Recentred barriers for model predictive control with state constraints

W. P. Heath* and A. G. Wills†

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Abstract

Model predictive control requires the minimization of a cost function at each control step. Recently the authors have proposed controlling plants with input constraints by including the constraints as a barrier with fixed weighting in the cost function. In effect the constrained model predictive control problem then requires an unconstrained nonlinear optimization at each control step. In particular, if the original cost function (without constraints) is smooth and convex, then a class of controllers is generated for the constrained problem where the cost to be minimized is also smooth and convex and whose gradient is zero at the optimal solution. In this paper we show that the idea may be straightforwardly generalised to plants with state constraints. We illustrate the idea with a simulation of a plant with rate constraints on the actuators.

1 Introduction

Model predictive control usually requires the minimisation of a cost function at each control step. In the case where system constraints are deemed important it

*Centre for Integrated Dynamics and Control, School of Electrical Engineering and Computer Science, University of Newcastle, NSW 2308, Australia. Email: wheath@ee.newcastle.edu.au; Tel: +61 2 4921 5997; Fax: +61 2 4960 1712.

†School of Electrical Engineering and Computer Science, University of Newcastle, NSW 2308, Australia.

is customary to include them in the optimization problem as inequality constraints (e.g. Mayne *et al.*, 2000). As an alternative it is possible to ensure that constraints are satisfied by including a convex barrier in the cost function. The idea is quite standard for interior point methods (Fiacco and McCormick, 1968) where a weighting is included on the barrier. As the weighting parameter tends to zero, the solution tends towards that of the original inequality constrained optimization problem.

Recently the authors proposed including a barrier for the constraints with a fixed weighting (Wills and Heath, 2002*a*); the resulting controller can be shown to have certain favourable dynamic properties. When the weighting parameter is sufficiently large the optimization problem can be solved using simple Newton steps. Alternatively, for linear systems with linear and/or convex quadratic constraints, the optimization problem can be solved extremely efficiently using simple modifications to state-of-the-art interior point machinery (Wills and Heath, 2002*b*,*c*).

Only static input constraints were considered in (Wills and Heath, 2002*a*), where a recentred barrier was introduced to ensure correct steady state behaviour. In this paper we extend the idea to convex state constraints, and give a necessary condition on the barrier to ensure correct steady state behaviour. Once again the condition is satisfied by a recentred barrier. If the model is linear we are able to ensure stability via the construction of a terminal constraint set analogous to that of Chen and Allgower (1998). We illustrate the idea for a plant with rate constraints on the actuators.

With non-zero weighting our proposed design transforms the limiting case control law (with inequality constraints) to an unconstrained non-linear control law. Recently Jadbabaie *et al.* (2001) have suggested that including terminal constraints for a similar class of control law is unnecessary. We discuss the implications for stability of constrained model predictive control.

2 Barrier functions and steady state behaviour

Consider a discrete system

$$x_{t+1} = f(x_t, u_t) \text{ with } x_t \in \mathbb{R}^n,\ u_t \in \mathbb{R}^m \tag{1}$$

subject to constraints

$$x_t \in \mathbb{X},\ u_t \in \mathbb{U} \text{ for all } t \tag{2}$$

where $\mathbb{X}$ and $\mathbb{U}$ are the convex sets

$$\begin{aligned}\mathbb{X} &= \{x \in \mathbb{R}^n \colon\ c_{x,i}(x) \leq 0,\ i = 1, \ldots, M^x\} \\ \mathbb{U} &= \{u \in \mathbb{R}^m \colon\ c_{u,i}(u) \leq 0,\ i = 1, \ldots, M^u\} \end{aligned} \tag{3}$$

We will impose as part of our control law the additional terminal constraint

$$x_{t+N} \in \mathbb{X}_f \subseteq \mathbb{X} \tag{4}$$

with

$$\mathbb{X}_f = \mathbb{X} \cap \{x \in \mathbb{R}^n \colon\ c_{f,i}(x) \leq 0,\ i = 1, \ldots, M^f\} \tag{5}$$

Let X_t and U_t be the sequences

$$\begin{aligned} X_t &= [x_{t+1}, x_{t+2}, \ldots, x_{t+N}] \\ U_t &= [u_t, u_{t+1}, \ldots, u_{t+N-1}] \end{aligned} \tag{6}$$

which are required to lie in the constraint sets $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$, the respective natural extensions of $\mathbb{X}$ (with $\mathbb{X}_f$) and $\mathbb{U}$

$$\begin{aligned} \bar{\mathbb{X}} &= \{X_t : x_{t+i} \in \mathbb{X}, i = 1, \ldots, N \text{ and } x_{t+N} \in \mathbb{X}_f\} \\ \bar{\mathbb{U}} &= \{U_t : u_{t+i} \in \mathbb{U}, i = 0, \ldots, N-1\} \end{aligned} \tag{7}$$

