quantity that measures the sensitivity of the spectrum of $X(u_t, \alpha)$ to changes in α.

The key point is that (3) illustrates the fact that not only does identifying a non-linearity entail an unavoidable cost on the accuracy of the linear system estimate, but this cost actually depends on the dynamics of the linear part. Compare this latter point with the expression (2) where the variability of $\widehat{G}$ is invariant to G itself.

2. PROBLEM FORMULATION

The non-linear systems studied in this paper are of the Hammerstein class (Narendra and Gallman 1966, Rangan *et al.* 1995, Tao and Kokotović 1996), which may be represented by a model structure of the form

$$y_t = G(q, \beta)x_t(\alpha) + e_t, \tag{5}$$

$$x_t(\alpha) = X(u_t, \alpha), \quad x_t'(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\alpha}X(u_t, \alpha) \tag{6}$$

where $X(\cdot, \alpha)$ is a memoryless non-linear function parameterised by the vector α and operating on the input $\{u_t\}$, $\{e_t\}$ is an i.i.d., zero mean white noise process that represents corruptions in the output measurement $\{y_t\}$, and $G(q, \beta)$ is an LTI strictly stable system parameterised in the forward shift operator q and by a vector β. The identification problem then becomes one of estimating the composite parameter vector ($n \triangleq \dim\{\beta\}$, $\ell \triangleq \dim\{\alpha\}$, $m = n + \ell \triangleq \dim\{\theta\}$)

$$\theta \triangleq [\beta^T, \alpha^T]^T \tag{7}$$

on the basis of observations of $\{y_t\}$ and $\{u_t\}$.

For the solution of this estimation problem, a prediction-error approach (Ljung 1987) may be taken, in which the mean square optimal one step ahead predictor $\widehat{y}_t(\theta)$ is calculated as

$$\widehat{y}_t(\theta) = G(q, \beta)x_t(\alpha), \tag{8}$$

with the associated prediction error

$$\varepsilon_t(\theta) = y_t - \widehat{y}_t(\theta) = y_t - G(q, \beta)x_t(\alpha).$$

In turn, this can be employed to define a "least-squares" estimation criterion of

$$V_N(\theta) = \frac{1}{2N}\sum_{t=1}^{N}\varepsilon_t^2(\theta),$$

with an associated prediction error estimate (the N subscript indicating dependence on N data samples)

$$\widehat{\theta}_N \triangleq \arg\min_{\theta\in\mathbf{R}^m} V_N(\theta). \tag{9}$$

Under the assumptions of $\mathbf{E}\{e_t\} = 0$, $\mathbf{E}\{{e_t}^2\} = \sigma^2 < \infty$ and $\mathbf{E}\{|e_t|^{4+\delta}\} < \infty$ for some $\delta > 0$, then the following strong consistency result is available (Ljung 1987, L.Ljung 1978)

$$\lim_{N\to\infty}\widehat{\theta}_N = \theta_\circ \text{ w.p.1}, \tag{10}$$

where

$$\theta_\circ \triangleq \arg\min_{\theta\in\mathbf{R}^m}\lim_{N\to\infty}\mathbf{E}\{V_N(\theta)\}. \tag{11}$$

Furthermore, the variability of the estimate $\widehat{\theta}_N$ (around the value $\theta_\circ$ to which it is converging) may be judged via the asymptotic distributional result (Ljung 1987, L.Ljung

and P.E.Caines 1979) (subscripts on matrices indicate their dimension)

$$\sqrt{N}(\widehat{\theta}_N - \theta_\circ) \xrightarrow{\mathcal{D}} \mathcal{N}(0, P_m). \tag{12}$$

The covariance matrix P_m in this expression is defined in terms of two other matrices R_m and Q_m as

$$P_m \triangleq R_m^{-1} Q_m R_m^{-1}$$

which themselves are defined as

$$\begin{aligned} R_m &\triangleq \lim_{N\to\infty} \frac{\mathrm{d}^2}{\mathrm{d}\theta\theta^T} \mathbf{E}\{V_N(\theta)\} \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N} \mathbf{E}\left\{\psi_t(\theta_\circ)\psi_t^T(\theta_\circ)\right\}, \qquad (13) \\ Q_m &\triangleq \lim_{N\to\infty} \mathbf{E}\left\{\frac{\mathrm{d}}{\mathrm{d}\theta}V_N(\theta)\left(\frac{\mathrm{d}}{\mathrm{d}\theta}V_N(\theta)\right)^T\right\} \\ &= \lim_{N\to\infty} \frac{1}{N}\sum_{t=1}^{N}\sum_{\ell=1}^{N} \mathbf{E}\left\{\psi_t\psi_\ell^T\varepsilon_t\varepsilon_\ell\right\} \qquad (14) \end{aligned}$$

