Barrier Function Methods For Model Predictive Control

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Abstract

We introduce a model predictive controller for input constrained linear systems based on a novel barrier function. As the barrier has fixed weight, the controller may be implemented in a highly efficient and numerically robust manner. We prove that the resulting closed loop behaviour is stable and converges to any specified target steady-state solution on the interior of the constraint set. In the case where the optimal steady-state solution may lie on the boundary of the constraint set, we propose a sub-optimal target computed with a traditional logarithmic barrier, but again with fixed weight. This guarantees a solution on the interior of the constraint set. The duality gap allows us to pre-specify a bound on the resulting degradation in system performance. In such a case the maximal invariant set may become arbitrarily small and this may be problematic in terms of feasibility. Such remarks are pertinent to any model predictive controller where stability results are based on invariant sets.

1 Introduction

The *raison d'être* for Model Predictive Control (MPC) is the implicit constraint handling capabilities, which includes both dynamic and steady-state constraints, Mayne *et. al.* [8]. An optimisation algorithm is usually employed to determine the optimal input sequence that regulates the process subject to these constraints. However, computational cost of such optimisation methods is often a deterrent when considering a model predictive control strategy. This issue has partly motivated the introduction of computationally efficient Interior Point methods, see Rao *et. al.* [11], Wright [14] and Bartlett *et. al.* [1]. Fundamentally linked to Interior Point methods is the concept of a barrier function, Ye [15], Wright [14], Fiacco and McCormick [4] and Nesterov and Nemirovskii [10]. In this paper, we introduce a novel weighted barrier function model predictive controller with analytic centre at the desired steady-state set-point. For a stable model predictive controller, this property guarantees convergence to the desired steady-state set-point. Furthermore, for a specific terminal constraint set X_f, we show that the closed-loop behaviour of this control strategy is stable.

Economic optimisation of process conditions often demands set-points on or near actuator constraints, Muske and Rawlings [9]. For this reason, constraints should be considered in both dynamic and steady-state operation. Most MPC stability proofs require the steady-state set-point to lie on the interior of the constraint set, Mayne *et. al.* [8]. We present a sub-optimal method for determining the steady-state set-point that strictly satisfies the input constraint polytope. This method is based on the standard weighted logarithmic barrier function and has several desirable properties, including the central path, see Ye [15]. For a fixed weighting parameter μ, we are guaranteed to lie on the central path, which allows us to exploit the duality gap and provide an upper bound on system performance degradation.

The terminal constraint set X_f, is fundamental to MPC stability proofs, see Mayne *et. al.* [8]. Indeed, a turning point for MPC stability was the introduction of this idea, see e.g. Mayne and Michalska [7], since feasibility may become impractical for a terminal equality constraint. Two common choices for

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the terminal constraint set are level sets of the Lyapunov function, see e.g. Chen and Allgöwer [2], and the Maximum Output Admissible set, Gilbert and Tan [5]. We show that the maximal terminal set may become arbitrarily small as the steady-state solution tends to the boundary. In fact, these remarks are pertinent to any model predictive controller where stability results are based on invariant sets.

The structure of the paper is as follows: In section 2 we introduce some notation and definitions relevant to receding horizon control and weighted barrier functions. In section 3 we introduces the novel barrier function and prove stability for the resulting controller in closed-loop. We explore steady-state set-points near constraints in section 4 and draw some concluding remarks in section 5.

2 Preliminaries

2.1 Model Predictive Control Preliminaries

This section, which is broadly based on Mayne *et. al.* [8], describes some preliminary notation and definitions used to describe the receding horizon control problem. In this paper we are concerned with linear time-invariant discrete-time state-space systems of the form,

$$\bar{\mathbf{x}}(k+1) = \mathbf{A}\bar{\mathbf{x}}(k) + \mathbf{B}\bar{\mathbf{u}}(k) \tag{1}$$

where $\bar{\mathbf{x}} \in \mathbb{R}^n$ and $\bar{\mathbf{u}} \in \mathbb{R}^m$. If the objective is to regulate the system about some point $\mathbf{x}_{ss} \in \mathbb{R}^n$, then we can shift the origin by replacing $\bar{\mathbf{x}}$ with $\mathbf{x} = \bar{\mathbf{x}} - \mathbf{x}_{ss}$ and $\bar{\mathbf{u}}$ with $\mathbf{u} = \bar{\mathbf{u}} - \mathbf{u}_{ss}$, where $\mathbf{u}_{ss}$ is the input corresponding to $\mathbf{x}_{ss}$ (Note: the point $\mathbf{u}_{ss}$ is not necessarily unique. For more details on this see Muske and Rawlings [9] where they provide a method for obtaining a unique $\mathbf{u}_{ss}$ for a general linear system). Using these substitutions we obtain,

$$\begin{aligned} \mathbf{x}(k+1) &= f(\mathbf{x}(k), \mathbf{u}(k)) \\ f(\mathbf{x}(k), \mathbf{u}(k)) &:= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \end{aligned} \tag{2}$$

such that $\mathbf{f}(\mathbf{x}_{ss}, \mathbf{u}_{ss}) = \mathbf{0}$. For prediction horizon N, we define the input sequence $\mathcal{U}_k$ as,

$$\mathcal{U}_k := \{\mathbf{u}(k), \mathbf{u}(k+1), \ldots, \mathbf{u}(k+N-1)\}$$

We define $\mathcal{O}$ as the sequence of $m \times 1$ zero vectors,

$$\mathcal{O} := \{\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}\}$$

and define the $N \times m$ vector $\mathbf{u}^*(\mathcal{U}_k)$ as,

$$\mathbf{u}^*(\mathcal{U}_k) := \begin{bmatrix} \mathbf{u}(k) \\ \mathbf{u}(k+1) \\ \vdots \\ \mathbf{u}(k+N-1) \end{bmatrix} \tag{3}$$