Suppose we desire x_t and u_t to reach x_{ss} and u_{ss} respectively in steady state. The natural constraints on x_{ss} and u_{ss} are

$$x_{ss} = f(x_{ss}, u_{ss}),\ x_{ss} \in \mathbb{X}_f,\ u_{ss} \in \mathbb{U} \tag{8}$$

We will impose the further restriction

$$x_{ss} \in \text{int}\mathbb{X}_f,\ u_{ss} \in \text{int}\mathbb{U} \tag{9}$$

and assume x_{ss} and u_{ss} exist. Similarly we will assume the interiors of $\bar{\mathbb{X}}$ and $\bar{\mathbb{U}}$ are non-empty.

Remark 1 *Often x_{ss} and u_{ss} are themselves computed via some separate constrained optimization (Muske and Rawlings, 1993). Suppose such a constrained optimization may be expressed as*

$$(x^*_{ss}, u^*_{ss}) = \arg\min_{u,x} J_{ss}(x, u) \ \textit{subject to (8)} \tag{10}$$

We propose to compute x_{ss} and u_{ss} as

$$(x_{ss}, u_{ss}) = \arg\min_{u,x} \{J_{ss}(x,u) + \mu_{ss} B_{ss}(x,u)\} \text{ subject to (8)} \tag{11}$$

where $B_{ss}(x,u)$ is an appropriate self-concordant barrier (Nesterov and Nemirovski-i, 1994) and $\mu_{ss} > 0$. The barrier ensures (9) is satisfied. Meanwhile, provided μ is small, the duality gap ensures that for a wide class of cost and barrier that $J_{ss}(x_{ss}, u_{ss}) - J_{ss}(x_{ss}^*, u_{ss}^*)$ is small.

We propose the following model predictive control law.

Algorithm: *For some suitable weighting matrices P, Q and R let*

$$J_t(X_t, U_t) = ||x_{t+N} - x_{ss}||_P^2 + \sum_{i=0}^{N-1} \left(||x_{t+i} - x_{ss}||_Q^2 + ||u_{t+i} - u_{ss}||_R^2 \right) \tag{12}$$

Let $B(X_t, U_t)$ be some suitable self-concordant barrier for the constraint set $\bar{\mathbb{X}} \times \bar{\mathbb{U}}$ and choose some fixed $\mu > 0$. We may write

$$X_t = \tilde{f}(x_t, U_t) \tag{13}$$

for some $\tilde{f}$ and hence define

$$\begin{aligned} J(X_t, U_t) &= \tilde{J}(x_t, U_t) \\ B(X_t, U_t) &= \tilde{B}(x_t, U_t) \end{aligned} \tag{14}$$

Then at each time step t, given x_t:

1. *Compute the sequence U_t^o as*

$$U_t^o(x_t) = \arg\min_{U_t} \left\{ \tilde{J}(x_t, U_t) + \mu \tilde{B}(x_t, U_t) \right\} \tag{15}$$

2. *Implement $u_t^o(x_t)$, the first term in $U_t^o(x_t)$*

Let U_{ss} be U_t evaluated at $u_{t+i} = u_{ss}$ for $i = 0, \ldots, N-1$, and let $\tilde{B}_{ss}(U_t)$ be $\tilde{B}(x_t, U_t)$ evaluated at $x_t = x_{ss}$. Then since $\tilde{J}(x_{ss}, U_t)$ attains its minimum at $U_t = U_{ss}$, and since U_{ss} lies on the interior of $\bar{\mathbb{U}}$, a necessary condition for the system to reach x_{ss} and u_{ss} in steady state is

$$\nabla \tilde{B}_{ss}(U_{ss}) = 0 \tag{16}$$

Let u^j_{t+i} denote the jth element of the vector u_{t+i}, and define x^l_{t+k} similarly. Then

$$\frac{\partial \tilde{B}(x_t, U_t)}{\partial u^j_{t+i}} = \frac{\partial B(X_t, U_t)}{\partial u^j_{t+i}} + \sum_{k=i+1}^{N} \sum_{l=1}^{n} \frac{\partial B(X_t, U_t)}{\partial x^l_{t+k}} \frac{\partial x^l_{t+j}}{\partial u^j_{t+i}} \tag{17}$$