where ψ_t is the "predictor gradient" defined as

$$\psi_t(\theta) \triangleq \frac{\mathrm{d}\widehat{y}_t(\theta)}{\mathrm{d}\theta}. \tag{15}$$

In principle then, the issue of evaluating the degree of measurement noise induced inaccuracy in an estimate $\widehat{\theta}_N$ may be addressed by assuming that N is large enough that the convergence in (12) has effectively occurred and then using P_m to generate confidence regions on $\widehat{\theta}_N$.

However, as shown by the formulation (13) and (14), the expression for P_m is quite complicated and it is difficult to obtain engineering insight from it. Furthermore, as precursed in the introduction, the error $G(e^{j\omega}, \widehat{\beta}_N) - G(e^{j\omega})$ in an estimate $G(e^{j\omega}, \widehat{\beta}_N)$ of any true underlying linear dynamics $G(e^{j\omega})$ may be decomposed as

$$G(\widehat{\beta}_N) - G = \left[G(\widehat{\beta}_N) - G(\beta_\circ)\right] + [G(\beta_\circ) - G]$$

where the last term (a so-called "bias error") is due to the estimation model structure (5) not being rich enough to encompass the true data generation mechanism, and the first term (the so-called "variance error") is due to the measurement noise.

As already mentioned, the quantification of the second term $G(e^{j\omega}, \beta_\circ) - G(e^{j\omega})$ has already been the subject of much research during the last decade (Koust *et al.* 1992, Milanese and Vicino 1991, Ninness and Goodwin 1995, Wahlberg and Ljung 1992). However, because of the challenging nature of the problem, it still appears to be open and hence the topic of ongoing investigation.

In recognition of this, the present work examines the following question: in situations where it is possible to extend the model structure richness so as to render the bias error $G(e^{j\omega}, \beta_\circ) - G(e^{j\omega})$ negligible, then what price is paid in terms of overall error increase by virtue of a possible increase in the variance error $G(e^{j\omega}, \widehat{\beta}_N) - G(e^{j\omega}, \beta_\circ)$?

For example, to continue the simulation example presented in the introduction, the bias error induced by ignoring a static input non-linearity is shown in Figure 1(b) to be very large but, by way of contrast, if the possibility of this non-linearity is included in the model structure so that it may be estimated (via (5)-(9)), then the results are shown in Figure 2.

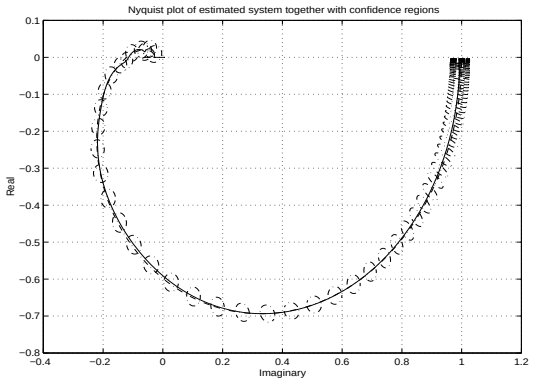


Fig. 2. *Nyquist plot of the true linear system within the Hammerstein system (dot-dashed line) and the linear part of the estimated system when identified with a suitable Hammerstein model set (solid line). 95% confidence intervals for the estimated system are represented as ellipses.*

Clearly, with the removal of the bias error, the overall error is now much smaller. But a price has been paid with increased variance error; that is, the 95% confidence ellipses that quantify the variance error are larger in Figure 2 than in Figure 1(b) where no non-linearity is present. The remainder of this paper is devoted to deriving a simple approximate expression of engineering relevance that quantifies this variance error, and which has already been presented in equation (3).

3. FREQUENCY DOMAIN FORMULATION

To analyse the question just introduced, the error quantification approach taken in this paper involves deriving frequency domain expressions, and this strategy has been inspired by its successful history (Hannan and Nicholls 1977, L.Ljung 1985, L.Ljung and Z.D.Yuan 1985).