The objective is to steer the system state to the origin subject to input constraints. We define the input constraint set as $\mathbb{U} := \{\mathbf{u} \in \mathbb{R}^m : \mathbf{L}\mathbf{u} \leq \mathbf{b}\}$, where $\mathbf{L}$ is an $M \times m$ matrix and $\mathbf{b}$ is an $M \times 1$ vector. In a similar manner, the input sequence constraint set $\mathbb{U}_s$ is defined as,

$$\mathbb{U}_s := \left\{\mathbf{u}^*(\cdot) \in \mathbb{R}^{N \times m} : \mathbf{L}_s\mathbf{u}^*(\cdot) \leq \mathbf{b}_s \text{ and } \mathbf{u}(\cdot) \in \mathbb{U}\right\}$$

where $\mathbf{L}_s$ is an $M_s \times (N \times m)$ matrix and b_s is an $M_s \times 1$ vector. It is assumed throughout this paper that $\mathbf{0} \in \text{int}\mathbb{U}$ and also that $\mathbf{0} \in \text{int}\mathbb{U}_s$. Then by definition, it is possible to choose $\mathbf{L}$, $\mathbf{b}$, $\mathbf{L}_s$ and $\mathbf{b}_s$ such that $\mathbb{U}$ and $\mathbb{U}_s$ are closed convex sets. We make the distinction between $\mathbb{U}$ and $\mathbb{U}_s$ to highlight the difference between a feasible input compared with a feasible control sequence. One of the themes of this paper is to consider the case where $\mathbf{u}_{ss}$ tends towards the boundary of $\mathbb{U}$ and in this sense it is important to distinguish the two.

Using system (2) we define the receding horizon control objective with initial state $\mathbf{x}(k) = \mathbf{x}_0$ as,

$$J_N(\mathbf{x}(k), \mathcal{U}_k) = \sum_{i=0}^{k+N-1} \left(\|\mathbf{x}(k+i)\|^2_{\mathbf{Q}} + \|\mathbf{u}(k+i)\|^2_{\mathbf{R}} \right) + \|\mathbf{x}(k+N)\|^2_{\mathbf{P}} \tag{4}$$

where $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{P}$ are positive definite and symmetric matrices. Then in spirit of Mayne *et. al.* [8] we define the optimal control sequence $\mathcal{U}_k^o$ as,

$$\mathcal{U}_k^o := \underset{\mathcal{U}_k}{\operatorname{argmin}} \; J_N(\mathbf{x}(k), \mathcal{U}_k) \text{ s.t. } \mathbf{u}^*(\mathcal{U}_k) \in \mathbb{U}_s \tag{5}$$

and we have $\mathcal{U}_k^o = \{\mathbf{u}^o(k), \mathbf{u}^o(k+1), \ldots, \mathbf{u}^o(k+N-1)\}$. Moreover, the optimal cost is defined as,

$$J_N^o(\mathbf{x}(k)) := J_N(\mathbf{x}(k), \mathcal{U}_k^o) \tag{6}$$

The receding horizon control strategy can be stated as: at time k, solve the open-loop minimisation problem (5) and apply the first control move $\mathbf{u}^o(k)$ to the system.

2.1.1 Terminal Constraint Set X_f

A key idea in MPC stability is the terminal constraint set X_f, which is invariant under a local controller $\kappa_f(\cdot)$ (see Mayne *et. al.* [8]). In what follows, we define X_f in a similar way to Chen and Allgöwer [2], but for the linear discrete case. This definition will form part of the terminal constraint set used in section 3.2.1.

If system (2) is stabilisable then there exists a linear controller $\mathbf{u} = -\mathbf{K}\mathbf{x}$ such that,

$$|\text{eig}(\mathbf{A} - \mathbf{B}\mathbf{K})| < 1$$

and in the spirit of Chen and Allgöwer [2], we have the following Lemma,

Lemma 2.1.1 *If the system (2) is stabilisable, then*

(a) For the stabilising controller $\mathbf{u} = -\mathbf{K}\mathbf{x}$ *and positive definite symmetric matrix* $\mathbf{Q}^*_{\mathbf{K}}$ *we can obtain the unique positive definite symmetric matrix* $\mathbf{P}$ *as the solution of the Lyapunov equation,*

$$(\mathbf{A} - \mathbf{B}\mathbf{K})^T \mathbf{P} (\mathbf{A} - \mathbf{B}\mathbf{K}) - \mathbf{P} = -\mathbf{Q}^*_{\mathbf{K}} \tag{7}$$

where $\mathbf{Q}^*_{\mathbf{K}} = \mathbf{Q} + \mathbf{K}^{\mathbf{T}}\mathbf{R}\mathbf{K}$.

(b) We can define a series of regions X_α *and a set* X_u *around the origin and a scalar* α_{max} *as*

$$X_\alpha := \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T \mathbf{P} \mathbf{x} < \alpha \right\} \tag{8}$$
$$X_u := \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{K}\mathbf{x} \in \mathbb{U}\} \tag{9}$$
$$\alpha_{max} := \max\{\alpha \in (0, \infty) : X_\alpha \subseteq X_u\} \tag{10}$$

(c) We can define an invariant terminal constraint set X_f *for the controller* $\mathbf{u} = -\mathbf{K}\mathbf{x}$ *as,*

$$X_f := X_{\alpha_{max}} \tag{11}$$

Proof. For $|\text{eig}(\mathbf{A} - \mathbf{B}\mathbf{K})| < 1$ and $\mathbf{Q}^*_{\mathbf{K}} > \mathbf{0}$ then by the general conditions for Lyapunov solvability there exists a unique positive definite symmetric matrix $\mathbf{P}$ that solves (7) and therefore (a) is true. Since $\mathbf{0} \in \text{int}\mathbb{U}$, we can always obtain sets in the form of (8) and (9), therefore $\alpha_{max} > 0$ and (b) is true. Since (a) is true and (b) is true, then the Lyapunov stability theorem says that the set $X_{\alpha_{max}}$ is invariant and (c) is therefore true.