Hence a sufficient condition for (16) is

$$\nabla B(X_{ss}, U_{ss}) = 0 \tag{18}$$

Note that the gradient in (16) is with respect to the elements of U_t, while the gradient in (18) is with respect to the elements of U_t and X_t. Given an arbitrary self-concordant barrier $F(X_t, U_t)$ for the constraint set $\bar{\mathbb{X}} \times \bar{\mathbb{U}}$, we may construct a recentred barrier $B(X_t, U_t)$ satisfying (18) as

$$B(X_t, U_t) = F(X_t, U_t) - \langle \nabla F(X_{ss}, U_{ss}), (X_t, U_t) \rangle \tag{19}$$

Remark 2 *It follows immediately that $B(X_t, U_t)$ is a convex barrier for $\bar{\mathbb{X}} \times \bar{\mathbb{U}}$ (see also Nesterov and Nemirovskii, 1994, p69). Furthermore, if complexity results for the barrier $F(X_t, U_t)$ are available, then complexity results for solving (15) can be obtained via subsuming the linear term $\langle \nabla F(X_{ss}, U_{ss}), (X_t, U_t) \rangle$ into the cost $J(X_t, U_t)$—see Wills and Heath (2002*b,c*).*

Example: the recentred logarithmic barrier

Suppose we choose $F(X_t, U_t)$ as the logarithmic barrier for $\bar{\mathbb{X}} \times \bar{\mathbb{U}}$

$$\begin{aligned} F(X_t, U_t) &= -\sum_{i=1}^{N} \sum_{j=1}^{M^x} \ln(-c_{x,j}(x_{t+i})) - \sum_{j=1}^{M^f} \ln(-c_{f,j}(x_{t+N})) \\ &\quad - \sum_{i=1}^{N} \sum_{j=1}^{M^u} \ln(-c_{u,j}(u_{t+i})) \end{aligned} \tag{20}$$

Then

$$\nabla F(X_t, U_t) = \Bigg[\left(-\sum_{j=1}^{M^x} \frac{\nabla c_{x,j}(x_{t+1})}{c_{x,j}(x_{t+1})}\right)^T, \ldots, \left(-\sum_{j=1}^{M^x} \frac{\nabla c_{x,j}(x_{t+N})}{c_{x,j}(x_{t+N})}\right)^T,$$
$$\left(-\sum_{j=1}^{M^f} \frac{\nabla c_{f,j}(x_{t+N})}{c_{f,j}(x_{t+N})}\right)^T, \left(-\sum_{j=1}^{M^u} \frac{\nabla c_{u,j}(u_{t+1})}{c_{u,j}(u_{t+1})}\right)^T, \ldots, \left(-\sum_{j=1}^{M^u} \frac{\nabla c_{u,j}(u_{t+N})}{c_{u,j}(u_{t+N})}\right)^T\Bigg]^T \tag{21}$$

and the recentred barrier is

$$\begin{aligned} B(X_t, U_t) &= \sum_{i=1}^{N} \left(\sum_{j=1}^{M^x} \frac{\nabla c_{x,j}(x_{ss})}{c_{x,j}(x_{ss})}\right)^T x_{t+i} + \left(\sum_{j=1}^{M^f} \frac{\nabla c_{f,j}(x_{ss})}{c_{f,j}(x_{ss})}\right)^T x_{t+N} \\ &\quad + \sum_{i=1}^{N} \left(\sum_{j=1}^{M^u} \frac{\nabla c_{u,j}(u_{ss})}{c_{u,j}(u_{ss})}\right)^T u_{t+i} + F(X_t, U_t) \end{aligned} \tag{22}$$

In particular, if a state constraint $c_{x,j}(x_{t+i})$ is linear with form

$$c_{x,j}(x_{t+i}) = a_{x,j}^T x_{t+i} - b_{x,j} \tag{23}$$

then

$$\left(\frac{\nabla c_{x,j}(x_{ss})}{c_{x,j}(x_{ss})}\right)^T x_{t+i} = \frac{a_{x,j}^T x_{t+i}}{a_{x,j}^T x_{ss} - b_{x,j}} \tag{24}$$

Figs 1 and 2 illustrate such recentring for a very simple state constraint.

3 Closed-loop stability via terminal constraint set

In the sequel we will restrict the system (1) to be linear

$$x_{t+1} = Ax_t + Bu_t \text{ with } x_t \in \mathbb{R}^n,\ u_t \in \mathbb{R}^m \tag{25}$$

In (Wills and Heath, 2002*a*) stability for MPC with a recentred barrier was demonstrated via the construction of a terminal constraint set similar to that of Chen and

Allgower (1998). Only linear and quadratic input constraints were considered, but the extension to more general convex constraints on both the inputs and the states is straightforward. We include a construction and stability proof for completeness.