Now, to this point, the discussion of the structure of the linear dynamics $G(q, \beta_N)$ has been focused on FIR models. The remainder of the paper will in fact address a generalisation expressible as

$$G(q, \beta_N) = \sum_{k=1}^{n} \beta_k \mathcal{B}_k(q), \tag{16}$$

where

$$\mathcal{B}_k(q) \triangleq \frac{\sqrt{1-|\xi_k|^2}}{(q-\xi_k)} \prod_{r=1}^{k-1}\left(\frac{1-\overline{\xi_r}q}{q-\xi_r}\right). \tag{17}$$

These latter transfer functions are orthonormal in the sense that

$$\langle \mathcal{B}_k, \mathcal{B}_\ell \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{B}_k(e^{j\omega}) \overline{\mathcal{B}_\ell(e^{j\omega})} \mathrm{d}\omega = \delta(k-l)$$

where δ is Kronecker's delta. This property turns out to be essential in the development of certain results that have been derived in (Ninness *et al.* 1999*a*, Ninness *et al.* 1999*b*) and are employed in the establishment of Theorem 3.1.

The terms $\xi_k \in \mathbf{D} \triangleq \{|z| \in \mathbf{C} : |z| < 1\}$ are user chosen fixed poles in the linear model structure, and the selection $\xi_k = 0, k = 1, 2, \ldots, m$ renders (16) as an FIR model as a special case. The use of such orthonormal model structures has a rich history, of which (Heuberger *et al.* 1995, P.M.J. Van den Hof *et al.* 1995, Wahlberg 1991, Wahlberg 1994) represents a recent part.

The following theorem provides a formal statement of the paper's main result.

Theorem 3.1. Consider the model structure defined by (5), and (16) together with the prediction error estimate $G(q, \widehat{\beta}_N)$ defined by (7), (8). Suppose that

(1) The innovations $\{e_t\}$ satisfy $\mathbf{E}\{e_t\} = 0$, $\mathbf{E}\{e_t^2\} \triangleq \sigma^2 < \infty$, $\mathbf{E}\{|e_t|^{4+\delta}\} < \infty$ for some $\delta > 0$.
(2) The signals $\{x_t\}$, $\{x'_t\}$ defined in (5), (6) are (jointly) quasi-stationary with spectra $\Phi_x(\omega)$, $\Phi_{x'}(\omega)$ and $\Phi_{xx'}(\omega)$ which are all Lipschitz continuous of some order $\alpha > 0$.
(3) The true system is in the model class.

Then

$$\lim_{N,n\to\infty} \frac{N}{K_n(\omega)} \mathbf{E}\left\{|G(e^{j\omega}, \widehat{\beta}_N) - G(e^{j\omega}, \beta_\circ)|^2\right\}$$
$$= \frac{\sigma^2}{\Phi_x(\omega)} + \left\| \frac{\mathrm{d}}{\mathrm{d}\alpha} \log \Phi_x(\omega) \right\|_\alpha^2 \lim_{n\to\infty} \frac{|G(e^{j\omega}, \beta_\circ)|^2}{K_n(\omega)} \quad (18)$$

where $\|\cdot\|_\alpha$ is defined in (4) and

$$P_\alpha^{-1} = \lim_{n\to\infty} \int_{-\pi}^{\pi} \frac{|G(\beta_\circ)|^2}{2\pi\sigma^2} \left(\frac{\Phi_x \Phi_{x'} - \Phi^\star_{xx'} \Phi_{xx'}}{\Phi_x} \right) \mathrm{d}\omega \quad (19)$$

$$K_n(\omega) = \sum_{k=1}^{n} \left|\mathcal{B}_k(e^{j\omega})\right|^2 = \sum_{k=1}^{n} \frac{\sqrt{1-|\xi_k|^2}}{|e^{j\omega} - \xi_k|^2}.$$

Proof: See (Ninness and Gibson 1999). ■

For the case in which α is a scalar, then the expression (18) may be interpreted as:

$$\mathbf{E}\left\{ \left| G(e^{j\omega}, \widehat{\beta}_N) - G(e^{j\omega}, \beta_\circ) \right|^2 \right\} \approx$$
$$\frac{1}{N} \left[\frac{\sigma^2 K_n}{\Phi_x} + \left(\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\alpha} \log \Phi_x \right)^2 |G(\beta_\circ)|^2 P_\alpha \right] \quad (20)$$

An immediately novel feature of the approximation (18) is that unlike pre-existing quantifications (such as (2)), it depends on the asymptotic estimated linear system frequency response $G(e^{j\omega}, \beta_\circ)$.