2.2 Logarithmic Barrier Preliminaries

Using the notation from section 2.1, we consider the general quadratic cost function,

$$J(\mathbf{u}^*(\mathcal{U})) = \frac{1}{2}\mathbf{u}^*(\mathcal{U})^T\mathbf{W}\mathbf{u}^*(\mathcal{U}) + \mathbf{f}^T\mathbf{u}^*(\mathcal{U}) \tag{12}$$

where $\mathbf{W} \in \mathbb{R}^{(N\times m)\times(N\times m)}$ is positive semi-definite and $\mathbf{f} \in \mathbb{R}^{N\times m}$. In many situations it is desirable to minimise (12) subject to linear inequality constraints on $\mathbf{u}^*(\mathcal{U})$. This is called the Quadratic Programming problem described as follows,

$$\mathcal{U}^o := \underset{\mathcal{U}}{\operatorname{argmin}}\, J(\mathbf{u}^*(\mathcal{U})) \quad \text{s.t. } \mathbf{u}^*(\mathcal{U}) \in \mathbb{U}_s \tag{13}$$

This minimisation problem can be solved using a number of different methods. Traditionally there are two classes of methods for solving (13); Active-Set (see Gill, Murray and Wright [6]), and Interior-Point (see Ye [15] and Wright [14]. A third possibility is mixed weight least squares (see Rossiter and Kouvaritakis [12]).

Fundamental to Interior-Point theory is the idea of barrier functions. Fiacco and McCormick [4] provide a detailed analysis of barrier function methods. One such Barrier function is the Logarithmic Barrier (see §3.2 of Nesterov and Nemirovskii [10] and §7.1 of Fiacco and McCormick [4] for more details). The logarithmic barrier function can be defined as follows:

$$\beta(\mathbf{u}^*(\mathcal{U})) := \begin{cases} -\sum_{i=1}^{M_s} \ln(s_i) & \text{for } \mathbf{u}^*(\mathcal{U}) \in \mathbb{U}_s, \\ \infty & \text{otherwise.} \end{cases} \tag{14}$$

where $s_i := b_s(i) - \ell_s(i)^T\mathbf{u}^*(\mathcal{U})$, $b_s(i)$ is the i'th element of $\mathbf{b}_s$ and $\ell_s(i)$ is the i'th column of $\mathbf{L}_s^T$. A new cost function, $J_\mu(\mathbf{u}^*(\mathcal{U}))$ can be expressed in terms of this barrier as follows,

$$J_\mu(\mathbf{u}^*(\mathcal{U})) = J(\mathbf{u}^*(\mathcal{U})) + \mu\beta(\mathbf{u}^*(\mathcal{U})) \tag{15}$$

where $\mu \in [0, \infty)$ is the associated weighting parameter for the barrier function. It is well known (see Fiacco and McCormick [4]) that the solution to the unconstrained minimisation problem,

$$\mathcal{U}_\mu^o = \underset{\mathcal{U}}{\operatorname{argmin}}\, J_\mu(\mathbf{u}^*(\mathcal{U})) \tag{16}$$

converges to the true solution $\mathcal{U}^o$ of (13) as $\mu \to 0$. Furthermore, the solution to (16) is guaranteed to satisfy the constraints, i.e. $\mathbf{u}^*(\mathcal{U}_\mu^o) \in \mathbb{U}_s$. In fact, if $\mu > 0$ then $\mathbf{u}^*(\mathcal{U}_\mu^o) \in \operatorname{int}\mathbb{U}_s$, and $\operatorname{int}\mathbb{U}_s = \{\mathbf{u}^*(\mathcal{U}) \in \mathbb{R}^{N\times m} \text{ s.t. } \mathbf{L}_s\mathbf{u}^*(\mathcal{U}) < \mathbf{b}_s\}$ denotes the interior of the convex constraint polytope. In subsequent analysis it will become desirable that $\beta(\mathbf{u}^*(\mathcal{U})) = 0$ at the origin. To achieve this, we make a simple transformation described as follows:

Let $\theta = [\operatorname{diag}(\mathbf{b}_s)]^{-1}$. Then the following transformation of $\mathbf{L}_s$ and $\mathbf{b}_s$ ensures that $\beta(\mathbf{u}^*(\mathcal{U})) = 0$ at the origin,

$$\mathbf{L}_s^\theta := \theta\mathbf{L}_s \tag{17}$$

$$\mathbf{b}_s^\theta := \theta\mathbf{b}_s \tag{18}$$

and we define $s_i^\theta := b_s^\theta(i) - \ell_s^\theta(i)^T\mathbf{u}^*(\mathcal{U})$, where $b_s^\theta(i)$ is the i'th element of $\mathbf{b}_s^\theta$ and $\ell_s^\theta(i)$ is the i'th column of $[\mathbf{L}_s^\theta]^T$. We assume without loss of generality that $b_s(i) \neq 0$, $\forall i$.

It has been shown that the logarithmic barrier is useful for determining the analytic centre of a constrained minimisation problem with fixed weighting parameter μ, and that as $\mu \to 0$, the analytic centre tends to the optimal solution (see Ye [15] §2.2). In fact, the analytic centre for $\mu \in (0, \infty)$ describes the central path.

3 MPC with a Novel Barrier

The computational cost of MPC has warranted efficient methods for solving the optimisation problem (5), see Wright [14], Bartlett *et. al.* [1] and Rao *et. al.* [11]. In this section we introduce a novel barrier function in place of existing Interior Point methods for a fixed weighting parameter $\mu > 0$.

3.1 Novel Barrier Function

We show in section 4 that the logarithmic barrier method with $\mu > 0$, has desirable properties in terms of determining the steady-state input and state. By contrast, in this section we show that if the MPC strategy does converge, then using the logarithmic barrier function with $\mu > 0$, does not guarantee convergence to the set-point. Thus, we propose a novel barrier function that guarantees convergence of the MPC strategy to the desired set-point, assuming that the MPC strategy is stable. In particular, this barrier function allows us to fix the weighting parameter μ, which has considerable advantages in terms of computational cost. Moreover, in section 3.2 we show stability of the model predictive control strategy using this barrier function.