Let Q and R be positive definite symmetric matrices. Further let $Q^* = \eta I + Q$ for some scalar $\eta \geq 0$. We consider the following algebraic Riccati equation:

$$\begin{aligned} P &= (A - BK)^T P (A - BK) + Q^* + K^T R K \\ K &= (B^T P B + R)^{-1} B^T P A \end{aligned} \tag{26}$$

If system (25) is stabilisable, then there exist both a unique positive definite symmetric matrix P which satisfies (26) and a linear stabilising control law $\kappa_f(x) := u_{ss} - K(x - x_{ss})$.

We will assume that the barrier $B(X_t, U_t)$ is constructed as

$$B(X_t, U_t) = \sum_{i=1}^{N} B_x(x_{t+i}) + B_f(x_{t+N}) + \sum_{i=0}^{N-1} B_u(u_{t+i}) \tag{27}$$

where B_x, B_f and B_u are suitably recentred barriers for $\mathbb{X}$, $\mathbb{X}_f$ and $\mathbb{U}$ respectively. We will further assume each barrier term is zero at its minimum. We have the following lemma:

Lemma 1 *For the stabilising controller $\kappa_f(x)$ determined above we can define a series of regions $\mathbb{X}_\alpha$, sets $\mathbb{X}_u$ and $\mathbb{X}_\eta$ and a scalar $\alpha_{\max} > 0$ as:*

$$\begin{aligned} \mathbb{X}_\alpha &= \{x \in \mathbb{R}^n : (x - x_{ss})^T P (x - x_{ss}) < \alpha\} \\ \mathbb{X}_\eta &= \{x \in \mathbb{R}^n : \eta ||x - x_{ss}||^2 \geq \mu[B_u(\kappa_f(x)) + B_x(Ax + B\kappa_f(x))]\} \\ \alpha_{\max} &= \max\{\alpha > 0 : \mathbb{X}_\alpha \subseteq \{\mathbb{X}_\eta \cap \mathbb{X}\}\} \end{aligned} \tag{28}$$

Furthermore we can define a terminal constraint set $\mathbb{X}_f = \mathbb{X}_{\alpha_{\max}}$ that is invariant under the control law $\kappa_f(x)$.

Proof: *Since $\eta > 0$ may be chosen to be arbitrarily large and $u_{ss} \in \text{int}\mathbb{U}$ we can always obtain non-empty sets satisfying (28), and hence $\alpha_{\max} > 0$. We have*

$$x \in \mathbb{X}_\eta \Rightarrow \{\kappa_f(x) \in \mathbb{U} \text{ and } Ax + B\kappa_f(x) \in \mathbb{X}\} \tag{29}$$

It follows immediately that $\mathbb{X}_{\alpha_{\max}}$ is invariant under the control law $\kappa_f(x)$ (see for example Chen and Allgower, 1998). □

We will choose $B_f(x)$ to be the logarithmic barrier for the constraint set $\mathbb{X}_f$ defined above:

$$B_f(x) \quad = \quad \ln(\alpha_{\max}) - \ln\left(\alpha_{\max} - (x - x_{ss})^T P (x - x_{ss})\right) \tag{30}$$

Since

$$\nabla B_f(x_{ss}) = 0 \tag{31}$$

the barrier does not require recentring.

Let $X_t^o(x_t)$ be the state evolution corresponding to the optimal control sequence $U_t^0(x_t)$ (15). Define also the control sequence $U_{t+1}^1(x_t)$ with corresponding state evolution $X_{t+1}^1(x_{t+1})$ as

$$\begin{aligned} U_{t+1}^1(x_t) &= [u_{t+1}^o(x_t), \ldots, u_{t+N-1}^o(x_t), u_{t+N}^1(x_t)] \\ X_{t+1}^1(x_t) &= [x_{t+2}^o(x_t), \ldots, x_{t+N}^o(x_t), x_{t+N+1}^1(x_t)] \end{aligned} \tag{32}$$

with

$$\begin{aligned} u_{t+N}^1(x_t) &= \kappa_f(x_{t+N}^o(x_t)) \\ x_{t+N+1}^1(x_t) &= A x_{t+N}^o(x_t) + B u_{t+N}^1(x_t) \end{aligned} \tag{33}$$

Note we have the relation

$$x_{t+N+1}^1(x_t) - x_{ss} = (A - BK)(x_{t+N}^o(x_t) - x_{ss}) \tag{34}$$

Let $\mathbb{X}_N$ be the set of states x for which $U_t^o(x)$ and $X_t^o(x)$ exist. Then we can say

Lemma 2 *If $x_t \in \mathbb{X}_N$ then (i) $U_{t+1}^1(x_t)$ is feasible, (ii) the corresponding state evolution $X_{t+1}^1(x_t)$ is feasible and (iii) the following inequality holds.*

$$\tilde{J}(x_{t+1}^o, U_{t+1}^1(x_t)) + \mu \tilde{B}(x_{t+1}^o, U_{t+1}^1(x_t)) \leq \tilde{J}(x_t^o, U_t^o(x_t)) + \mu \tilde{B}(x_t^o, U_t^o(x_t)) \tag{35}$$

Proof:

(i) Since $x_t \in \mathbb{X}_N$ the first $N-1$ terms in $U_{t+1}^1(x_t)$ each lie in $\mathbb{U}$. Furthermore the state $x_{t+N}^o(x_t) \in \mathbb{X}_f$. From (28) we have $u_{t+N}^1(x_t) \in \mathbb{U}$ and hence $U_{t+1}^1(x_t) \in \bar{\mathbb{U}}$.