To understand the reason behind this, note that at a relatively fundamental level, the well known and pre-existing approximation (2) can be interpreted as an assertion of a principle that relative estimation errors depend upon relative sensitivities of model predictions to model parameters. That is, (using rather loose notation for the purposes of motivation) by denoting relative estimation error as $\mathrm{d}G/G$ and relative sensitivity of model predictions as $\mathrm{d}y/y$, then one could hypothesise that the well known approximation (2) is a manifestation of a more fundamental principle which ensures that the following equation holds:

$$N \frac{\mathrm{Var}\{\widehat{G}\}}{|G|^2} \equiv N \frac{|\mathrm{d}G|^2}{|G|^2} = \frac{(\mathrm{d}y)^2}{y^2}.$$

As a consequence of this hypothesis and noting that $y_t = G(q)x_t$,

$$N|\mathrm{d}G|^2 = \frac{(\mathrm{d}y)^2}{y^2}|G|^2 = \frac{(\mathrm{d}y)^2}{|G|^2 x^2}|G|^2 = \frac{(\mathrm{d}y)^2}{x^2}.$$

Now, in the Hammerstein case, there are two factors contributing to $\mathrm{d}y$ - sensitivities of $G(q,\beta)$ to β, and sensitivities of $X(\cdot,\alpha)$ to α, so that $(\mathrm{d}y)^2 = (\mathrm{d}G)^2 x^2 + |G|^2 (\mathrm{d}x)^2$. Therefore, after recognising that $(\mathrm{d}\beta)^2 \equiv \mathrm{Var}\widehat{\beta} \propto \sigma^2/\Phi_x$,

$$\frac{(\mathrm{d}y)^2}{x^2} = \frac{1}{x^2}\left[(\mathrm{d}G)^2 x^2 + |G|^2 (\mathrm{d}x)^2\right]$$
$$= \frac{1}{x^2}\left[\left(\frac{\mathrm{d}G}{\mathrm{d}\beta}\right)^2 (\mathrm{d}\beta)^2 x^2 + |G|^2 \left(\frac{\mathrm{d}x}{\mathrm{d}\alpha}\right)^2 (\mathrm{d}\alpha)^2 \right]$$
$$\equiv K_n(\omega)\frac{\sigma^2}{\Phi_x(\omega)} + |G|^2 \frac{1}{4x^4}\left(\frac{\mathrm{d}x^2}{\mathrm{d}\alpha}\right)^2 (\mathrm{d}\alpha)^2$$
$$\equiv K_n(\omega)\frac{\sigma^2}{\Phi_x(\omega)} + |G|^2 P_\alpha \left(\frac{1}{2}\log \Phi_x(\omega)\right)^2,$$

which is the quantification (20).

Therefore, the presence of the (perhaps unexpected) term $|G(e^{j\omega}, \beta_\circ)|^2$ reflects the fact that uncertainties in the nature of the non-linearity $X(\cdot,\alpha)$ are mapped through $G(q,\beta_\circ)$ before manifesting themselves in the output prediction error.

Another interesting aspect of Theorem 3.1 is that it provides insight into when the non-linearity parameterised by α is identifiable. Specifically, the spectral weighting in the definition (19) for P_α shows that identifiability is lost whenever $\Phi_x(\omega)\Phi_{x'}(\omega) = \Phi^\star_{xx'}(\omega)\Phi_{xx'}(\omega)$ for all ω. For example, if there is a linear relationship $x'_t(\alpha) = K(q)x_t(\alpha)$ for some LTI system $K(q)$, then for the case in which α is scalar,

$$|\Phi_{x'x}|^2 = |K|^2 \Phi_x^2 = \left(|K|^2 \Phi_x\right)\Phi_x = \Phi_{x'}\Phi_x.$$

Therefore, the weighting

$$\frac{\Phi_x(\omega)\Phi_{x'}(\omega) - |\Phi_{x'x}(\omega)|^2}{\Phi_x(\omega)}$$

can be interpreted as a measure of the non-linearity of the relationship between $x_t(\alpha)$ and $x'_t(\alpha)$.

