Technical Report EE03025 - Barrier Function Based Model Predictive Control

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Abstract

Barrier functions are considered within the context of constrained model predictive control (MPC). A new class of controller is developed by including a weighted barrier function in the objective that ensures inequality constraints are satisfied. Fixing the barrier weighting term to be some positive value – possibly much greater than zero – has interesting effects on controller dynamics, particularly near constraint boundaries. When the barrier weighting term is close to zero, the corresponding dynamic behaviour resembles that of standard MPC. The new class may be seen as a generalisation of standard MPC; in particular, standard MPC is subsumed within the new class as a special limiting case. Conditions are determined for the barrier such that correct steady-state behaviour is guaranteed; a barrier satisfying these conditions is called a recentred barrier and consequently the new controller class is called recentred barrier function MPC (abbreviated as r-MPC). Nominal closed-loop stability is shown for this class of controllers. This relies on bounding the recentred barrier function from above using a convex quadratic term. We point out that this property is satisfied using gradient recentred self-concordant barrier functions, which are known to exist for all closed convex domains.

1 Introduction

We define a new class of model predictive control (MPC) algorithm where inequality constraints are guaranteed via the inclusion of a weighted barrier in the cost function. Conventional MPC is subsumed within the new class as a limiting case as the weighting tends to zero. The barriers result in a more cautious control action when operating near constraints, whilst having little effect on the dynamic behaviour (compared with conventional MPC) when far from constraints.

The concept is closely related to interior point methods for solving constrained optimisation problems. The analysis of Nesterov and Nemirovskii (1994) allows the application of such controllers to a wide class of convex problem. If the barrier function is self-concordant barriers, we construct invariant sets and associated Lyapunov functions (for nominal stability) in an elegant manner. The optimization problem may also be solved efficiently on-line via simple modifications of existing interior point algorithms (Wills and Heath, 2002b; Wills, 2003; Wills and Heath, 2003).

The requirement for correct steady state behaviour restricts the class of barrier functions that may be used. We determine conditions on the barrier such that correct steady state is guaranteed, and define the class of barriers that satisfy such conditions *recentred barriers*. We also discuss one particular construction which we term the *gradient recentred barrier*.

1.1 Model Predictive Control

Model predictive control (MPC) is a popular control strategy in industry. It is often said that MPC is the only advanced control strategy to have a substantial impact on industrial control engineering (see e.g. Maciejowski, 2002). In the literature, MPC is usually considered as a strategy which results in a sequence of discrete control actions; the strategy takes the current state of the system (or observation thereof) as the initial state and computes the next control action by solving, *on-line*, an optimal control problem. The objective function involved in this optimal control problem typically penalises deviations of the states and inputs from their respective steady-state values.

A number of comprehensive surveys exist for MPC including those by Clarke et al. (1987), Garcia et al. (1989), Muske and Rawlings (1993), Qin and Badgwell (1997), Chen and Allgöwer (1998), Mayne et al. (2000), and Maciejowski (2002). In particular, Mayne et al. (2000) offer an authoritative account of MPC from a theoretical perspective. MPC literature appears in three separate strands of control: process control, adaptive control and optimal control.

It is generally accepted that the heuristics of MPC were developed in the process industries. Early propagation of MPC is attributed to the works of Richalet et al. (1976), Richalet et al. (1978), Cutler and Ramaker (1980), Cutler et al. (1983). For an authoritative summary of industrial MPC see Qin and Badgwell (1997).

The work of Clarke et al. (1987) on Generalised Predictive Control (GPC) has received much attention, both theoretically and practically. GPC was developed within the adaptive control setting. The GPC model structure is input-output and is more general than the impulse and step response models used by Richalet et al. (1976) and Cutler and Ramaker (1980) respectively. It is interesting to note that Clarke et al. (1987) initially compared GPC with other adaptive control strategies and that GPC did not consider (until later) inequality constraints on the inputs. Further, Bitmead et al. (1990) showed that unconstrained MPC is adequately subsumed within Linear Quadratic Gaussian (LQG) control.

From an optimal control perspective, MPC may be interpreted as a numerical implementation of the classical Receding Horizon Control (RHC) strategy. Mayne et al. (2000) make reference to cases in the literature where this is made explicit. In particular, they point out that the idea of MPC was present in the literature as early as 1967 with Lee and Markus (1967). Rather than derive an optimal control law, MPC returns the optimal control action at each time interval.

An important advantage of MPC is the ability to handle input and state constraints in a natural manner. Such constraints are usually included as side conditions in the MPC optimisation problem and are therefore treated as separate to the objective function; in fact, aside from soft constraint formulations of MPC, this seems exclusively to be the case. The resultant constrained optimisation problem is then solved using an appropriate algorithm.

1.2 Interior-point methods

Interior-point methods are a popular approach used in solving constrained optimisation problems. There have been in excess of 3000 papers published in the area of interior-point methods subsequent to the publication of Karmarker's ground-breaking paper in 1984 Glineur (2000). A number of reviews and surveys exist, including books by Nesterov and Nemirovskii (1994), Roos et al. (1997), Ye (1997), Wright (1997b), Renegar (2001). Some of the pioneering work of interior-point methods is discussed by Fiacco and McCormick (1968). A recent survey by Forsgren et al. (2002) gives a broad perspective of interior-point methods with particular attention to nonlinear and non-convex problems.

The strategy of interior-point methods is to generate a series of "unconstrained" auxiliary problems whose limiting case solution coincides with an optimal solution to the original constrained problem. In some respects, creating a series of unconstrained problems is no different from other popular methods; e.g. active-set approaches for quadratic programming create a series of unconstrained sub-problems by projecting the cost function onto the active constraints at each iteration. Interior-point methods differ in that all constraints are considered when obtaining a search direction and a boundary point is approached only in the limit. The benefit of creating a series of "unconstrained" problems is that more traditional optimisation techniques may be used, such as Newton's method. Typically, this series of auxiliary problems is parameterised by a positive scalar term where the limiting case occurs as the scalar itself approaches a limit, e.g. zero.

One way to generate a family of auxiliary problems is to include a barrier function for the constraint set into the objective. Frisch's logarithmic barrier Frisch (1956) is perhaps the most famous example of such a barrier function. Renegar (2001) refers to this barrier as the primordial barrier function of interior-point methods. A different but related approach for generating a family of auxiliary problems is the method of centres originally developed by Lieu and Huard (1966). A detailed exposition of barrier generated auxiliary problems is given in Section 3 of Nesterov and Nemirovskii (1994). A summary of early activities is provided by Fiacco and McCormick (1968).

During the 1970's these classical interior-point methods (also known as interior penalty schemes (see e.g. Ben-Tal and Nemirovskii, 2001)) fell out of popularity in the optimisation community; *inter-alia*, they suffer from an inherent ill-conditioning problem near the solution and alternative methods were thought to be more efficient (see e.g. Wright, 1992).

Karmarkar (1984) showed that linear-programming problems could be solved using an interior-point strategy with excellent practical results. Furthermore, the worst case complexity bound was polynomial-time. Prior to this, the Simplex method had been generally considered the best practical approach for linear-programming (see e.g. Wright, 1992; Roos et al., 1997), albeit that its worst-case complexity bound is exponential. Hence Karmarker's result was promising from both a theoretical and an applied perspective. Gill et al. (1986) showed a connection between Karmarker's algorithm and the classical interior penalty methods.

The theory of polynomial-time interior-point methods was unified and generalised by Nesterov and Nemirovskii (1994). Their framework provides a means to develop polynomial-time algorithms for a broad class of convex programming problems. Their work also introduces a new conic formulation of the convex programming problem. Nesterov and Todd (1997) exploit this formulation with their introduction of self-scaled cones and barriers, which play an integral role in state-of-the-art interior-point methods.

1.3 Interior-point methods and MPC

Interior-point methods have been considered by several authors within the framework of MPC. Polak et al. (1990) presented an algorithm based on barrier functions (in particular, they use a method of centres - see e.g. Lieu and Huard (1966)) for a general optimal control problem with control and state inequality constraints. Their paper and references therein present applications of classical interior-point methods to constrained optimal control problems. However, Wright (1993) appears to be the first to have explicitly discussed new polynomial-time interior-point methods for the Quadratic Programs (QP's) and Sequential Quadratic Programs (SQP's) associated with MPC. These ideas are developed further in Wright (1997a), where infeasible-start interior-point methods are used. Lim et al. (1996) also study constrained control from an interior-point framework.

Under certain formulations, the MPC optimisation problem has exploitable structure. Exploiting this structure can, in certain cases, lead to efficient algorithms. An example of this is shown by Dunn and Bertsekas (1989) who consider unconstrained MPC and exploit the coupling between successive intervals in terms of the states and co-states. They develop a recursion algorithm with complexity that grows linearly in the horizon. This is a common theme in the literature on MPC and interior-point methods.

Wright (1993) expresses the optimality conditions on an interval-by-interval basis which leads to a block-banded system of equations; such a system must be factored and solved (at least once) at each iteration of an interior-point method. Factorisation of this matrix grows linearly in the prediction horizon. Rao et al. (1998) exploit a Riccati recursion approach within the interior-point framework for solving this block-banded matrix. This also results in an algorithm whose complexity grows linearly in the prediction horizon.

Gopal and Biegler (1998) offer an authoritative account of large-scale optimisation and control. Their analysis and observations are primarily from the perspective of SQP – see also Albuquerque et al. (1997). Gopal and Biegler (1998) also point out that certain formulations of MPC result in structure that can be exploited and consider this structure in the context of SQP.