(ii) Since $x_t \in \mathbb{X}_N$ the first $N-1$ terms of $X^1_{t+1}(x_t)$ each lie in $\mathbb{X}$. Since $\mathbb{X}_f$ is invariant under control law κ_f and $x^o_{t+N}(x_t) \in \mathbb{X}_f$ it follows that $x^1_{t+N+1}(x_t)$ also lies in $\mathbb{X}_f$. Hence $X^1_{t+1}(x_t) \in \bar{\mathbb{X}}$.

(iii) For conciseness we will drop dependence on x_t from our notation. From (26) we have

$$\begin{aligned} &\tilde{J}(x^o_{t+1}, U^1_{t+1}) - \tilde{J}(x_t, U^o_t) \\ &= ||x^1_{t+N+1} - x_{ss}||^2_P + ||x^0_{t+N} - x_{ss}||^2_Q + ||u^1_{t+N} - u_{ss}||^2_R \\ &\quad -||x^o_{t+N} - x_{ss}||^2_P - ||x_t - x_{ss}||^2_Q - ||u^o_t - u_{ss}||^2_R \\ &= -\eta||x^o_{t+N} - x_{ss}||^2 - ||x_t - x_{ss}||^2_Q - ||u^o_t - u_{ss}||^2_R \end{aligned} \tag{36}$$

Similarly

$$\begin{aligned} \tilde{B}(x^o_{t+1}, U^1_{t+1}) - \tilde{B}(x_t, U^o_t) &= B_x(x_{t+N+1}) - B_x(x^o_{t+1}(x_t)) \\ &+ B_f(x_{t+N+1}) - B_f(x^o_{t+N}(x_t)) + B_u(u_{t+N}) - B_u(u^o_t(x_t)) \end{aligned} \tag{37}$$

From the definition of $\mathbb{X}_\eta$ we thus find

$$\begin{aligned} \tilde{J}(x^o_{t+1}, U^1_{t+1}) + \mu\tilde{B}(x^o_{t+1}, U^1_{t+1}) - \tilde{J}(x_t, U^o_t) - \mu\tilde{B}(x_t, U^o_t) \\ \leq \mu[B_f(x^1_{t+N+1}) - B_f(x^o_{t+N})] \end{aligned} \tag{38}$$

and it suffices to observe

$$\begin{aligned} B_f(x^1_{t+N+1}) - B_f(x^o_{t+N}) &= \ln\left(\frac{\alpha_{\max} - ||x^o_{t+N} - x_{ss}||^2_P}{\alpha_{\max} - ||x^1_{t+N+1} - x_{ss}||^2_P}\right) \\ &= \ln\left(\frac{\alpha_{\max} - ||x^o_{t+N} - x_{ss}||^2_P}{\alpha_{\max} - ||x_{t+N} - x_{ss}||^2_{P-Q^*-K^TRK}}\right) \\ &\leq 0 \end{aligned} \tag{39}$$

□

We may now state:

Result 1 *Suppose we construct a barrier according to (27) with B_u and B_x appropriate convex recentred barriers and B_f given by (30) for the terminal constraint set $\mathbb{X}_f$ given by (28). Then for $x_t \in \mathbb{X}_N$ the model predictive control law is stable for the linear plant (25).*

Proof: *It suffices to show that*

$$\begin{aligned}&\tilde{J}(x^o_{t+1}, U^o_{t+1}(x^o_{t+1})) + \mu\tilde{B}(x^o_{t+1}, U^o_{t+1}(x^o_{t+1}))\\ &\qquad \leq \tilde{J}(x_t, U^o_t(x_t)) + \mu\tilde{B}(x_t, U^o_t(x_t))\end{aligned} \tag{40}$$