At first glance the observation that α is only identifiable if the sensitivity of x_t to changes in α is not linear might seem to be a trivial conclusion, but as an example of its application it provides a simple and direct means to indicate that PRBS signals cannot be used for the identification of dead-zones or saturations.

4. SIMULATION EXAMPLES

Having derived the variance approximation (18), it is of course important to test its accuracy. While some theoretical analysis involving convergence rates in the supporting Theorem 3.1 is possible by using the results in (Ninness *et al.* 1999*b*), this will not provide any hard bounds on the accuracy of (18) for finite model order n.

Therefore, in order to study this issue, empirical testing seems the only viable option. Consider then the motivational example provided in the introduction wherein the system (1), has its input-output response observed in the case where the output $\{y_t\}$ is corrupted by i.i.d. zero mean Gaussian noise of standard deviation $\sigma = 0.01$, and with input $\{u_t\}$ being an i.i.d. zero mean and uniformly distributed process with $\sigma_u^2 = \mathbf{E}\{u_t^2\} = 1$. Consider further the situation introduced around Figure 2 where a symmetric saturation of level $\alpha = 0.5$ affects the input, and which is estimated together with a 25^{th} order FIR model for the linear dynamics $G(q)$.

Then the true variability $\mathbf{E}\left\{|G(e^{j\omega}, \widehat{\beta}_N) - G(e^{j\omega}, \beta_\circ)|^2\right\}$ can be estimated via a Monte-Carlo simulation over different measurement noise and input realisations. This was done for 1000 such realisations each over $N = 3000$ data points and the results are displayed in Figure 3 with the ensuing sample variability shown as a solid line.

Also shown in that figure as a dash-dot line is the approximation (18) for this variability, calculated as (20). The dashed line on the same figure is the approximation

$$\mathrm{Var}\{\widehat{G}\} \approx \frac{n}{N}\frac{\sigma^2}{\Phi_x(\omega)}$$

that, as was discussed in the introduction, might at first glance be expected to be appropriate. However, as is clear from Figure 3, this is not the case, and it is in fact vital to use the extra term that exists in (18); clearly the 'low-pass' effect invoked by the presence of the $|G(e^{j\omega}, \beta_\circ)|^2$ term dominates the approximation in such a way as to render it qualitatively as well as quantitatively "correct".

For the situation where $X(u_t, \alpha)$ is a symmetric dead-zone, with $\alpha_1 = -\alpha_2 = 0.5$ then the corresponding Monte-Carlo simulation results are shown in Figure 4. Again, excellent agreement between the Monte-Carlo estimate of the variability (solid line) and the new approximation (3) is illustrated.

To complete this simulation study section, then continuing with the first case of a (symmetric) saturation non-linearity of size α, then if $\{u_t\}$ is i.i.d. with uniform density of support $[-\lambda, \lambda]$, then it is straight-forward to calculate that for $\lambda > \alpha$

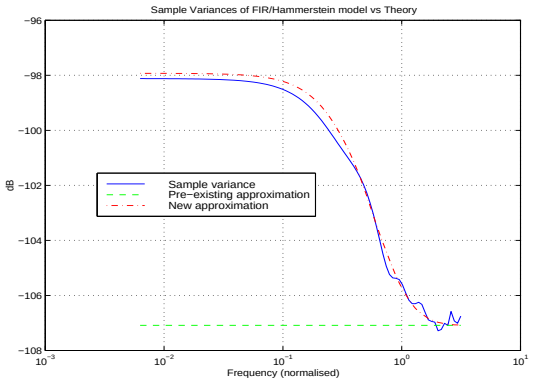


Fig. 3. *Variability of an 25^{th} order FIR estimate of the linear part of a Hammerstein system with saturation non-linearity. Solid line is Monte–Carlo estimate of true variability, dashed line is the pre-existing approximation (2) and the dash-dotted line is the new approximation (3).*