From a practical perspective, active-set methods offer an alternative to interior-point methods for the case of linear MPC. The topic of active-set methods has been addressed by Bartlett et al. (2000) who offer one comparison between these alternatives. Of interest is that active-set methods can also exploit the MPC structure; Bartlett et al. (2000) refer to a Schur-Complement approach which allows for efficient updating of the linear sub-system solved at each iteration. Glad and Jonson (1984) also exploit the recursive nature of the MPC structure in an active-set framework.

Interior-points have been used in other areas of MPC, most notably in the areas of steady-state reference calculation and robustness. Kassmann et al. (2000) study steady-state reference calculations where the model is subject to uncertainty. Their analysis utilises the second-order cone programming framework. Hansson (2000) studies robust MPC also in a conic programming framework. Vandenberghe et al. (2002) look at robust linear programming for optimal control and incorporate the Riccati recursion approach of Rao et al. (1998) for solving the linear system of equations associated with their algorithm.

1.4 Contribution

In this paper, we adopt the barrier philosophy used in interior-point methods and include a weighted barrier function directly into the MPC cost. This is done in order to change the controller dynamics near constraint boundaries. The approach differs from interior-point methods in that the barrier weighting parameter is fixed and possibly much greater than zero. In fact, we require a careful choice of barrier function to ensure correct steady-state behaviour. The so-called *recentred barrier function* is used for this purpose. The resulting controller class is called *recentred barrier function MPC* (abbreviated as r-MPC). This class of controllers is a generalisation of standard MPC since the latter is included in the former as a special limiting case. Some preliminary results can be found in Wills and Heath (2002a); Heath and Wills (2002); Wills (2003).

Including the barrier term in the objective produces a smooth cost function where the cost increases rapidly for points approaching constraint boundaries. In fact, the barrier function has a singularity on the constraint boundary which results in infinite cost at such points. Hence, inequality constraints are inherently satisfied using the barrier approach (provided the constraint set has an interior). The weighting parameter determines the rate at which the barrier term increases for points approaching the boundary. Meanwhile, points away from the constraint boundary receive negligible cost from the barrier. This approach allows the control engineer to determine a smooth transition between interior and boundary points.

When the weighting parameter is large, the associated optimisation problem may be solved using almost standard unconstrained minimisation techniques. However, when the weighting parameter is small, this approach is likely to fail unless care is taken. Algorithms for solving the associated optimisation problem for *any* value of weighting parameter have been developed in Wills (2003). These algorithms are closely related to standard interior-point algorithms, but are modified to handle the case where the weighting parameter does not necessarily approach zero in the limit.

Modifying the barrier function so that its minimum lies at the steady-state setpoint produces a recentred barrier function which guarantees correct steady-state behaviour. The steady-state setpoints are necessarily on the interior of the constraint sets. This is ensured by minimising in turn a steady-state cost function which also includes a weighted barrier function.

In the spirit of Chen and Allgöwer (1998) and after Mayne et al. (2000) we show nominal closed-loop stability for r-MPC. An important component of the stability analysis concerns a terminal constraint set which is invariant under some local controller. We develop an ellipsoidal terminal constraint set in a standard manner using the level set of an appropriately chosen Lyapunov function. An underlying assumption for this proof is that the barrier function can be locally bounded from above using a convex quadratic function. We point out that this assumption is satisfied for gradient recentred self-concordant barrier functions. Nesterov and Nemirovskii (1994) showed that every closed convex domain which does not contain an affine sub-space admits a self-concordant barrier function.

This paper is organised as follows. Relevant notation, definitions and some properties of recentred barrier functions and r-MPC are given in Section 2. The important issue of correct steady-state behaviour and the need for recentred barrier functions is discussed in Section 3. An ellipsoidal invariant set is constructed in Section 4, which is pertinent to the stability analysis given in Section 5. We outline some important properties of self-concordant functions and how they can be recentred using the gradient in Section 6; further, we point out that a gradient recentred self-concordant barrier function satisfies certain assumptions used in showing stability. This section is important from both a practical (algorithmic) and theoretical (nominal stability) perspective. A simulation example for linear system models is given in Section 7.

We stress that Sections 3 and 6 contain the main thrust of this paper. Section 3 explains why recentred barrier functions are necessary and Section 6 presents one practical construction of such a barrier function.

2 Problem statement

Consider the following time-invariant, discrete-time system with integer t representing the current discrete time event,

$$\begin{aligned} x(t+1) &= f(x(t), u(t)), \\ y(t) &= h(x(t)). \end{aligned} \tag{1}$$

In the above, $u(\cdot) \in \mathbb{R}^m$ is the system input, $y(\cdot) \in \mathbb{R}^p$ is the system output and $x(\cdot) \in \mathbb{R}^n$ is the system state. The mapping $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is assumed to be continuously differentiable on $\mathbb{R}^n \times \mathbb{R}^m$. It is assumed throughout this paper that

$$f(0,0) = 0.$$

Let the positive integer N denote the *prediction horizon*. Define a sequence $\mathscr{X}$ (called a *state sequence*) with $N+1$ elements where each element is a vector of dimension n, i.e.

$$\mathscr{X} = \{x_0, \ldots, x_N\}.$$

Similarly let $\mathscr{U}$ denote a sequence (called an *input sequence*) with N elements where each element is a vector of dimension m,

$$\mathscr{U} = \{u_0, \ldots, u_{N-1}\}.$$

Given some initial state x, then the mapping f can be used to *predict* future states for a sequence of inputs. Let $\mathscr{X}(x, \mathscr{U})$ denote the *predicted state evolution* for initial state x under the control sequence $\mathscr{U}$,

$$\mathscr{X}(x, \mathscr{U}) \quad = \quad \{x_0(x, \mathscr{U}), \ldots, x_N(x, \mathscr{U})\}, \tag{2}$$

where the elements are defined by the recursion,

$$x_0(x, \mathscr{U}) \quad = \quad x, \qquad x_{i+1}(x, \mathscr{U}) \quad = \quad f(x_i(x, \mathscr{U}), u_i), \; i = 0, \ldots, N-1. \tag{3}$$

State and input constraints are expressed mathematically as a restriction of the state $x(\cdot)$ and input $u(\cdot)$ to the sets $\mathbb{X}$ and $\mathbb{U}$ respectively. Furthermore, the final state $x_N(x, \mathscr{U})$ may be further restricted to lie in a *terminal* constraint set $\mathbb{X}_F \subseteq \mathbb{X}$. The state constraint set $\mathbb{X}$ is assumed to be convex and closed. The input constraint set $\mathbb{U}$ is assumed to be convex and compact. Further, each set is assumed to have a *non-empty* interior denoted by $\mathbb{X}^o$, $\mathbb{X}_F^o$ and $\mathbb{U}^o$ respectively. Define the feasible set $\mathcal{F}(x)$ as (also known as the set of admissible control sequences),

$$\begin{aligned} \mathcal{F}(x) \quad = \quad \Big\{ \mathscr{U} : \; & u_i \in \mathbb{U}, \; i = 0, \ldots, N-1, \\ & x_i(x, \mathscr{U}) \in \mathbb{X}, \; i = 1, \ldots, N-1, \\ & x_N \in \mathbb{X}_F \Big\}. \end{aligned}$$

Denote the interior of $\mathcal{F}(x)$ by $\mathcal{F}^o(x)$. Note that since $\mathbb{U}$ is compact then $\mathcal{F}(x)$ is compact.

Let J_θ denote the time-invariant MPC objective function defined as

$$J_\theta(x, \mathscr{U}) \quad = \quad F_\theta(x_N(x, \mathscr{U})) + \sum_{i=0}^{N-1} \ell_\theta(x_i(x, \mathscr{U}), u_i). \tag{4}$$

where $x_i(x, \mathscr{U})$ is given by (3). The subscript θ denotes co-efficients for the analytical structure of F_θ and ℓ_θ. For brevity, θ is henceforth omitted.

In the above, F is called the *terminal cost function* and ℓ is called the *stage cost function*. In what follows we consider one possible construction of the terminal cost F and stage cost ℓ which includes a weighted recentred barrier function. This structure will be assumed throughout the remainder of this paper.

Let the terminal and stage cost functions assume the following form,

$$F(x) \quad = \quad \|x\|_P^2 + \mu B_F(x), \tag{5}$$

$$\ell(x, u) \quad = \quad \|x\|_Q^2 + \|u\|_R^2 + \mu B_x(x) + \mu B_u(u), \tag{6}$$

In the above, B_F, B_x and B_u are each assumed to be recentred barrier functions (defined below) about the origin. The matrices P, Q and R are assumed to be symmetric and positive definite. The scalar μ which weights the barrier terms is assumed to be positive.

Define the class of (recentred) barrier functions as follows.

Definition 2.1 *Let Z be a closed convex and let $\bar{z} \in Z^o$. A function $g : Z^o \to \mathbb{R}$ will be called a barrier function about $\bar{z}$ if*

($\mathcal{B}.1$) *g is continuous and strictly convex on Z^o, and*

($\mathcal{B}.2$) *For every sequence $\{z_k \in Z^o\}$ such that $\lim_{k\to\infty} z_k = z_\infty \in \partial Z$ then $\lim_{k\to\infty} g(z_k) \to \infty$.*

The function g is called a recentred barrier function about $\bar{z}$ if g is a barrier function which also satisfies

($\mathcal{B}.3$) *$g(\bar{z}) = 0$ and $g(z) > 0$ for all $z \in Z^o$ with $z \neq \bar{z}$.* □

The graph in Figure 1 is a recentred barrier function about $\bar{z} = -2$.