But since $U^o_{t+1}(x^o_{t+1})$ is chosen optimally

$$\begin{aligned}&\tilde{J}(x^o_{t+1}, U^o_{t+1}(x^o_{t+1})) + \mu\tilde{B}(x^o_{t+1}, U^o_{t+1}(x^o_{t+1}))\\ &\qquad \leq \tilde{J}(x^o_{t+1}, U^1_{t+1}(x_t)) + \mu\tilde{B}(x^o_{t+1}, U^1_{t+1}(x_t))\end{aligned} \tag{41}$$

and the result follows. □

4 Illustration: plant with rate constraints on the actuators

Consider a linear SISO (single input single output) system

$$\tilde{x}_{t+1} = \tilde{A}\tilde{x}_t + \tilde{B}u_t \tag{42}$$

with absolute and rate constraints on the inputs

$$\begin{aligned} u_t &\leq \upsilon_{\max}\\ u_t &\geq \upsilon_{\min}\\ u_t - u_{t-1} &\leq \Delta\upsilon_{\max}\\ u_t - u_{t-1} &\geq \Delta\upsilon_{\min}\end{aligned} \tag{43}$$

Let the desired steady state input be u_{ss}. Note that even for a regulation problem u_{ss} will not in general be zero due to plant disturbances.

A choice of recentred logarithmic barrier for the absolute constraints would be

$$B_u(u_t) = \ln\left(\frac{\upsilon_{\max} - u_{ss}}{\upsilon_{\max} - u_t}\right) + \ln\left(\frac{u_{ss} - \upsilon_{\min}}{u_t - \upsilon_{\min}}\right) - \frac{u_t - u_{ss}}{\upsilon_{\max} - u_{ss}} + \frac{u_t - u_{ss}}{u_{ss} - \upsilon_{\min}} \tag{44}$$

When $\upsilon_{\min} = -\upsilon_{\max}$ this reduces to

$$B_u(u_t) = \ln\left(\frac{\upsilon^2_{\max} - u^2_{ss}}{\upsilon^2_{\max} - u^2_t}\right) + \frac{2u_{ss}(u_{ss} - u_t)}{\upsilon^2_{\max} - u^2_{ss}} \tag{45}$$

In steady state $u_{t+1}-u_t=0$ so a choice of logarithmic barrier for the rate constraints would be

$$\begin{aligned} B_\Delta(u_{t+1},u_t) &= \ln\left(\frac{\Delta v_{\max}}{\Delta v_{\max}-u_{t+1}+u_t}\right)+\ln\left(\frac{-\Delta v_{\min}}{u_{t+1}-u_t-\Delta v_{\min}}\right) \\ &\quad -\frac{u_{t+1}-u_t}{\Delta v_{\max}}-\frac{u_{t+1}-u_t}{\Delta v_{\min}} \end{aligned} \tag{46}$$

When $\Delta v_{\min}=-\Delta v_{\max}$ this reduces to

$$B_\Delta(u_{t+1},u_t) = \ln\left(\frac{\Delta v_{\max}^2}{\Delta v_{\max}^2-(u_{t+1}-u_t)^2}\right) \tag{47}$$

and no recentring is necessary.

If we consider the rate constraints as input constraints they do not appear in the form we originally assumed (2,3). Rather we must augment the plant and consider them as state constraints. The augmented system

$$x_{t+1}=Ax_t+Bu_t \tag{48}$$

is formed with

$$A=\begin{bmatrix}\tilde{A} & 0 & 0\\ 0 & 0 & 0\\ 0 & -1 & 0\end{bmatrix},\ B=\begin{bmatrix}\tilde{B}\\ 1\\ 1\end{bmatrix} \tag{49}$$

We may then express the rate constraint as a constraint on the nth state x_t^n:

$$B_x(x_t) = \ln\left(\frac{\Delta v_{\max}}{\Delta v_{\max}-x_t^n}\right)+\ln\left(\frac{-\Delta v_{\min}}{x_t^n-\Delta v_{\min}}\right)-\frac{x_t^n}{\Delta v_{\max}}-\frac{x_t^n}{\Delta v_{\min}} \tag{50}$$

and when $\Delta v_{\min}=-\Delta v_{\max}$ this reduces to

$$B_x(x_t) = \ln\left(\frac{\Delta v_{\max}^2}{\Delta v_{\max}^2-(x_t^n)^2}\right) \tag{51}$$

It is now straightforward to construct a terminal constraint set according to (28).

Figs 3 to 6 illustrate results from a specific simulation example. The plant is

$$\begin{aligned} \tilde{x}_{t+1} &= \begin{bmatrix}1.1161 & -0.2780\\ 1 & 0\end{bmatrix}\tilde{x}_t+\begin{bmatrix}1\\ 0\end{bmatrix}u_t \\ y_t &= \begin{bmatrix}0.3331 & 0.2176\end{bmatrix}\tilde{x}_t \end{aligned} \tag{52}$$

An receding horizon controller is implemented with a 20 sample horizon. The state weighting matrices Q and R are given by

$$\begin{aligned} Q &= 0.01 \times C^T C \\ R &= 1 \end{aligned} \tag{53}$$

while P is found by solving the appropriate Riccati equation. A simple 'DMC scheme' observer is used (Maciejowski, 2002) with integral action incorporated via an additional disturbance observer. In all cases the absolute value of u_t is constrained to $|u_t| < 0.5$. No terminal constraints are introduced.