Using a suitable change of co-ordinants, the standard linear regulator problem can be cast in the form of (2), (4) and (5). Then the unconstrained minimum of (4) occurs at $(\mathbf{0}, \mathcal{O})$ and $J_N(\mathbf{0}, \mathcal{O}) = 0$. Using a weighted logarithmic barrier we can define a combined cost function as,

$$J_\beta(\mathbf{x}(k), \mathcal{U}_k) := J_N(\mathbf{x}(k), \mathcal{U}_k) + \mu\beta(\mathbf{u}^*(\mathcal{U}_k)) \tag{19}$$

Then for $\mu > 0$ we define the minimum of (19) as

$$\mathcal{U}^o_{k,\mu,\beta} := \underset{\mathcal{U}_k}{\operatorname{argmin}} \; J_{N,\beta}(\mathbf{x}(k), \mathcal{U}_k) \tag{20}$$

Since both (4) and $\mu\beta(\mathbf{u}^*(\mathcal{U}_k))$ are convex functions and $\mu > 0$, then $\mathcal{U}^o_{k,\mu,\beta} \neq \mathcal{U}^o_k$ unless $\mu\beta(\mathbf{u}^*(\mathcal{U}_k))$ has its minimum at $\mathbf{u}^*(\mathcal{O})$. In general, the analytic centre of the constraint polytope $\mathbb{U}_s$ is not equal to $\mathbf{u}^*(\mathcal{O})$. In fact, this condition would only be satisfied if the constraints were symmetric about the origin, after a change of co-ordinants. Therefore, with $\mu > 0$ we are not guaranteed to converge to the desired set-point at steady-state.

Definition 1 *We define the novel barrier function $\phi(\mathbf{u}^*(\mathcal{U}_k))$ as,*

$$\phi(\mathbf{u}^*(\mathcal{U}_k)) := \beta(\mathbf{u}^*(\mathcal{U}_k)) + \sum_{i=1}^{M_s} \gamma_i \tag{21}$$

where the additional terms γ_i, are defined as,

$$\gamma_i := \sum_{j=1}^{N\times m} \left.\frac{\partial[\ln(s_i)]}{\partial u^*_j}\right|_{u^*_j = u^*_{ss,j}} \times u^*_j$$

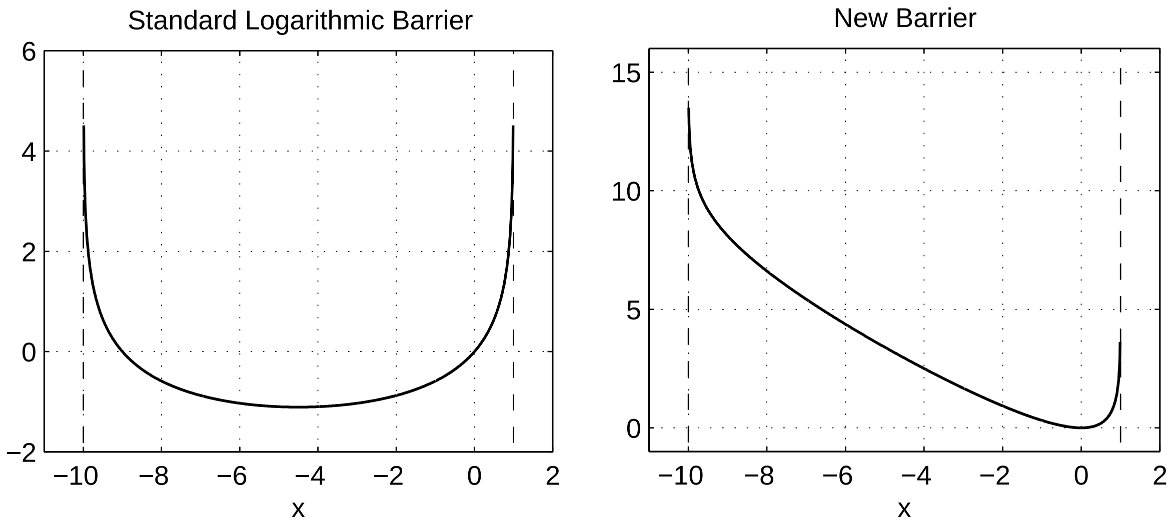
where u^*_j is the j'th element of $\mathbf{u}^*(\mathcal{U}_k)$ and $u^*_{ss,j}$ is the j'th element of the $N\times m$ vector $u^*_{ss} := [\mathbf{u}^T_{ss}\ \mathbf{u}^T_{ss}\ \ldots\ \mathbf{u}^T_{ss}]^T$ for $j = 1 \ldots N \times m$.

See figure 1 for a comparison of the two different barrier functions.

Remark 3.1.1 *Since $\beta(\mathbf{u}^*(\mathcal{U}_k))$ and $\sum_{i=1}^{M} \gamma_i$ are convex functions, then the sum is also convex. Furthermore, by construction the minimum of $\phi(\mathbf{u}^*(\mathcal{U}_k))$ is at $\mathbf{u}^*(\mathcal{O})$ which ensures, for a stable MPC strategy, that the MPC algorithm converges to the origin in steady-state.*

3.2 MPC Stability Using weighted Barrier Approach

In this section, we describe an MPC algorithm that uses the barrier function described in (21). Stability of this algorithm is shown for a particular choice of terminal constraint set X_f defined in the following subsection.



(a) Standard Logarithmic Barrier Function: This is the standard logarithmic barrier function for $-10 \leq x \leq 1$. Note that the minimum does not occur at the origin.

(b) New Barrier Function: This is the new barrier function for $-10 \leq x \leq 1$. Note that the minimum occurs at the origin. Also, the function is zero at the origin.

Figure 1: Comparison of logarithmic and new barrier functions.

3.2.1 Stability of Barrier Function MPC

The receding horizon control problem using a weighted barrier function can be stated as follows. For the new barrier function introduced in section 3.1, we may construct the combined cost function,

$$J_\phi(\mathbf{x}(k), \mathcal{U}_k) := J_N(\mathbf{x}(k), \mathcal{U}_k) + \mu\phi(\mathbf{u}^*(\mathcal{U}_k)) + \mu_f \phi_f^\alpha(\mathbf{x}(k+N), \mathbf{P}) \tag{22}$$

where the barrier

$$\phi_f^\alpha(\mathbf{x}(k+N), \mathbf{P}) := -(\ln(\alpha - \|\mathbf{x}(k+N)\|_{\mathbf{P}}^2))$$

ensures that $\mathbf{x}(k+N) \in X_f$ and μ_f is the associated weighting parameter. In the sequel, we assume that $\phi_f^\alpha(\mathbf{x}(k+N), \mathbf{P})$ has been normalised such that $\alpha = 1$. Then for $\mu, \mu_f > 0$ we define the minimum of (22) as

$$\mathcal{U}_{k,\mu,\phi}^o := \underset{\mathcal{U}_k}{\operatorname{argmin}}\ J_\phi(\mathbf{x}(k), \mathcal{U}_k) \tag{23}$$

which is well defined since $\mathbf{x}(k+N)$ is a function of $\mathcal{U}_k$. For this control sequence we have a corresponding state evolution described as,

$$\mathcal{X}_{k,\mu,\phi}^o := \left\{\mathbf{x}(k), \mathbf{x}_{\mu,\phi}^o(k+1), \ldots, \mathbf{x}_{\mu,\phi}^o(k+N)\right\} \tag{24}$$