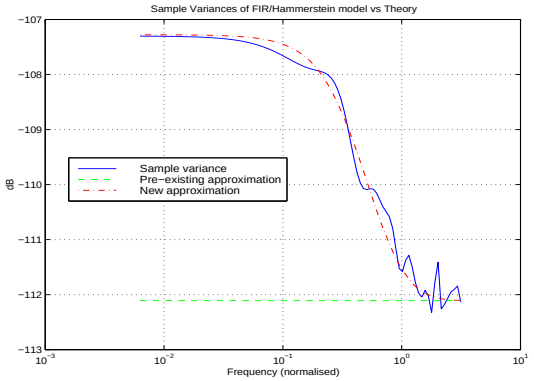


Fig. 4. *Variability of a 25^{th} order FIR estimate of the linear part of a Hammerstein system with dead-zone non-linearity. Solid line is Monte–Carlo estimate of true variability, dashed line is the pre-existing approximation (2) and the dash-dotted line is the new approximation (3).*

$$\Phi_x = \frac{\alpha^2}{3\lambda}(3\lambda - 2\alpha),\ \Phi_{xx'} = \alpha\left(\frac{\lambda-\alpha}{\lambda}\right),\ \Phi_{x'} = \frac{\lambda-\alpha}{\lambda}.$$

Substituting this into (20) then provides the approximation

$$\mathrm{Var}\{G(\widehat{\beta}_N)\} \approx \left[\frac{3\sigma^2}{\alpha^2 N}\right]\frac{\lambda}{(3\lambda-2\alpha)}\times \left[K_n + \frac{3|G(\beta_\circ)|^2}{\|G\|^2}\left(\frac{\lambda-\alpha}{\alpha}\right)\right], \qquad (21)$$

where $\|G\|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|G(e^{j\omega}, \beta_\circ)|^2 \mathrm{d}\omega$. An interesting conclusion that can be drawn from (21) is that it indicates that an optimal input excitation power exists.

That is, in the linear case, the estimation accuracy is proportional to $1/\Phi_u(\omega)$ so that the more input excitation power, the better. However, as shown by (21), in the non-linear case with input saturation, (neglecting the common constants) the overall variability consists of two terms, one of which

$$\frac{K_n(\omega)\lambda}{(3\lambda-2\alpha)}$$

decreases monotonically with increasing input power $\Phi_u(\omega) = \lambda^2/3$, and the second term of which

$$\frac{3\lambda}{(3\lambda - 2\alpha)} \left(\frac{\lambda - \alpha}{\alpha}\right) \frac{|G(e^{j\omega}, \beta_\circ)|^2}{\|G\|^2}$$

increases monotonically with increasing $\Phi_u(\omega) = \lambda^2/3$. As a result, for every frequency ω there is an optimal input power (in the sense of minimising the variability of the estimate) given by the solution of

$$(3t^2 - 4t + 2) = \frac{2}{3} \frac{\|G\|^2}{|G(e^{j\omega}, \beta_\circ)|^2} K_n(\omega) \qquad (22)$$

where $t = \lambda/\alpha$.

At first, this principle that an optimal input power exists, and hence that increasing the input power may lead to a degrade estimate, might seem counter-intuitive. However, once exposed via (18) and (21), it is undersood to arise from the fact that as λ is increased, the probability of a realisation of u_t being below the saturation level is *decreased* and hence an increased λ leads to an input sequence $\{u_t\}$ which provides less 'exploration' of the non-linearity. As a result, the variability of the estimate is increased. Vice-versa, if λ is too small (only slightly larger than α), then again the non-linearity is not explored since there is a low probability that u_t will hit the saturation limit α.

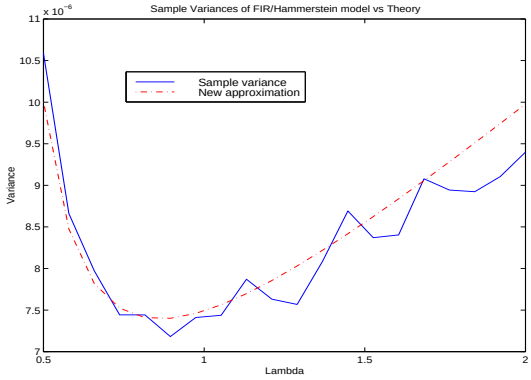


Fig. 5. *Variability of a 25^{th} order FIR estimate of the linear part of a Hammerstein system with saturation non-linearity at a single frequency but at a range of input spectral powers. Solid line is Monte–Carlo estimate of true variability and the dash-dotted line is the new approximation (21).*

The presence of this optimum is illustrated once again via Monte-Carlo simulation where, under the same conditions as previously reported for the saturation non-linearity case (and for $n = 25$), the true variability of $G(e^{j\omega}, \widehat{\beta}_N)$ is estimated over 500 realisations and at various input powers $\Phi_u = \lambda^2/3$. This is shown as the solid line in Figure 5, and is profiled against the approximation (18) which in this case appears as (21). The presence of an optimal power around $\lambda = 0.86$ is clear.