Fig 3 shows unit step responses (up and down) for the case where rate constraints are symmetric $|u_{t+1} - u_t| < 0.1$. Three responses are shown: (i) with a limiting case inequality constrained model predictive control law (with the corresponding optimization problem solved via an active set method at each control step), (ii) with a barrier included in the cost function with weighting $\mu = 10^{-6}$ and (iii) with a barrier included in the cost function and a weighting $\mu = 1$. To the eye cases (i) and (ii) are indistinguishable, while the response for (iii) is considerably smoother.

Fig 4 illustrates responses under similar conditions, but this time the rate constraints are given by $-0.05 < u_{t+1} - u_t < 0.1$. The barriers for cases (ii) and (iii) are not recentred, and significant steady state error occurs for case (iii). Fig 5 illustrates responses where the barriers are recentred. This time the steady state behaviour is correct. Finally Fig 6 illustrates the evolutions of $u_{t+1} - u_t$ under conditions corresponding to those of Fig 5.

5 Discussion: constrained MPC posed as unconstrained non-linear optimization

Model predictive control is inherently non-linear. It has been observed that if both the plant and constraints are linear then a model predictive control law with a quadratic cost may be piecewise affine (Bemporad *et al.*, 2000, 2002; Seron *et al.*, 2000)—the result follows immediately from the KKT (Karush-Kuhn-Tucker) conditions for optimality, and is indeed exploited in active set methods for solving quadratic programs (e.g. Gill *et al.*, 1981). But the idea cannot be extended to more general plants or constraints (nor indeed to more general convex cost functions). Kouvaritakis *et al.* (2002) also question the practicality of using the observation to trade off on-line complexity with off-line computation. Note in particular that convex state constraints can be usefully incorporated to ensure robustness (Kothare *et al.*, 1996; Hansson, 2000).

Interior point algorithms offer an alternative to active set methods for solving constrained optimization problems, and may be applied efficiently to a wide class of convex problem (Nesterov and Nemirovskii, 1994). In (Wills and Heath, 2002*a*) we proposed a novel class of model predictive controller inspired by interior point methods. In particular we suggested that stopping short on the central path (equivalently fixing a lower bound on the barrier weighting) could have beneficial effects in terms of both computational efficiency and closed loop behaviour for control problems with input constraints. In this paper we have shown the idea generalises straightforwardly to control problems with state constraints.

One aspect of such a control scheme is that the optimization problem to be solved at each step is the minimization of an unconstrained convex cost function—in the sense that the solution always lies on the interior of the constraint set $\mathbb{U}$ and the gradient of the cost function is zero at the solution. Indeed this requirement motivated the introduction of the recentred barrier. Recently Jadbabaie *et al.* (2001) have shown that it may be possible to demonstrate stability of unconstrained nonlinear model predictive controllers without the introduction of terminal constraints. A natural question would be whether the proposed class of controllers might be analysed in such a fashion. Certainly the simulation of the previous section indicates that it is not *necessary* to introduce a terminal constraint set (this is, of course, well known for more conventional linear model predictive controllers). The class of controller we have proposed includes more conventional model predictive control as a limiting case (as the barrier weighting $\mu \to 0$). This suggests a possible new avenue for the analysis of model predictive control.

6 Conclusion

In (Wills and Heath, 2002*a*) we proposed a novel class of model predictive control for plants with input constraints. The idea is to include a barrier in the cost function and find the solution for a fixed weighting on the barrier. A necessary condition for correct steady state behaviour is that the gradient of the cost function is zero at the desired steady state value.

In this paper we have shown that the idea generalises in a straightforward manner to systems with state constraints. We have illustrated the result for a plant with rate constraints on the actuators, and shown both how stability can be guaranteed using terminal constraints, and also demonstrated the closed loop behaviour with a simple simulation example.

One interesting observation is that such model predictive controllers may be thought of as unconstrained, in the sense that the solution always lies on the interior of the constraint set. We postulate that such an observation may be useful for the analysis of closed loop stability of model predictive control.