The terminal and stage cost functions defined above produce an objective function whose natural domain is the interior of $\mathcal{F}(\cdot)$. In particular, the objective associates high (ultimately infinite) penalties for points approaching a constraint boundary. Moreover, when away from constraints the objective behaves in a similar manner to the unconstrained case.

The following r-MPC strategy computes the optimal input trajectory at each time interval and applies the first control move of this trajectory to the system. Lemma 2.1 which follows, shows that by including a barrier into the cost we effectively ensure against constraint violations.

Algorithm 2.1 ***r-MPC****: At each time interval t, given the initial state $x(t)$, compute the following if $\mathcal{F}(x(t)) \neq \emptyset$,*

1. *Solve*
$$\mathscr{U}^*(x(t)) \quad = \quad \arg\min_{\mathscr{U}} \; \{J(x(t), \mathscr{U}) : \mathscr{U} \in \mathcal{F}(x(t))\}\,. \tag{7}$$
2. *Apply $u_0^*(x(t))$ (the first element of $\mathscr{U}^*(x(t))$) to the system.*

□

If F and ℓ are recentred barrier functions, then the optimal input sequence is strictly feasible. Specifically:

Lemma 2.1 *Suppose that $\mathcal{F}^o(x) \neq \emptyset$ for some x and assume that F is a recentred barrier function for $\mathbb{X}_F$ about 0 and ℓ is a recentred barrier function for $\mathbb{X} \times \mathbb{U}$ about $(0, 0)$. Then there exists a finite value $J^*(x)$ and a point $\mathscr{U}^* \in \mathcal{F}^o(x)$ such that*
$$J^*(x) \quad = \quad J(x, \mathscr{U}^*) \quad = \quad \min_{\mathscr{U}} \{J(x, \mathscr{U}) : \mathscr{U} \in \mathcal{F}^o(x)\}. \tag{8}$$

Proof. (After Section 3.3. of Fiacco and McCormick (1968)). Define $\bar{J}(\mathscr{U}) = J(x, \mathscr{U})$ and let $\mathscr{U}_0 \in \mathcal{F}^o(x)$. Define the set
$$W \quad = \quad \{\mathscr{U} \in \mathcal{F}^o(x) : \bar{J}(\mathscr{U}) \leq \bar{J}(\mathscr{U}_0)\}.$$
If $\{\mathscr{U}_k \in W\}$ is a convergent sequence with $\lim_{k\to\infty} \mathscr{U}_k = V$, then $V \in \mathcal{F}(x)$ because $\mathcal{F}(x)$ is compact. So either $V \in \mathcal{F}^o(x)$ or $V \in \partial\mathcal{F}(x)$.

Assume that $V \in \partial\mathcal{F}(x)$. Then by ($\mathcal{B}$.1) in Definition 2.1, $\exists k < \infty$ such that $\mathscr{U}_k \notin W$ - a contradiction. Therefore $V \in W$.

Furthermore, since $\bar{J}(\mathscr{U}_k) \leq \bar{J}(\mathscr{U}_0)$ for all k then the continuity of $\bar{J}(\cdot)$ on $\mathcal{F}^o(x)$ implies that $\bar{J}(V) \leq \bar{J}(\mathscr{U}_0)$ and therefore W is closed and bounded, hence compact. Equation (8) follows since
$$\inf_{\mathscr{U} \in \mathcal{F}^o(x)} \bar{J}(\mathscr{U}) \quad = \quad \min_{\mathscr{U} \in W} \bar{J}(\mathscr{U}).$$

□

3 Recentred barriers and steady-state behaviour

In the MPC formulation of Muske and Rawlings (1993) feedback is provided via a state observer. In particular, integral action is incorporated via a disturbance estimator, and the computation at each control step of a steady state pair (x_{ss}, u_{ss}). Co-ordinates are then shifted so that (x_{ss}, u_{ss}) becomes the origin. As the pair is computed via a separate quadratic program, it usually lies on the constraint boundary (Qin and Badgwell, 1997; Rao and Rawlings, 1999).

A requirement of r-MPC is that the origin lies on the *interior* of $\mathbb{X}_F$ and $\mathbb{U}$. Indeed this is a common assumption for the analysis of MPC in the literature (see e.g. Mayne et al., 2000). In this section we discuss one approach which guarantees this. We replace the steady-state quadratic program of Muske and Rawlings (1993) with an optimisation problem which includes its own weighted barrier function in the cost. Although this results in a degradation of nominal steady state performance, the degradation is bounded by the duality gap (Nesterov and Nemirovskii, 1994) if a logarithmic barrier is chosen. The degradation can be made arbitrarily small by choosing a sufficiently small weighting on the barrier.

Let (x_r, u_r) denote some reference pair. Let B_{xs} and B_{us} denote barrier functions for $\mathbb{X}_F$ and $\mathbb{U}$ respectively.

Construct a steady-state cost function J_{ss} as follows.

$$J_{ss} : \mathbb{X}_F^o \times \mathbb{U}^o, \quad J_{ss}(x,u) \;=\; \tfrac{1}{2}\|x - x_r\|^2_{Q_{ss}} + \tfrac{1}{2}\|u - u_r\|^2_{R_{ss}} + \mu_{ss} B_{xs}(x) + \mu_{ss} B_{us}(u).$$

The steady-state optimisation problem is given by

$$(x_{ss}, u_{ss}) \;=\; \arg\min_{x,u} \{ J_{ss}(x,u) : x = f(x,u) \}.$$

In the spirit of Section 3.3 in Fiacco and McCormick (1968), it follows immediately that $x_{ss} \in \mathbb{X}_F^o$ and $u_{ss} \in \mathbb{U}^o$ (see also Lemma 2.1). Hence we can make a shift of co-ordinates using (x_{ss}, u_{ss}) such that $(0,0) \in \mathbb{X}_F^o \times \mathbb{U}^o$.

In Section 2 we restrict the terminal cost F and stage cost ℓ to be recentred barrier functions about the origin. This restriction means that if system (1) reaches steady-state under the control of Algorithm 2.1 then it reaches the origin. This property is not guaranteed for a more general choice of barrier function.

Fianlly we show the optimal solution to the r-MPC optimisation problem given in Equation 7 is indeed the origin as desired.

Lemma 3.1 *If $\mathcal{F}^o(0) \neq \emptyset$, then*

$$0 = \arg\min_{\mathscr{U}} \; \{ J(0, \mathscr{U}) : \mathscr{U} \in \mathcal{F}^o(0) \}. \tag{9}$$

Proof. Since $\mathscr{X}(0,0) = \{0, \ldots, 0\}$ and $(0,0) \in \mathbb{X}_F^o \times \mathbb{U}^o$ then $0 \in \mathcal{F}^o(0)$. The objective evaluated at $\mathscr{U} = 0$ is given by

$$\begin{aligned} J(0,0) &= F(0) + N\ell(0,0) \\ &= 0. \end{aligned}$$

From $(\mathcal{B}.3)$ in Definition 2.1, this is the global minimum. □

4 Invariant set

Invariant sets play an important role in the analysis of constrained MPC (see e.g. Gilbert and Tan, 1991; Michalska and Mayne, 1993; Chen and Allgöwer, 1998; Blanchini, 1999; Mayne et al., 2000).

In particular, Blanchini (1999) surveys invariant sets for control, reiterating the two prominent invariant set structures as being ellipsoidal and polytopic. Chen and Allgöwer (1998) provide an example of ellipsoidal invariant sets while Gilbert and Tan (1991) developed the polytopic Maximal Output Admissible Set. Blanchini (1999) concludes that ellipsoidal sets are more conservative than polytopic sets while the latter are usually more complicated.

In this section we construct an ellipsoidal invariant set which is used in Section 5 where nominal closed-loop stability is discussed. The development here is based on Chen and Allgöwer (1998) albeit that they consider continuous-time systems.

Consider the Jacobian linearisation of system (1) about the origin,

$$x(t+1) = Ax(t) + Bu(t). \tag{10}$$

Assumption 4.1 *The linear system described in* (10) *is assumed to be stabilisable.* □

Under the above assumption there exists a matrix K such that system (10) is asymptotically stable under the control law

$$u(t) = -Kx(t).$$

Let A_K be defined as

$$A_K = A - BK.$$

Let X_K denote the set of states such that state constraints are satisfied inside X_K and $-Kx$ satisfies input constraints in X_K, i.e.

$$X_K = \{x \in \mathbb{X} : -Kx \in \mathbb{U}\}.$$

Let X_K^o denote the interior of X_K. Since convexity is preserved under affine transformation (see e.g. Rockafellar, 1970), we may define a recentred barrier function B_K for X_K as

$$B_K(x) = B_x(x) + B_u(-Kx).$$

The following assumption bounds the recentred barrier function B_K by a quadratic term. This property is not trivially satisfied. In Section 6 we discuss self-concordant barrier functions introduced by Nesterov and Nemirovskii (1994); it turns out that a suitably recentred self-concordant barrier function satisfies the following assumption.

Let $W_M(r)$ denote an ellipsoid of radius $r \geq 0$ given by $W_M(r) = \{x \in \mathbb{R}^n : \|x\|_M \leq r\}$, where $M \succ 0$.