5. REFERENCES

Hannan, E. and D.F. Nicholls (1977). 'The estimation of the prediction error variance'. *J. Amer. Stat. Assoc* **72**(360, part 1), 834–840.

Heuberger, P., P.M.J. Van den Hof and O.H. Bosgra (1995). 'A generalized orthonormal basis for linear dynamical systems'. *IEEE Trans. Auto. Cont.* **AC-40**(3), 451–465.

Koust, R., M.K Lau and S.P. Boyd (1992). 'Set-membership identification of systems with parametric and nonparametric uncertainty'. *IEEE Trans. Auto. Cont* **AC-37**(7), 929–941.

Ljung, L. (1987). *System Identification: Theory for the User*. Prentice-Hall, Inc.

Ljung, L. and Lei Guo (1997). 'The role of model validation for assessing the size of the unmodeled dynamics'. *IEEE Trans. Auto. Cont* **AC-42**(9), 1230–1240.

L.Ljung (1978). 'Convergence analysis of parametric identification methods'. *IEEE Trans. Auto. Cont* **AC-23**(5), 770–783.

L.Ljung (1985). 'Asymptotic variance expressions for identified black-box transfer function models'. *IEEE Trans. Auto. Cont* **AC-30**(9), 834–844.

L.Ljung and P.E.Caines (1979). 'Asymptotic normality of prediction error estimators for approximate system models'. *Stochastics* **3**, 29–46.

L.Ljung and Z.D.Yuan (1985). 'Asymptotic properties of black-box identification of transfer functions'. *IEEE Trans. Auto. Cont* **30**(6), 514–530.

Milanese, M. and A. Vicino (1991). 'Optimal estimation theory for dynamic systems with set membership uncertainty:An overview'. *Automatica* **27**(6), 997–1009.

Narendra, K. and P.G. Gallman (1966). 'In iterative method for the identification of nonlinear systems using a Hammerstein model'. *IEEE Trans. Auto. Cont* **11**(7), 546–550.

Ninness, B. and Graham Goodwin (1995). 'Estimation of model quality'. *Automatica* **31**(12), 32–74.

Ninness, B., Håkan Hjalmarsson and Fredrik Gustafsson (1999*a*). 'The fundamental role of general orthonormal bases in system identification'. *IEEE Trans. Auto. Cont* **44**(8), pp.

Ninness, B., Håkan Hjalmaxrsson and Fredrik Gustafsson (1999*b*). 'Generalised Fourier and Toeplitz results for rational orthonormal bases'. *SIAM J. Cont & Optim* **37**(2), 429–460.

Ninness, B. and S. Gibson (1999). 'Quantification of Estimation Error of Hammerstein Model Structure'. *Tech. Report 9952, Dept EE, Uni. Newcastle*.

P.M.J. Van den Hof, P.S.C. Heuberger and J. Bokor (1995). 'System identification with generalized orthonormal basis functions'. *Automatica* **31**(12), 1821–1834.

Rangan, S., Greg Wolodkin and Kameshwar Poolla (1995). New results for hammerstein system identification. In 'Proc. 34th CDC'. pp. 697–702.

Smith, R. and M. Dahleh (editors) (1994). *The Modelling of Uncertainty in Cont Systems*. Springer Verlag.

(1992). *IEEE Trans. Auto. Cont*.

Tao, G. and Petar V. Kokotović (1996). *Adaptive Cont of Systems with Actuator and Sensor Nonlinearities*. Adaptive and Learning Systems for Signal Processing, Communications and Cont. John Wiley.

Wahlberg, B. (1991). 'System identification using Laguerre models'. *IEEE Trans. Auto. Cont* **AC-36**(5), 551–562.

Wahlberg, B. (1994). 'System identification using Kautz models'. *IEEE Trans. Auto. Cont* **AC-39**(6), 1276–1282.

Wahlberg, B. and L. Ljung (1992). 'Hard frequency-domain model error bounds from least-squares like identification techniques'. *IEEE Trans. Auto. Cont* **37**(7), 900–912.