We also define the optimal cost at time k, as $J_\phi^o(\mathbf{x}(k)) := J_\phi(\mathbf{x}(k), \mathcal{U}_{k,\mu,\phi}^o)$. Then at each time interval we solve (23) and apply the first control move to the system. Therefore, the implicit receding horizon control law is defined as

$$\kappa_N(\mathbf{x}(k)) := \mathbf{u}_{\mu,\phi}^o(k). \tag{25}$$

Using a similar approach to Mayne *et. al.* [8] we define a terminal constraint set X_f, a terminal controller $\kappa_f(\mathbf{x}) = -\mathbf{K}\mathbf{x}$ and a terminal penalty function $F(\mathbf{x})$ and show that the cost function (22) is a Lyapunov function, i.e. we wish to show that

$$J_\phi^o(\mathbf{x}_{\mu,\phi}^o(k+1)) - J_\phi^o(\mathbf{x}(k)) \leq 0.$$

Since

$$J_\phi^o(\mathbf{x}_{\mu,\phi}^o(k+1)) \leq J_\phi(\mathbf{x}_{\mu,\phi}^o(k+1), \tilde{\mathcal{U}})$$

for any feasible control sequence $\tilde{\mathcal{U}}$, it suffices to show that

$$J_\phi(\mathbf{x}^o_{\mu,\phi}(k+1),\tilde{\mathcal{U}}) - J^o_\phi(\mathbf{x}(k)) \leq 0$$

Let $\mathbf{A_K}$ denote the closed-loop system matrix given by $\mathbf{A_K} = \mathbf{A} - \mathbf{BK}$. Also, let $\mathbf{Q}^* = \mathbf{Q}+n\mathbf{I}$ for some $n \in [0,\infty)$. Then we can obtain $\mathbf{P}$ and $\mathbf{K}$ from the solution of the Discrete Arithmatic Riccati Equation:

$$\mathbf{P} = \mathbf{A_K}^T\mathbf{P}\mathbf{A_K} + \mathbf{Q}^* + \mathbf{K}^T\mathbf{R}\mathbf{K} \tag{26}$$

Then for the system described in (2), with initial condition $\mathbf{x}(k) = \mathbf{x}_0$, we have the following result,

Result 3.1 *If problem (23) has a feasible solution then the receding horizon controller described in (22), (23) and (25) is stable for the terminal constraint set X_f defined by,*

$$\begin{aligned}
X_\alpha &:= \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|^2_\mathbf{P} \leq \alpha\} \\
X_n &:= \left\{\mathbf{x} \in \mathbb{R}^n : n\|\mathbf{x}\|^2 \geq \mu\phi(\mathbf{u}^*(\tilde{\mathcal{U}})) + \mu_f\phi^\alpha_f(\mathbf{A_K}\mathbf{x}(k+N),\mathbf{P})\right\} \\
X_u &:= \{\mathbf{x} \in \mathbb{R}^n : -\mathbf{K}\mathbf{x} \in \mathbb{U}\} \\
\bar{\alpha} &:= \max\{\alpha \in (0,\infty) : X_\alpha \subseteq [X_n \cap X_u]\} \\
X_f &:= X_{\bar{\alpha}}
\end{aligned}$$

Proof. Let $\tilde{\mathcal{U}}$ be constructed as follows,

$$\tilde{\mathcal{U}} := \{\mathbf{u}^o_{\mu,\phi}(k+1),\ \mathbf{u}^o_{\mu,\phi}(k+2),\ \ldots\ ,\ \mathbf{u}^o_{\mu,\phi}(k+N-1),\ -\mathbf{K}\mathbf{x}^o_{\mu,\phi}(k+N)\}$$

Feasibility gives that $\mathbf{x}^o_{\mu,\phi}(k+N) \in X_f$ and that the truncated control sequence $\{\mathbf{u}^o_{\mu,\phi}(k+1),\ \ldots\ ,\ \mathbf{u}^o_{\mu,\phi}(k+N-1)\}$ takes the initial state $\mathbf{x}^o_{\mu,\phi}(k+1)$ to the state $\mathbf{x}^o_{\mu,\phi}(k+N) \in X_f$. Therefore, in order that $\tilde{\mathcal{U}}$ is a feasible control sequence, we require $\tilde{\mathcal{U}} \in \mathbb{U}_s$ and that $\mathbf{A_K}\mathbf{x}^o_{\mu,\phi}(k+N) \in X_f$. Since X_f is a level set of the Lyapunov function $\mathbf{x}^T\mathbf{P}\mathbf{x}$ then by design, X_f is an invariant set. It is possible to write the cost function associated with $\tilde{\mathcal{U}}$ as,

$$\begin{aligned}
&J_\phi(\mathbf{x}^o_{\mu,\phi}(k+1),\tilde{\mathcal{U}}) - J^o_\phi(\mathbf{x}(k)) \\
&= \|\mathbf{x}^o_{\mu,\phi}(k+N)\|^2_\mathbf{Q} + \|\mathbf{x}^o_{\mu,\phi}(k+N)\|^2_{\mathbf{K}^T\mathbf{R}\mathbf{K}} + \|\mathbf{x}^o_{\mu,\phi}(k+N)\|^2_{\mathbf{A_K}^T\mathbf{P}\mathbf{A_K}} \\
&\quad +\mu\phi(\mathbf{u}^*(\tilde{\mathcal{U}})) + \mu_f\phi^{\bar{\alpha}}_f(\mathbf{A_K}\mathbf{x}^o_{\mu,\phi}(k+N),\mathbf{P}) \\
&\quad -\|\mathbf{x}(k)\|^2_\mathbf{Q} - \|\mathbf{u}^o_{\mu,\phi}(k)\|^2_\mathbf{R} - \|\mathbf{x}^o_{\mu,\phi}(k+N)\|^2_\mathbf{P} - \mu\phi(\mathbf{u}^*(\mathcal{U}^o_{k,\mu,\phi})) - \mu_f\phi^{\bar{\alpha}}_f(\mathbf{x}^o_{\mu,\phi}(k+N),\mathbf{P}) \\
&\leq -\|\mathbf{x}^o_{\mu,\phi}(k+N)\|^2_{n\mathbf{I}} + \mu\phi(\mathbf{u}^*(\tilde{\mathcal{U}})) + \mu_f\phi^{\bar{\alpha}}_f(\mathbf{A_K}\mathbf{x}^o_{\mu,\phi}(k+N),\mathbf{P})
\end{aligned}$$