Assumption 4.2 *Assume there exists a positive real number β and positive definite matrices M and H such that*

$$W_M(\beta) \subseteq \left\{x \in X_K^o : \mu B_K(x) \leq \|x\|_H^2\right\}.$$

□

Corollary 4.1 *For any β, M and H which satisfy Assumption 4.2 it follows immediately that*

$$W_M(\beta) \subset X_K.$$

□

Let Q and R be given as in Equation (6) and assume that $A_K = (A - BK)$ is discrete Hurwitz for some K. Define the positive definite matrix Q^* as

$$Q^* \;=\; Q + K^T R K,$$

Since A_K is discrete Hurwitz then there exists a positive definite matrix P for any positive real number η such that

$$A_K^T P A_K - P \;=\; -(1+\eta)Q^* - H. \tag{11}$$

Since P is positive definite and $\beta > 0$ then there exists an $\alpha > 0$ such that

$$W_P(\alpha) \;\subseteq\; W_M(\beta).$$

Let α^* denote the largest value of α such that the above holds. It follows that α^* is given by

$$\alpha^* \;=\; \sqrt{\frac{\beta^2}{\lambda}}$$

where λ denotes the maximum eigenvalue of $P^{-1}M$.

In the following lemma, we show that there exists an $\alpha \in (0, \alpha^*]$ such that the set $W_P(\alpha)$ is invariant for system (1) with the local controller $u(t) = -Kx(t)$. In particular, if we define a function V as

$$V(x) \;=\; \|x\|_P^2,$$

then V decreases under the local control law.

Lemma 4.1 *There exists $\alpha \in (0, \alpha^*]$ such that*

$$x \in W_P(\alpha) \;\;\Rightarrow\;\; V(f(x, -Kx)) - V(x) \leq -\|x\|_{Q^*}^2 - \|x\|_H^2.$$

Proof. Define a function $r(\cdot)$ as

$$r(x) \;=\; f(x, -Kx) - A_K x.$$

So

$$\begin{aligned} V(f(x,-Kx)) - V(x) &= V(A_K x + r(x)) - V(x) \\ &= \|x\|_{A_K^T P A_K}^2 - \|x\|_P^2 + 2x^T A_K^T P r(x) + \|r(x)\|_P^2. \end{aligned}$$

Since

$$\begin{aligned} 2x^T A_K^T P r(x) &\leq 2\|A_K P\|\,\|x\|\,\|r(x)\|, \\ &\leq \frac{2\|A_K P\|}{\sqrt{\lambda_{\min}(P)\lambda_{\min}(Q^*)}}\|x\|_{Q^*}\,\|r(x)\|_P, \end{aligned}$$

it follows immediately that

$$2x^T A_K^T P r(x) + \|r(x)\|_P^2 \;\leq\; c\|x\|_{Q^*}\|r(x)\|_P + \|r(x)\|_P^2,$$

where $c = \frac{2\|A_K P\|}{\sqrt{\lambda_{\min}(P)\lambda_{\min}(Q^*)}}$. Since f is continuously differentiable it follows that for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\|x\|_M \leq \delta \;\;\Rightarrow\;\; \|r(x)\|_N \leq \epsilon\|x\|_M,$$

where M and N are positive definite matrices. Hence for the particular case where $M = Q^*$ and $N = P$ it follows that for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|x\|_{Q^*} \leq \delta \;\;\Rightarrow\;\; c\|x\|_{Q^*}\|r(x)\|_P + \|r(x)\|_P^2 \leq (c\epsilon + \epsilon^2)\|x\|_{Q^*}^2.$$

Furthermore, there exists a value of $\epsilon > 0$ such that $c\epsilon + \epsilon^2 = \eta$. It follows immediately that there exists $\alpha > 0$ such that

$$\|x\|_P^2 \leq \alpha^2 \quad \Rightarrow \quad 2x^T A_K^T P r(x) + \|r(x)\|_P^2 \leq \eta \|x\|_{Q^*}^2.$$

Hence from the definition of $W_P(\alpha)$ and Equation (11) there exists $\alpha \in (0, \alpha^*]$ such that

$$x \in W_P(\alpha) \quad \Rightarrow \quad V(f(x, -Kx)) - V(x) \leq -\|x\|_{Q^*}^2 - \|x\|_H^2,$$

as desired. □

5 Stability

A recent survey of stability for MPC is offered in Mayne et al. (2000). They stipulate sufficient conditions for nominal stability of MPC as follows.

1. $\mathbb{X}_F \subset \mathbb{X}$ is a non-empty terminal constraint set.
2. A strictly feasible input sequence $\mathscr{U}$ exists for the initial state x, i.e. $\mathcal{F}^o(x) \neq \emptyset$.
3. $u(t) = -Kx(t)$ is a local stabilising controller about $(0, 0) \in \mathbb{X}_F^o \times \mathbb{U}^o$.
4. $\mathbb{X}_F$ is positively invariant under this local control law.
5. The following inequality holds,

$$J(x^+, \mathscr{U}^*(x^+)) - J(x, \mathscr{U}^*(x)) \quad \leq \quad 0,$$

   where $x^+ = f(x, u_0^*(x))$. The optimal control sequence $\mathscr{U}^*(x)$ is the solution to (7) with initial state x and $\mathscr{U}^*(x^+)$ is the solution to (7) with initial state x^+.

The following discussion concerns stability of system (1) under the control of Algorithm 2.1. In particular, we show that the above sufficient conditions for nominal stability also hold for the case of r-MPC.

Choose $\alpha \in (0, \alpha^*]$ such that Lemma 4.1 holds for that value. The terminal constraint set $\mathbb{X}_F$ is defined as

$$\mathbb{X}_F \quad = \quad W_P(\alpha).$$

Let B_F be given by

$$B_F(x) \quad = \quad \ln\left(\alpha^2\right) - \ln\left(\alpha^2 - \|x\|_P^2\right).$$

Then B_F is a recentred barrier function for $\mathbb{X}_F$ about the origin. The terminal cost F is constructed as follows (*c.f.* (6)),

$$F(x) \quad = \quad \|x\|_P^2 + \mu B_F(x).$$

Lemma 5.1 *If $x \in \mathbb{X}_F^o$ then*

$$B_F(f(x, -Kx)) - B_F(x) \leq 0.$$

Proof. From Lemma 4.1 we know that for $x \in W_P(\alpha)$

$$\|f(x, -Kx)\|_P^2 \leq \|x\|_P^2 - \|x\|_{Q^*}^2 - \|x\|_H^2.$$

So

$$\begin{aligned} B_F(f(x, -Kx)) &= \ln\left(\alpha^2\right) - \ln\left(\alpha^2 - \|f(x, -Kx)\|_P^2\right), \\ &\leq \ln\left(\alpha^2\right) - \ln\left(\alpha^2 - \|x\|_P^2 + \|x\|_{Q^*}^2 + \|x\|_H^2\right). \end{aligned}$$

Hence

$$\begin{aligned} B_F(f(x,-Kx)) - B_F(x) &\leq -\ln\left(\alpha^2 - \|x\|_P^2 + \|x\|_{Q^*}^2 + \|x\|_H^2\right) + \ln\left(\alpha^2 - \|x\|_P^2\right), \\ &\leq 0. \end{aligned}$$

□

Suppose that x is the current state and that $\mathcal{F}^o(x)$ is non-empty. Let $\mathscr{X}(x, \mathscr{U}^*(x))$ be the state evolution (see (2)) corresponding to the optimal control sequence $\mathscr{U}^*(x)$ given by (7). By definition the optimal control move $u_0^*(x)$ steers x to the successor state $x^+ = x_1^*(x, \mathscr{U}^*(x)) = f(x, u_0^*(x))$. Let $\mathscr{U}^*(x^+)$ denote the optimal control sequence for initial state x^+. The remainder of this section is intended to show that

$$J(x^+, \mathscr{U}^*(x^+)) \leq J(x, \mathscr{U}^*(x)), \tag{12}$$

which shows that Algorithm 2.1 stabilises system (1) with initial state x. The approach taken is broadly based on Mayne et al. (2000); construct a control sequence $\tilde{\mathscr{U}}$ and show that $J(x^+, \tilde{\mathscr{U}}) \leq J(x, \mathscr{U}^*(x))$ which implies Inequality 12 since $J(x^+, \mathscr{U}^*(x^+)) \leq J(x^+, \tilde{\mathscr{U}})$.

For brevity let $\mathscr{U}^*(x)$ be abbreviated as $\mathscr{U}^* = \{u_0, \ldots, u_{N-1}\}$ and let $\mathscr{X}(x, \mathscr{U}^*(x))$ be abbreviated by $\mathscr{X}^* = \{x, \ldots, x_N\}$. Define a control sequence $\tilde{\mathscr{U}}$ with corresponding state evolution $\tilde{\mathscr{X}} = \mathscr{X}(x^+, \tilde{\mathscr{U}})$ as

$$\begin{aligned} \tilde{\mathscr{U}} &= \{u_1, \ldots, u_{N-1}, -Kx_N\}, \\ \tilde{\mathscr{X}} &= \{x_1, \ldots, x_N, x_N^+\}, \end{aligned}$$

where $x_N^+ = f(x_N, -Kx_N)$.

Proposition 5.1 *If $\mathcal{F}^o(x)$ is non-empty then (i) $\mathcal{F}^o(x^+)$ is non-empty with $\tilde{\mathscr{U}} \in \mathcal{F}^o(x^+)$ and (ii) the following inequality holds.*

$$J(x^+, \tilde{\mathscr{U}}) \leq J(x, \mathscr{U}^*)$$

Proof.