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Figures

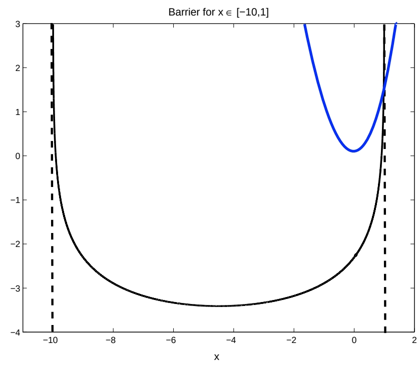

Figure 1: Illustration of a barrier for state constraint $-10 \leq x - x_{ss} < 1$. The cost function without barrier has its minimum at $x - x_{ss} = 0$, but the barrier has its minimum at $x = 0$.

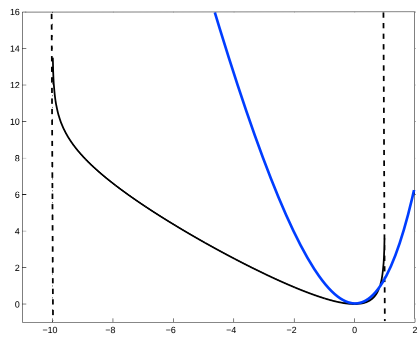

Figure 2: Illustration of a recentred barrier for state constraint $-10 \leq x - x_{ss} < 1$. Both the cost function (without barrier) and the barrier itself have their minima at $x - x_{ss} = 0$.

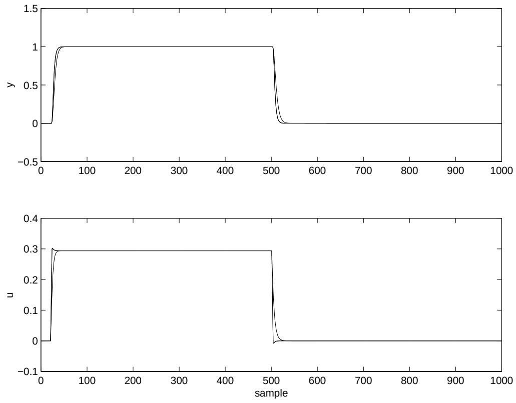

Figure 3: Simulation example with (i) no barrier, (ii) $\mu = 10^{-6}$ and (iii) $\mu = 1$, and symmetric rate constraints. Cases (i) and (ii) are indistinguishable to the eye, but the response for case (iii) is smoother.

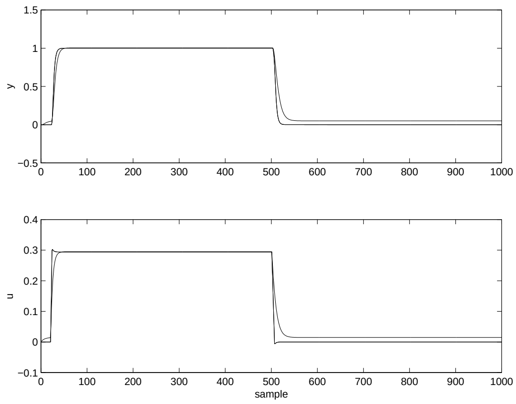

Figure 4: Simulation example under similar conditions to those for Fig 3, but with asymmetric rate constraints. There is no recentring, and the steady state behaviour for case (iii) is significantly awry.

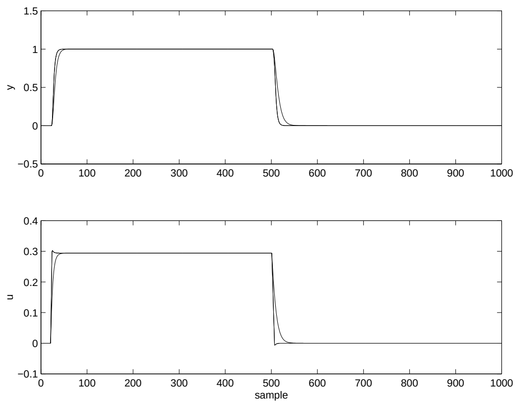

Figure 5: Simulation example under similar conditions to those for Fig 4. The barriers are recentred.

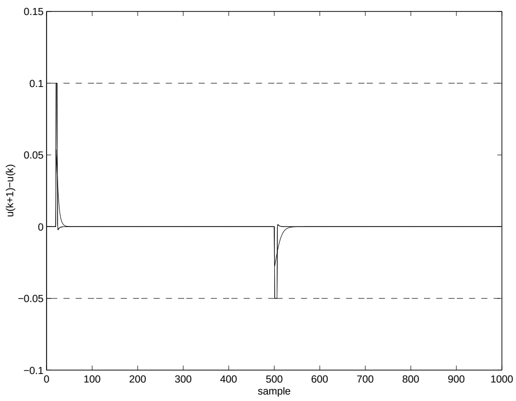

Figure 6: Evolution of $u_{t+1} - u_t$ under conditions corresponding to those of Fig 5.