since $\mathbf{P} - \mathbf{A_K}^T\mathbf{P}\mathbf{A_K} - \mathbf{Q} - \mathbf{K}^T\mathbf{R}\mathbf{K} = n\mathbf{I}$. So we require

$$\mu\phi(\mathbf{u}^*(\tilde{\mathcal{U}})) + \mu_f\phi^\alpha_f(\mathbf{A_K}\mathbf{x}^o_{\mu,\phi}(k+N),\mathbf{P}) - n\|\mathbf{x}^o_{\mu,\phi}(k+N)\|^2 \leq 0$$

which is satisfied in X_f, and hence

$$J^o_\phi(\mathbf{x}^o_{\mu,\phi}(k+1) - J^o_\phi(\mathbf{x}(k)) \leq 0 \quad \square$$

Remark 3.2.1 *For given n, we can ensure stability by choosing μ sufficiently small. Moreover, for μ sufficiently small, the above converges to the optimal quadratic programming case which is well studied and known to be stable Mayne et. al. [8].*

Remark 3.2.2 *Suppose as $n \to \infty$, the solution of (26) converges to finite $\mathbf{K}$, then for finite μ we can find an n sufficiently large, such that the conditions for stability can be guaranteed.*

Remark 3.2.3 *The above proof is highly conservative. Since $\mu\phi(\mathbf{u}^*(\tilde{\mathcal{U}})) - \mu\phi(\mathbf{u}^*(\mathcal{U}^o_{k,\mu,\phi}))$ is dependent on the input sequence constraint polytope, which usually has some shift invariance properties, then it will be possible to remove most of the terms in this expression.*

Remark 3.2.4 *The above barrier function falls into a class of* a-self-concordant *barrier functions considered by Nesterov and Nemirovskii [10]. For such functions, they provide an upper bound on the number of iterations required to converge to the optimal solution. They use Newton iterations with pre-defined step lengths for different regions, and combine this with a descending weight approach for this proof. It is accepted in practice, that this bound is highly conservative. By contrast, we are fixing the weighting parameter μ, and using Newton iterations to converge to the solution. Moreover, for $\mu > 0$ it can be shown, following a similar procedure to Nesterov and Nemirovskii [10], that the resulting Hessian for equation (22) is positive definite, which ensures a unique solution. Furthermore, since the solution to (23) is guaranteed to be on the interior of $\mathbb{U}_s$, then this barrier approach may be an excellent candidate for "hot-starting" algorithms. Lastly, the weighting parameter can be used as a tuning-knob for optimality versus computational cost.*

4 Active Constraints in Steady-State

The MPC algorithm described in section 3.2.1 has considerable computational advantages. However, this method requires explicit calculation of $\mathbf{x}_{ss}$ and $\mathbf{u}_{ss}$; this is a standard method for including integral action into MPC, see Muske and Rawlings [9]. Moreover, in the situation where constraints are active in steady-state, we require the solution of a quadratic program to determine the optimal set-point; albeit one of considerably less dimension than the corresponding constrained dynamic optimisation. This is in contrast to the General Predictive Control approach, see Clarke [3], where integration is introduced implicitly via weighting of control moves. Furthermore, our stability proof (and indeed the majority in the literature, Mayne *et. al.* [8]) requires $\mathbf{u}_{ss} \in \text{int}\mathbb{U}$. We propose a sub-optimal method for determining the set-point using a logarithmic barrier function, which for $\mu > 0$, ensures that $\mathbf{u}_{ss} \in \text{int}\mathbb{U}$. Moreover, the Duality Gap theorem allows us to measure system performance degradation for selected μ. We also consider stability as $\mu \to 0$, i.e. as $\mathbf{u}_{ss} \to \delta\mathbb{U}$. In fact, the remarks we make are pertinent to stability results for MPC using invariant terminal constraint sets in general.

4.1 Steady-State Input Determination and the Central Path

The central path is integral to Interior-Point optimisation methods. Associated with the central path are a number of desirable analytical properties including the concept of the Duality Gap. It will be shown that for any point on the central path, the Duality Gap provides an upper bound on system performance degradation. Our proposal then, is to provide a sub-optimal solution to (27) by stopping a predefined distance along the central path. This approach is conceptually different from conventional Interior-Point methods because of the sub-optimal objective. Thus, we require an Interior-Point method with the capability of stopping at any distance along the central path. It turns out that the logarithmic barrier function satisfies these conditions.

We state here for reference the optimal steady-state cost function as,

$$J_{ss}(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{W}_{ss}\mathbf{u} + \mathbf{f}_{ss}^T\mathbf{u}$$

where, for the system matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$

$$\mathbf{W}_{ss} = \left[\mathbf{C}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]^T\left[\mathbf{C}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right] \quad \text{and} \quad \mathbf{f}_{ss} = \left[\mathbf{C}(\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right]^T\mathbf{y}_{ss}$$

We may introduce a logarithmic barrier function as,

$$J_{ss,\mu}(\mathbf{u}) = \frac{1}{2}\mathbf{u}^T\mathbf{W}_{ss}\mathbf{u} + \mathbf{f}_{ss}^T\mathbf{u} + \mu\beta(\mathbf{u})$$

and for $\mu > 0$ we have the unconstrained minimisation problem,

$$\mathbf{u}_{ss,\mu}^o = \operatorname*{argmin}_{\mathbf{u}\in\mathbb{R}^m} J_{ss,\mu}(\mathbf{u}) \tag{27}$$

Moreover, the solution $\mathbf{u}^o_{ss,\mu}$ is guaranteed to be on the interior of the input constraint polytope. We have the following standard result, see Ye [15], which we interpret as measure of system performance for the weighting parameter μ,

$$J_{ss}(\mathbf{u}^o_{ss,\mu}) - J_{ss}(\mathbf{u}^o) \leq \mu M_s \tag{28}$$

where M_s is the number of linear constraints and $\mathbf{u}^o$ is the optimal steady-state input.