(i) Lemma 2.1 gives that $\mathscr{U}^* \in \mathcal{F}^o(x)$. Therefore $x_N \in \mathbb{X}_F^o$ and $-Kx_N \in \mathbb{U}^o$ since $x_N \in \mathbb{X}_F^o$ and therefore $\tilde{\mathscr{U}} \in \bar{\mathbb{U}}^o$. It suffices to show that $\tilde{\mathscr{X}} \in \bar{X}^o$ which reduces to showing that $x_N^+ \in \mathbb{X}_F^o$ since the first N elements of $\tilde{\mathscr{X}}$ are strictly feasible. This follows immediately since X_F is invariant.

(ii) We may write $J(x^+, \tilde{\mathscr{U}}) - J(x, \mathscr{U}^*)$ as follows by cancelling common terms.

$$\begin{aligned} J(x^+, \tilde{\mathscr{U}}) - J(x, \mathscr{U}^*(x)) &= F(x_N^+) - F(x_N) + \ell(x_N, -Kx_N) - \ell(x, u_0^*), \\ &= \|x_N^+\|_P^2 - \|x_N\|_P^2 + \mu B_F(x_N^+) - \mu B_F(X_N) \\ &\quad + \|x_N\|_Q^2 + \|Kx_N\|_R^2 + \mu B_x(x_N) + \mu B_u(-Kx_N) \\ &\quad - \ell(x, u_0^*). \end{aligned}$$

Since $x_N \in \mathbb{X}_F^o$ it follows from Lemma 4.1 that

$$\|x_N^+\|_P^2 - \|x_N\|_P^2 \leq -\|x_N\|_Q^2 - \|Kx_N\|_R^2 - \|x_N\|_H^2.$$

So

$$\begin{aligned} J(x^+, \tilde{\mathscr{U}}) - J(x, \mathscr{U}^*) &\leq \mu B_F(x_N^+) - \mu B_F(x_N) \\ &\quad - \|x_N\|_H^2 + \mu B_x(x_N) + \mu B_u(-Kx_N) \\ &\quad - \ell(x, u_0^*). \end{aligned}$$

Furthermore, by Lemma 5.1 it follows that

$$\begin{aligned} J(x^+, \tilde{\mathscr{U}}) - J(x, \mathscr{U}^*) &\leq -\|x_N\|_H^2 + \mu B_x(x_N) + \mu B_u(-Kx_N) \\ &\quad -\ell(x, u_0^*). \end{aligned}$$

Since $B_x(x_N) + B_u(-Kx_N) = B_K(x_N)$, it follows from Assumption 4.2 that

$$J(x^+, \tilde{\mathscr{U}}) - J(x, \mathscr{U}^*) \leq -\ell(x, u_0^*),$$

which completes the proof since $\ell(x, u_0^*) \geq 0$.

□

6 Gradient recentred self-concordant barrier functions

In this section we discuss self-concordant barrier functions; these were originally introduced by Nesterov and Nemirovskii (1994). Using the gradient of such a barrier we construct a gradient recentred self-concordant barrier function which satisfies Definition 2.1 and Assumption 4.2.

Fundamental to the development of self-concordant barrier functions are the class of strongly self-concordant functions. Loosely speaking, if a convex function is self-concordant then, *inter-alia*, its second order differential changes in a "predictable" manner (at least locally). The implication is that Newton's method applied to self-concordant functions behaves in a predictable manner (at least locally). We are not so interested in Newton's method in this paper, but rather we exploit certain properties of gradient recentred self-concordant barrier functions to satisfy key assumptions used in proving stability of r-MPC.

A particularly elegant exposition of Newton's method and interior-point methods from the context of self-concordant functions and barrier functions can be found in Renegar (2001).

Let E denote a finite dimensional real vector space and Q denote an open subset of E. Let f be a function from Q to $\mathbb{R}$ which is d-th continuously Fréchet differentiable on Q, denoted by $f \in \mathcal{C}^k$. Let $\nabla^k f(x)[h_1, \dots, h_k]$ denote the k'th differential of f about $x \in Q$ along the collection of directions h_k for $k = 1, \dots, d$ (see Definition 2.1.1 in Nesterov and Nemirovskii (1994)). Self-concordant functions are defined as follows. It is worth noting that although the definition includes a third differential, it is not used beyond the proof of Theorem 6.1 (see Nesterov and Nemirovskii, 1994). Indeed, Renegar (2001) uses Theorem 6.1 as a definition of self-concordance.

Definition 6.1 ***Self-concordance.*** *Let Q be an open nonempty subset of a finite dimensional real vector space E and let $f : Q \to \mathbb{R}$ be a function and ν be a non-negative scalar. If $f \in \mathcal{C}^3$ and for all $x \in Q$ and all $h \in E$ the following inequality is satisfied*

$$|\nabla^3 f(x)[h, h, h]| \leq 2\nu^{-1/2}(\nabla^2 f(x)[h, h])^{3/2},$$

then f is said to be ν-self-concordant on Q. Furthermore, if the sets $\{x \in Q : f(x) \leq t\}$ are closed in E for all $t \in \mathbb{R}$ then f is called strongly ν-self-concordant on Q. Alternatively, if $f(x_k) \to \infty$ for every sequence $\{x_k \in Q\}$ approaching a boundary point of Q, then f is strongly ν-self-concordant on Q. (Note that if a function f is ν-self-concordant then $\nu^{-1} f$ is 1-self-concordant). □

Define a set E_f as $E_f = \{h \in E : \nabla^2 f(x)[h, h] = 0\}$. If $E_f = \{0\}$ then f is called non-degenerate. For the purposes of this paper it suffices to consider non-degenerate strongly 1-self-concordant functions.

Given an open non-empty convex set $Q \subset E$ and a non-degenerate strongly 1-self-concordant function $f : Q \to \mathbb{R}$, define a local norm about the point $x \in Q$ as

$$\|h\|_{x,f} = \left(\nabla^2 f(x)[h, h]\right)^{1/2}, \quad \text{for } h \in E.$$

The Dikin's ellipsoid of radius r Nesterov and Nemirovskii (1994) about $x \in Q$ is given by

$$W_{x,f}(r) \quad = \quad \{y \in Q : \|x - y\|_{x,f} \leq r\}.$$

Two important properties of strongly self-concordant functions are given below (this is a modified version of Theorem 2.1.1 in Nesterov and Nemirovskii (1994)).

Theorem 6.1 *Let f be a non-degenerate strongly 1-self-concordant function on the open non-empty convex set $G^o \subset E$ (the interior of G) and let $x \in G^o$. Then every Dikin's ellipsoid centred on x of radius $r < 1$ is contained in G^o, i.e.*

$$r \in [0,1) \quad \Rightarrow \quad W_{x,f}(r) \subset G^o.$$

Moreover, if $r = 1$ then $W_{x,f} \subset G$. Further, let $r \in [0,1)$, then for each $y \in W_{x,f}(r)$ the following holds for all $h \in E$,

$$(1-r)^2 \nabla^2 f(x)[h,h] \quad \leq \quad \nabla^2 f(y)[h,h] \quad \leq \quad \frac{1}{(1-r)^2} \nabla^2 f(x)[h,h].$$

□

We see from Definition 6.1 that strongly self-concordant functions exhibit properties $(\mathcal{B}.1)$ and $(\mathcal{B}.2)$ of Definition 2.1. In fact, functions which satisfy these two properties are often called barrier functions. Nesterov and Nemirovskii (1994) define a class of self-concordant barrier functions, which in addition to being 1-strongly self-concordant, satisfy a Lipschitz condition using the local norm defined by the second order differential.

A self-concordant barrier function satisfies the following.

Definition 6.2 ***Self-concordant barrier.*** *Let G be a closed convex subset of a finite dimensional real vector space E and ϑ be a non-negative scalar. Let G^o denote the interior of G. A function $f : G^o \to \mathbb{R}$ is called a ϑ-self-concordant barrier for G if f is strongly 1-self-concordant on G^o and*

$$\sup\left\{\lambda^2(f,x) : x \in G^o\right\} \quad \leq \quad \vartheta,$$

where $\lambda(f,x)$ is given by

$$\lambda(f,x) \quad = \quad \inf\left\{\alpha \geq 0 : |\nabla f(x)[h]| \leq \alpha \left(\nabla^2 f(x)[h,h]\right)^{1/2} \quad \forall h \in E\right\}.$$

The value of ϑ is called the barrier parameter. □

In order to satisfy property $(\mathcal{B}.3)$ from Definition 2.1 we use the gradient of a self-concordant barrier function about the origin and construct a gradient recentred barrier function as follows.

Lemma 6.1 *Let G be a closed convex set which contains the origin in its interior G^o. Let f be a self-concordant barrier function for G. Define a function g as*

$$g : G^o \to \mathbb{R}, \quad g(x) \quad = \quad f(x) - f(0) - \nabla f(0)[x]. \tag{13}$$

Then g is a recentred barrier function for G about the origin.

Proof. Note that g trivially satisfies properties $(\mathcal{B}.1)$ and $(\mathcal{B}.2)$ since f does. Furthermore, since f is strictly convex and continuously differentiable on G^o then (see e.g. Boyd and Vandenberghe, 2002),

$$f(x) \quad \geq \quad f(0) + \nabla f(0)[x],$$

with equality only at $x = 0$, hence g satisfies property $(\mathcal{B}.3)$. $\square$

Hence we can construct a gradient recentred self-concordant barrier function given any self-concordant barrier function. In the following lemma we show that a gradient recentred self-concordant barrier function satisfies Assumption 4.2.

Lemma 6.2 *Let f be a gradient recentred self-concordant barrier function for the closed convex set G about the origin, then f satisfies Assumption 4.2 with the following choices for $\beta > 0$, $M \succ 0$ and $H \succ 0$,*

$$\beta \in (0,1), \qquad M = \nabla^2 f(0), \qquad H = \frac{\mu}{2(1-\beta)^2} M,$$

where $\nabla^2 f(0)$ denotes the Hessian matrix of f at the origin.

Proof. Using the exact form of Taylor's Theorem, there exists a point z on the line segment between the origin and x such that

$$f(x) = f(0) + \nabla f(0)[x] + \tfrac{1}{2}\nabla^2 f(z)[x,x].$$

Since f is a gradient recentred barrier it follows by definition that $f(0) = 0$ and $\nabla f(0)[x] = 0$. Furthermore, since f is 1-strongly self-concordant it follows from Theorem 6.1 that for any $\beta \in (0,1)$

$$x \in W_{0,f}(\beta) \ \Rightarrow \ \nabla^2 f(z)[x,x] \ \leq \ \frac{1}{(1-\beta)^2}\nabla^2 f(0)[x,x].$$

Hence for any $\beta \in (0,1)$ it follows that

$$x \in W_{0,f}(\beta) \ \Rightarrow \ \mu f(x) \ \leq \ \frac{\mu}{2(1-\beta)^2}\nabla^2 f(0)[x,x].$$

$\square$

From the above Lemma, it follows that if β is fixed then H can be chosen arbitrarily small using μ. However, for μ fixed even the smallest values of β will not reduce H arbitrarily.

A remarkable result from Nesterov and Nemirovskii (1994) is the existence of an $O(n)$-self-concordant barrier function for an arbitrary closed convex set $G \subset \mathbb{R}^n$. Unfortunately, the universal barrier function does not in general have easily computable gradient vectors and Hessian matrices. Hence, algorithms which use this information in computing search directions for minimisation are not practical in this case. However, in many important cases self-concordant barriers exist with easily computable gradient vectors and Hessian matrices.

Nesterov and Nemirovskii (1994) provide a barrier calculus in Chapter 5 of their book. In addition they stipulate self-concordant barrier functions for some important convex sets. For example, let G be a polytope given by

$$G = \{x \in \mathbb{R}^n : \langle a_i, x\rangle \leq b_i, \ i = 1, \ldots, m\}.$$

If $G^o \neq \emptyset$ then the barrier B defined by

$$B(x) = -\sum_{i=1}^{m} \ln\left(b_i - \langle a_i, x\rangle\right),$$

is an m-self-concordant barrier for G. Let G be defined by the union of quadratic domains

$$G = \{x \in \mathbb{R}^n : f_i(x) \leq 0, \ i = 1, \ldots, m\},$$

where each f_i is a convex quadratic function. Again if $G^o \neq \emptyset$ then the barrier B defined by

$$B(x) \;=\; -\sum_{i=1}^{m} \ln\left(-f_i(x)\right),$$

is an m-self-concordant barrier for G. Finally, let G be given by the cone of positive semi-definite matrices

$$G \;=\; \{x \in \mathbb{S} : \lambda_i(x) \geq 0, i = 1, \ldots, n\},$$

where $\mathbb{S}$ denotes the space of $n \times n$ symmetric matrices and $\lambda_i(\cdot)$ denotes the i'th eigenvalue of x. The barrier B defined by

$$B(x) \;=\; -\ln(\det(x))$$

is an n-self-concordant barrier function for G. Rules for combining sets and associated barriers can be found in Proposition 2.3.1 and Chapter 5 of Nesterov and Nemirovskii (1994). The gradient recentred self-concordant barrier function used in this section relies on an underlying self-concordant barrier function. Recall that a self-concordant barrier function is a 1-strongly self-concordant function which satisfies a further Lipschitz condition with constant ν. Including the recentring term into the barrier increases the value of ν, which results in slower algorithms (at least theoretically). This technical difficulty may be overcome by separating the recentring term from the underlying barrier and grouping it with the cost function. More formally, suppose $B : X \to \mathbb{R}$ is a ν-self-concordant barrier function for the open convex set $X \subset \mathbb{R}^n$ and $R : X \to \mathbb{R}$ is a gradient recentred barrier function about the point $\bar{x} \in X$ constructed by

$$R(x) \;=\; B(x) - B(\bar{x}) - \nabla B(\bar{x})^T(x - \bar{x}).$$

Further suppose there is a cost $J : \mathbb{R}^n \to \mathbb{R}$ and a solution to the following problem is sought for some positive value of μ.

$$(\mathcal{R}_\mu): \qquad \min_x \tfrac{1}{\mu} J(x) + R(x),$$

Let $J_R : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$J_R(x) \;=\; J(x) - \mu\left(B(\bar{x}) + \nabla B(\bar{x})^T(x - \bar{x})\right).$$

It is immediate to see that the solution to

$$(\mathcal{S}_\mu): \qquad \min_x \tfrac{1}{\mu} J_R(x) + B(x),$$

coincides with the solution to $(\mathcal{R})_\mu$. Furthermore, solving $(\mathcal{S})_\mu$ is equivalent to finding the point on the central path of

$$(\mathcal{S}): \qquad \begin{array}{ll} \min\limits_x & J_R(x) \\ \text{s.t.} & x \in X, \end{array}$$

for the same value of μ.

7 Illustration for a linear system

When the system is linear certain aspects of the above discussion simplify. In particular, the value of η in (11) can be set to zero and Lemma 4.1 holds for $\alpha = \alpha^*$. Furthermore, the r-MPC optimisation problem from Section 2 and the steady-state optimisation problem from Section 3 are both convex. Therefore, polynomial-time interior-point methods can be employed to solve these problems.

The following simulation example is intended to illustrate r-MPC for a linear system with absolute and rate constraints on the input. Since the rate constraints must be represented as absolute state constraints (see below), then this example covers both input and state constraints. A terminal constraint set $\mathbb{X}_F$ is constructed in the same manner used in Section 5.

7.1 Simulation example

Consider a linear SISO (single input single output) system

$$\begin{aligned}\tilde{x}(t+1) &= \tilde{A}\tilde{x}(t) + \tilde{B}u(t),\\ y(t) &= \tilde{C}x(t).\end{aligned}$$

with absolute and rate constraints on the inputs

$$\begin{aligned}u(t) &\leq \upsilon_{\max},\\ u(t) &\geq \upsilon_{\min},\\ u(t)-u(t-1) &\leq \Delta\upsilon_{\max},\\ u(t)-u(t-1) &\geq \Delta\upsilon_{\min}.\end{aligned}$$

Assume that $(0,0) \in \mathbb{X}^o \times \mathbb{U}^o$. The gradient recentred logarithmic barrier for the absolute constraints would be

$$B_u(u(t)) = \ln\left(\frac{\upsilon_{\max}}{\upsilon_{\max}-u(t)}\right) + \ln\left(\frac{-\upsilon_{\min}}{u(t)-\upsilon_{\min}}\right) - \frac{u(t)}{\upsilon_{\max}} - \frac{u(t)}{\upsilon_{\min}}.$$

The rate constraints given above are not in the correct form. Rather they must be considered as state constraints for a suitably augmented plant,

$$\begin{aligned}x(t+1) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned}$$

with

$$A = \begin{bmatrix}\tilde{A} & 0 & 0\\ 0 & 0 & 0\\ 0 & -1 & 0\end{bmatrix}, \qquad B = \begin{bmatrix}\tilde{B}\\ 1\\ 1\end{bmatrix}, \qquad C = \begin{bmatrix}\tilde{C} & 0 & 0\end{bmatrix}.$$

The input rate constraints are expressed as absolute constraints on the nth component of the state as denoted by $x^n(t)$ (this should not be confused with the ith state of a state evolution which is denoted by $x_i(t) \in \mathbb{R}^n$). The gradient recentred logarithmic barrier function for this constraint is given by,

$$B_x(x(t)) = \ln\left(\frac{\Delta\upsilon_{\max}}{\Delta\upsilon_{\max}-x^n(t)}\right) + \ln\left(\frac{-\Delta\upsilon_{\min}}{x^n(t)-\Delta\upsilon_{\min}}\right) - \frac{x^n(t)}{\Delta\upsilon_{\max}} - \frac{x^n(t)}{\Delta\upsilon_{\min}}$$

Furthermore, when $\Delta\upsilon_{\min} = -\Delta\upsilon_{\max}$ this reduces to

$$B_x(x(t)) = \ln\left(\frac{\Delta\upsilon^2_{\max}}{\Delta\upsilon^2_{\max} - (x^n(t))^2}\right)$$

Figures 2 to 4 illustrate results from a specific simulation example. The plant is

$$\tilde{x}(t+1) = \begin{bmatrix}1.0759 & 0.1382\\ 0.1036 & 1.0068\end{bmatrix}\tilde{x}(t) + \begin{bmatrix}0.1036\\ 0.0051\end{bmatrix}u(t), \tag{14}$$

$$y(t) = \begin{bmatrix}-3 - 6\end{bmatrix}\tilde{x}(t) \tag{15}$$

Algorithm 2.1 is implemented with a 20 sample prediction horizon. The state weighting matrix Q and input weighting matrix R are given by

$$Q = 0.01 \times C^T C, \qquad R = 1.$$

The final state weighting matrix P is given by the solution to

$$P = A^T PA - A^T PB \left[B^T PB + R\right]^{-1} B^T PA + Q.$$

In all cases the absolute value of $u(t)$ is constrained to $|u(t)| \leq 0.5$.