Remark 4.1.1 *We propose to solve (27) with μ fixed. In the case where μM_s is required to be very small, iterations of μ may be necessary, see Ye [15] and Nesterov and Nemirovskii [10].*

4.2 Properties of Elliptical Invariant Sets Near the Boundary

In general we would like to consider the set X_f for an arbitrary choice of set-point $\mathbf{x}_{ss}$ such that $-\mathbf{K}\mathbf{x}_{ss} \in \text{int}\mathbb{U}$, but where $-\mathbf{K}\mathbf{x}_{ss}$ tends the boundary of the input constraint polytope.

Observation 1 *As the point $-\mathbf{K}\mathbf{x}_{ss}$ approaches the boundary of $\mathbb{U}$, then the terminal sets X_f defined in Lemma 2.1.1, in Result 3.1 and also in Chen and Allgöwer [2], reduce to the point $\mathbf{x}_{ss}$.*

Proof. For $\epsilon > 0$ there exists an equilibrium point $\mathbf{x}_{ss}$ for the system (2) such that $d(-K\mathbf{x}_{ss}, \delta\mathbb{U}) < \epsilon$ where $\delta\mathbb{U}$ is defined as the boundary of $\mathbb{U}$ and

$$d(\mathbf{u}, \delta\mathbb{U}) := \inf_{\tilde{\mathbf{u}} \in \delta\mathbb{U}} \{\|\mathbf{u} - \tilde{\mathbf{u}}\|_2\} \text{ with } \mathbf{u} \in \text{int}\mathbb{U}$$

Now if we let $\epsilon \to 0$ then this implies $-\mathbf{K}\mathbf{x}_{ss} \to \delta\mathbb{U}$ and consequently $\alpha_{max} \to 0$. The above observation suggests that as the equilibrium point approaches the boundary (in some sense), then the terminal set X_f tends to the point $\mathbf{x}_{ss}$.

Remark 4.2.1 *The feasibility of constrained receding horizon control problems is a pivotal concept in MPC stability [13][8]. It was suggested in Mayne and Michalska [7] that considerable demands are placed on the optimisation algorithm when using a terminal equality constraint. In the same paper they offer an alternative relaxed terminal constraint set which leads to improved feasibility conditions. Therefore in light of the above observation, feasibility is an issue when constraints are active in steady-state. However, this is not true in general. In the above formulation of X_f we have selected a particular Lyapunov function. It may be possible, however, to construct a terminal constraint set that has volume > 0 on the boundary. For more detail on this see section 4.3 below.*

Remark 4.2.2 *Defining the terminal constraint set as the equilibrium point $\mathbf{x}_{ss}$ has some practical disadvantages [7]. From corollary 1 we see that this condition is imposed rather than defined. Avoiding constraints in steady-state is a candidate remedy for infeasibility. This, however, may contradict the optimal control objective. Therefore, it may be necessary to compromise performance for feasibility.*

4.3 Maximal invariant sets near the boundary

We have shown that the volume of elliptical invariant sets tend to 0 as $d(\mathbf{u}_{ss}, \delta\mathbb{U}) \to 0$. Gilbert and Tan [5] show (by constructing a simple example) that there may exist an invariant set with finite, but non-zero volume as $d(\mathbf{u}_{ss}, \delta\mathbb{U}) \to 0$. In general the existence of such an invariant set depends on both the properties of $\mathbf{A_K}$ and the constraint boundary. We illustrate this by giving sufficient conditions both for such a set to exist, and, for there to be no such set. The latter case is problematic for stability proofs, as discussed in remarks 4.2.1 and 4.2.2.

We will find it convenient to paramaterize a continuous path such that $\mathbf{u}_{ss} \in \text{int}\mathbb{U}$ for all $\mu > 0$ but such that $\mathbf{u}_{ss} \in \delta\mathbb{U}$ in the limit as $\mu \to 0$. One such path is, of course, the central path. We will also shift co-ordinates so that $\mathbf{u}_{ss} = 0$ for all μ and hence we may consider $\mathbb{U}$ as a function of μ, i.e. $\mathbb{U} = \mathbb{U}(\mu)$. For clarity of exposition, we have included two diagrams for the following result, see figure 2.

Result 4.1 *If all the eigenvalues of* $\mathbf{A_K} = \mathbf{A} - \mathbf{BK}$ *are real, and on the interval* $(0, 1)$*, and* $\mathbf{u}_{ss}$ *is not on a vertex of* $\delta\mathbb{U}(0)$*, then we can define an invariant set* $X_f(\mu)$ *such that*

$$\text{vol}\,[X_f(\mu)] \geq V_{min} > 0$$

for some V_{min} *independent of* μ*.*

Proof. Suppose $\mathbf{A_K} = \mathbf{A} - \mathbf{KB}$ has all its eigenvalues real and on the interval $(0, 1)$. Denote the corresponding eigenvectors as $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$. The n hyperplanes defined by

$$H_j = \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \sum_{i \neq j} x_i \mathbf{e}_i, \quad x_i \in \mathbb{R} \right\} \text{ for } j = 1, 2, \ldots, n_a$$

are each invariant under $\mathbf{A_K}$. The set of hyperplanes divides $\mathbb{R}^n$ into 2^{n_a} cones C_j, and it follows that each of these cones is also invariant under $\mathbf{A_K}$. Assume without loss of generality that $\mathbf{K}$ has full row rank. We define X_α^K as

$$X_\alpha^K := \{\mathbf{u} \in \mathbb{R}^m : \exists \mathbf{x} \in X_\alpha \text{ with } \mathbf{u} = -\mathbf{Kx}\}$$

for X_α as defined in (8). We define H_j^K and C_j^K similarly for the hyperplane H_j and cone C_j respectively. Observe that X_α^K is an ellipsoid containing $\mathbf{u}_{ss}$, and similarly H_j^K is a hyperplane containing $\mathbf{u}_{ss}$. The set of such hyperplanes divide $\mathbb{R}^m$ into the cones C_j^K.