Figure 2 shows a unit step response from 1 to 0 for the case where rate constraints are symmetric $|u(t+1) - u(t)| \leq 0.04$. Two responses are shown: (i) with the barrier weighting parameter $\mu = 10^{-6}$ and (ii) with $\mu = 1$. Note that case (ii) is considerably smoother.

Figure 3 illustrates responses under similar conditions, but this time the rate constraints are given by $-0.04 \leq u(t+1) - u(t) \leq 0.02$. The barriers are not recentred, and significant steady state error results. Figure 4 illustrates responses where the barriers are recentred. This time the steady state behaviour is correct.

Consider the case where the input is constrained to lie between -0.5 and 1 but there are no rate constraints. This may be represented as

$$Lu(t) \leq b,$$

with

$$L = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}.$$

Let Q and R be given by

$$Q = \begin{bmatrix} 0.9 & 1.8 \\ 1.8 & 3.6 \end{bmatrix}, \qquad R = 1.$$

We can construct a local stabilising controller $u(t) = -Kx(t)$ by firstly solving the following equation for P_K,

$$P_K = A^T P_K A - A^T P_K B \left[B^T P_K B + R\right]^{-1} B^T P_K A + Q,$$

and then set K according to

$$K = \left[B^T P_K B + R\right]^{-1} B^T P_K A.$$

The set $\mathbb{X}_K$ defined in Section 4 is thus given by

$$\mathbb{X}_K = \left\{x \in \mathbb{R}^2 : -LKx \leq b\right\}.$$

The gradient recentred logarithmic barrier function for $\mathbb{X}_K$ is given by

$$B_K(x(t)) = \ln\left(\frac{b_1}{b_1 + L_1 K x(t)}\right) + \ln\left(\frac{b_2}{b_2 + L_2 K x(t)}\right) + \frac{L_1 K}{b_1} x(t) + \frac{L_2 K}{b_2} x(t),$$

where b_i denotes the i'th element of b and likewise L_i denotes the i'th element of L. Note that B_K is a strongly 1-self-concordant function (see Section 2.3 of Nesterov and Nemirovskii (1994)). The Hessian of B_K about the origin is given by

$$\nabla^2 B_K(0) = K^T L^T \begin{bmatrix} b_1^{-2} & 0 \\ 0 & b_2^{-2} \end{bmatrix} LK.$$

Set $\mu = 1$. We consider two values for β. Let $\beta = \sqrt{0.99}$ and choose M and H according to Lemma 6.2, i.e. $M = \nabla^2 B_K(0)$ and $H = \frac{\mu}{2(1-\beta)^2} M$. Determine P from the following Lyapunov equation (see Section 4),

$$A_K^T P A_K - P = -Q - K^T RK - H.$$

From Section 4 it follows that $\alpha^* = 210.5424$. Therefore $W_P(\alpha^*)$ is an invariant set under the control $u(t) = -Kx(t)$. The set $\mathbb{X}_K$ and ellipse $W_P(\alpha^*)$ are shown in Figure 5. If the above process is repeated for $\beta = \sqrt{0.5}$ then a smaller ellipse is generated as shown in Figure 6.

8 Conclusion

We have developed a new class of model predictive control based on the recentred barrier function. The simulations illustrate that control action becomes cautious near constraint boundaries. We conjecture that such behaviour may be desirable in physical systems where uncertainty exists near constraint boundaries. The degree of caution is directly related to the positive weighting parameter which characterises the controller class. Moreover it is possible to specify separate parameter values for the dynamic and steady-state cases. The successful application of such a controller to a 3-input 3-output edible oil refining plant is discussed by Wills (2003).

The control algorithms can be implemented efficiently using simple modifications to existing interior point algorithms, and applied to a wide class of convex problem. We have exploited the theory of self-concordant functions introduced by Nesterov and Nemirovskii (1994) to construct an ellipsoidal invariant set in a particularly elegant manner. The associated geometry can also be used to demonstrate nominal stability via the incorporation of a terminal constraint set and an argument based on that of Mayne et al. (2000). We stress that this is subject to the usual caveats about feasibility and robustness. The simulations demonstrate that stability can be achieved without terminal constraint sets (this is, of course, well known for more conventional model predictive controllers).

One aspect of such a control scheme is that the optimization problem to be solved at each step is the minimization of an unconstrained convex cost function—in the sense that the solution always lies on the interior of the constraint set and the gradient of the cost function is zero at the solution. Indeed this requirement motivated the introduction of the recentred barrier. Recently Jadbabaie et al. (2001) have shown that it may be possible to demonstrate stability of unconstrained nonlinear model predictive controllers without the introduction of terminal constraints. A natural question would be whether the proposed class of controllers might be analysed in such a fashion. As the class of controller we have proposed includes more conventional model predictive control as a limiting case, this may suggest a new avenue for the analysis of constrained model predictive control.

References

Albuquerque, J., Gopal, V., Staus, G., Biegler, L. T., Ydstie, B. E., 1997. Interior point SQP strategies for structured process optimization problems. Computers & Chemical Engineering 2, S853–S859.

Bartlett, R. A., Wachter, A., Biegler, L. T., 2000. Active set vs. interior point strategies for model predictive control. In: Proceedings of the American Control Conference. Chicago, Illinois, pp. 4229–4233.

Ben-Tal, A., Nemirovskii, A., 2001. Lectures on Modern Convex Optimization. SIAM, Philadelphia.

Bitmead, R. R., Gevers, M., Wertz, V., 1990. Adaptive Optimal Control -The Thinking Man's GPC. Prentice Hall, Inc., Englewood Cliffs, New Jersey.

Blanchini, F., 1999. Set invariance in control. Automatica 35 (11), 1747–1767.

Boyd, S., Vandenberghe, L., December 2002. Convex optimization, to appear. `http://www.stanford.edu/~boyd/cvxbook.html`.

Chen, H., Allgöwer, F., 1998. A quasi-infinite nonlinear model predictive control scheme with guaranteed stability. Automatica 34 (10), 1205–1217.

Clarke, D. W., Mohtadi, C., Tuffs, P. S., 1987. Generalized predictive control, parts 1 and 2. Automatica 23 (2), 137–148.

Cutler, C. R., Ramaker, B. L., 1980. Dynamic matrix control - a computer control algorithm. In: Proceedings Joint American Control Conference. San Francisco, CA.

Cutler, D. W., Morshedi, A., Haydel, J., 1983. An industrial perspective on advanced control. AIChE Annual Meeting .

Dunn, J. C., Bertsekas, D. P., 1989. Efficient dynamic programming implementations of Newton's method for unconstrained optimal control problems. Journal of Optimization Theory and Applications 63, 23–38.

Fiacco, A. V., McCormick, G. P., 1968. Nonlinear Programming: Sequential Unconstrained Minimization Techniques. John Wiley & Sons Inc., New York.

Forsgren, A., Gill, P. E., Wright, M. H., October 2002. Interior methods for nonlinear optimization. SIAM Review 44 (4), 525–597.

Frisch, R., 1956. La resolution des problemes de programme linearie par la methode du potential logarithmique. Cahiers du Seminaire D'Econometrie 4, 7–20.

Garcia, C. E., Prett, D. M., Morari, M., 1989. Model predictive control: Theory and practice - A survey. Automatica 25 (3), 335–348.

Gilbert, E. G., Tan, K. T., September 1991. Linear systems with state and control constraints: The theory and application of maximal output admissible sets. IEEE Transactions on Automatic Control 36 (9), 1008–1020.

Gill, P. E., Murray, W., Saunders, M. A., Tomlin, J. A., Wright, M. H., 1986. On projected Newton barrier methods for linear programming and an equivalence to Karmarkar's projective method. Mathematical Programming 36, 183–209.

Glad, T., Jonson, H., July 1984. A method for state and control constrained linear quadratic control problems. In: Proceedings of the 9th IFAC World Congress. Budapest, Hungary.

Glineur, F., December 2000. Topics in convex optimization: Interior-point methods, conic duality and approximations. Ph.D. thesis, Faculté Polytechnique de Mons, http://mathro.fpms.ac.be/~glineur/.

Gopal, V., Biegler, L. T., 1998. Large scale inequality constrained optimization and control. IEEE Control Systems Magazine 18 (6), 59–68.

Hansson, A., September 2000. A primal-dual interior-point method for robust optimal control of linear discrete-time systems. IEEE Transactions on Automatic Control 45 (9), 1639–1655.

Heath, W. P., Wills, A. G., 2002. Recentred barriers for model predictive control with state constraints. Non-linear Predictive Control Workshop at the University of Oxford, 18-19th July .

Jadbabaie, A., Yu, J., Hauser, J., May 2001. Unconstrained receding horizon control on nonlinear systems. IEEE Transactions on Automatic Control 46 (5), 776–783.

Karmarkar, N., 1984. A new polynomial time algorithm for linear programming. Combinatorica 4 (4), 373–395.

Kassmann, D. E., Badgwell, T. A., Hawkins, R. B., May 2000. Robust steady-state target calculation for model predictive control. AIChE Journal 46 (5), 1007–1024.

Lee, E. B., Markus, L., 1967. Foundations of Optimal Control Theory. John Wiley & Sons, Inc., New York.

Lieu, B. T., Huard, P., 1966. La methode des centres dans un espace topologique. Numerische Mathematik 8, 56–67.

Lim, A., Moore, J., Faybusovich, L., July 1996. Linearly constrained LQ and LQG optimal control. In: Proceedings of the 13th IFAC World Congress. San Francisco, California.