If $\mathbf{u}_{ss}$ is not on a vertex of $\delta\mathbb{U}(0)$, then $\delta\mathbb{U}(\mu)$ may be characterised locally as a set of parallel hyperplanes $H_u(\mu)$ in $\mathbb{R}^m$ that do *not* contain $\mathbf{u}_{ss}$ for $\mu > 0$. Define $\mathbb{H}$ as the half-space defined by $H_u(0)$ with $H_u(\mu) \cap \mathbb{H} = \emptyset$ for $\mu > 0$. Since $\mathbf{K}$ has full row rank the projected eigenvectors span $\mathbb{R}^m$ and there exists some j such that $\bar{C}^K \subset \mathbb{H}$ with $\bar{C} = C_j$, where $\bar{C}^K$ denotes the projection of $\bar{C}$ onto $\mathbb{R}^m$ under $\mathbf{K}$.

Define $\bar{\alpha}_{max}(\mu)$ as the maximum value of α such that

$$X_\alpha^K \cap \bar{C}^K \subseteq \mathbb{U}(\mu)$$

and $\bar{\alpha}$ as the minimum value of $\bar{\alpha}_{max}(\mu)$ over the interval $\mu \in (0, M]$ for some $M > 0$.

By construction $\mathbf{u}_{ss} \in X_{\bar{\alpha}}^K \cap \bar{C}^K$ for all $\mu \in (0, M]$ with $\bar{\alpha}$ non-zero and independent of μ. We can then define

$$X_f(\mu) = X_{\alpha_{max}}(\mu) \cup \left(X_{\bar{\alpha}} \cap \bar{C}\right)$$

Then

$$\mathbf{u}_{ss} \in \text{int} X_f^K(\mu) \text{ for all } \mu \in (0, M]$$

and

$$\text{vol}\,[X_f(\mu)] \geq \text{vol}\left[X_{\bar{\alpha}} \cap \bar{C}\right]$$

with the volume on the right hand side independent of μ.

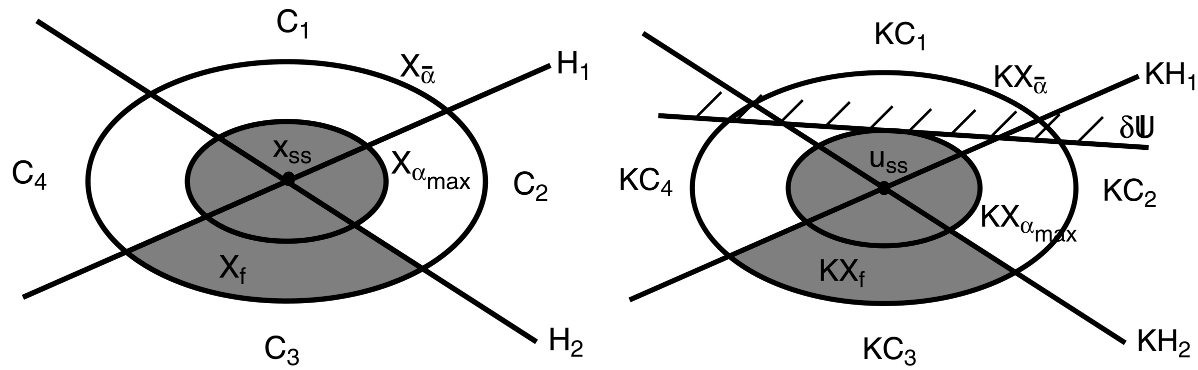
Remark 4.3.1 *The set we have constructed is not convex. If this is a requirement, we may simply take* $X_f(\mu) = \left[X_{\bar{\alpha}} \cap \bar{C}\right]$*. Note that* $\mathbf{x}_{ss}$ *lies on the boundary of such an invariant set.*

Result 4.2 *If all the eigenvalues of* $\mathbf{A_K} = \mathbf{A} - \mathbf{BK}$ *are complex, then*

$$vol[X_f(\mu)] \to 0$$

for any *choice of invariant set* $X_f(\mu)$*.*

Proof. Suppose $\mathbf{A_K} = \mathbf{A} - \mathbf{KB}$ has all its eigenvalues complex and we have a densely filled $X_f(\mu)$ whose volume is non-zero as $\mu \to 0$. Then by linearity there must be an invariant cone C with $X_f(0) \subset C$ with its vertex at $\mathbf{x}_{ss} = 0$. Again by linearity the negative extension of C is also invariant, and hence the reflecting hyperplane between C and its negative extension is also invariant. But the existence of such a hyperplane implies that $\mathbf{A_K}$ has a real eigenvalue, contradicting our original assumption.



(a) State Space

(b) Input Space

Figure 2: Invariant set for result 4.1

5 Conclusion

In this paper we have introduced a novel barrier function model predictive controller that is guaranteed stable and to converge to any pre-specified steady-state point on the interior of the constraint set. This approach is conceptually distinguishable from traditional Interior Point methods since we have fixed the weighting parameter $\mu > 0$. This may have implications for hot-starting since for $\mu > 0$, and $\mathbf{u}_{ss} \in \text{int}\mathbb{U}$ we are guaranteed to lie on the interior of the constraint polytopes $\mathbb{U}$ and $\mathbb{U}_s$. Moreover, this method also provides a convenient framework for analysing performance versus computational efficiency in model predictive control. We have re-addressed the issue of active steady-state constraints with the introduction of a sub-optimal approach that ensures that $\mathbf{u}_{ss} \in \text{int}\mathbb{U}$. Furthermore, using this approach we are able to pre-define a bound on system performance degradation. In some cases setting this bound too small may be detrimental to feasibility. Finally, we have illustrated sufficient conditions for maximal invariant terminal constraint sets to have volume greater than zero and equal to zero as $\mathbf{u}_{ss}$ approaches the boundary of $\mathbb{U}$.

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