Maciejowski, J. M., 2002. Predictive Control with Constraints. Pearson Education Limited, Harlow, Essex.

Mayne, D. Q., Rawlings, J. B., Rao, C. V., Scokaert, P. O. M., 2000. Constrained model predictive control: Stability and optimality. Automatica 36, 789–814.

Michalska, H., Mayne, D. Q., November 1993. Robust receding horizon control of constrained nonlinear systems. IEEE Transactions on Automatic Control 38 (11), 1623–1633.

Muske, K. R., Rawlings, J. B., 1993. Model predictive control with linear models. AIChE Journal 39 (2), 262–287.

Nesterov, Y., Nemirovskii, A., 1994. Interior-point Polynomial Algorithms in Convex Programming. SIAM, Philadelphia.

Nesterov, Y. E., Todd, M. J., April 1997. Self-scaled cones and interior–point methods in nonlinear programming. Mathematics of Operations Research 22, 1–42.

Polak, E., Yang, T. H., Mayne, D. Q., December 1990. A method of centers based on barrier functions for solving optimal control problems with continuum state and control constraints. In: Proceedings of the 29th Conference on Decision and Control. Vol. 4. Honolulu, Hawaii, pp. 2327–2332.

Qin, S. J., Badgwell, T. A., 1997. An overview of industrial model predictive control technology. AIChE Symposium Series, 5th International Symposium on Chemical Process Control 93, 232–256.

Rao, C. V., Rawlings, J. B., 1999. Steady states and constraints in model predictive control. AIChE Journal 45 (6), 1266–1278.

Rao, C. V., Wright, S. J., Rawlings, J. B., 1998. Application of interior point methods to model predictive control. Journal of Optimization Theory and Applications 99 (3), 723–757.

Renegar, J., 2001. A Mathematical View of Interior-Point Methods in Convex Optimization. SIAM, Philadelphia.

Richalet, J., Rault, A., Testud, J. L., Papon, J., 1976. Algorithmic control of industrial processes. In: Proceedings of Fourth IFAC symposium on identification and system parameter estimation. pp. 1119–1167.

Richalet, J., Rault, A., Testud, J. L., Papon, J., 1978. Model predictive heuristic control: Applications to industrial processes. Automatica 14, 413–428.

Rockafellar, R. T., 1970. Convex Analysis. Princeton University Press, Princeton, New Jersey.

Roos, C., Terlaky, T., Vial, J.-P., 1997. Theory and Algorithms for Linear Optimization. John Wiley and Sons Inc., New York.

Vandenberghe, L., Boyd, S., Nouralishahi, M., July 2002. Robust linear programming and optimal control. In: 15th Triennial World Congress of the International Federation of Automatic Control. Barcelona.

Wills, A. G., 2003. Barrier function based model predictive control. Ph.D. thesis, School of Electrical Engineering and Computer Science, submitted March 2003.

Wills, A. G., Heath, W. P., 2002a. A recentred barrier for constrained receding horizon control. In: Proceedings of the 2002 American Control Conference. Vol. 5. Anchorage, Alaska, pp. 4177–4182.

Wills, A. G., Heath, W. P., July 21-26 2002b. Using a modified predictor-corrector algorithm for model predictive control. In: Proceedings of the 15th IFAC World Congress on Automatic Control. Barcelona, Spain.

Wills, A. G., Heath, W. P., 2003. Technical Report - EE03016 Interior-Point Methods for Linear Model Predictive Control. Tech. rep., School of Electrical Engineering and Computer Science, http://www.eecs.newcastle.edu.au/cgi-bin/report_db/new_search.cgi.

Wright, M. H., 1992. Interior methods for constrained optimization. In: Iserles, A. (Ed.), Acta Numerica 1992. Cambridge University Press, New York, USA, pp. 341–407.
URL citeseer.nj.nec.com/wright92interior.html

Wright, S. J., 1993. Interior-point methods for optimal control of discrete-time systems. Journal of Optimization Theory and Applications 77, 161–187.

Wright, S. J., 1997a. Applying new optimization algorithms to model predictive control. Chemical Process Control-V, CACHE, AIChE Symposium Series 93 (316), 147–155.

Wright, S. J., 1997b. Primal-Dual Interior-Point Methods. SIAM, Philadelphia.

Ye, Y., 1997. Interior Point Algorithms - Theory and Analysis. John Wiley & Sons Inc., New York.

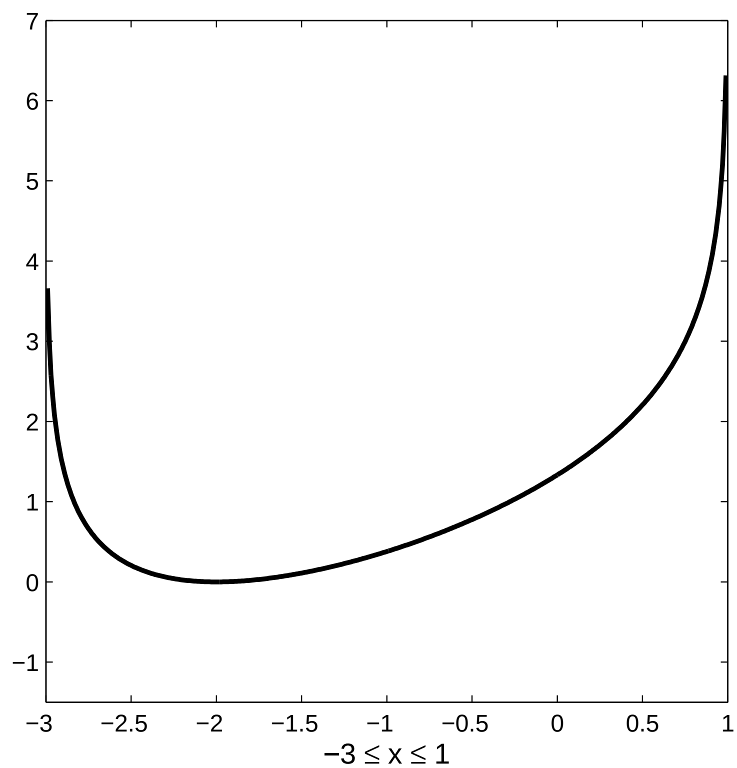

Figure 1: The graph of a recentred barrier function for $-3 \leq x \leq 1$ which is centred about -2. Note that the minimum occurs at $x = -2$ with the barrier being zero at this point.

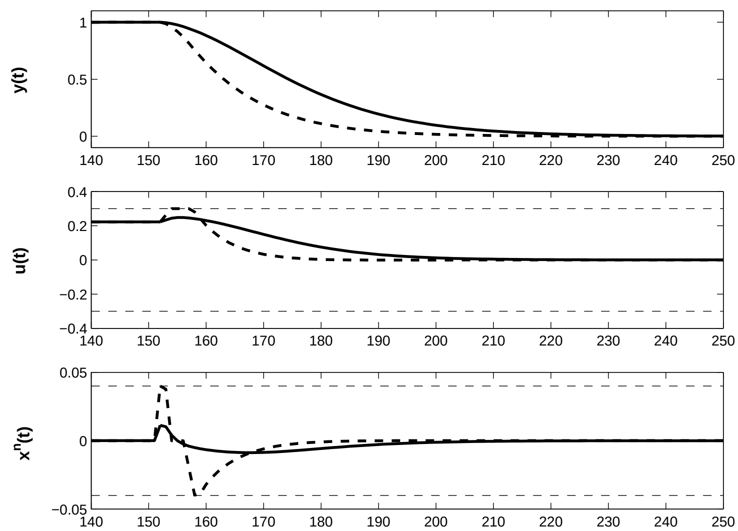

Figure 2: Simulation example with symmetric rate constraints. There are two cases: (i) $\mu = 10^{-6}$ shown as a dashed line and (ii) $\mu = 1$ shown as a solid line.

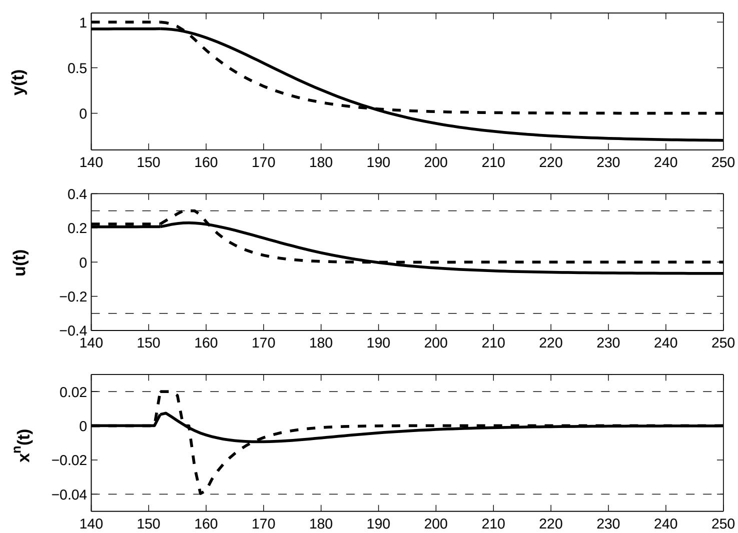

Figure 3: Simulation example with asymmetric rate constraints. There are two cases: (i) $\mu = 10^{-6}$ shown as a dashed line and (ii) $\mu = 1$ shown as a solid line. The barrier function for rate constraints is not recentred and the steady-state behaviour for case (ii) is significantly awry.

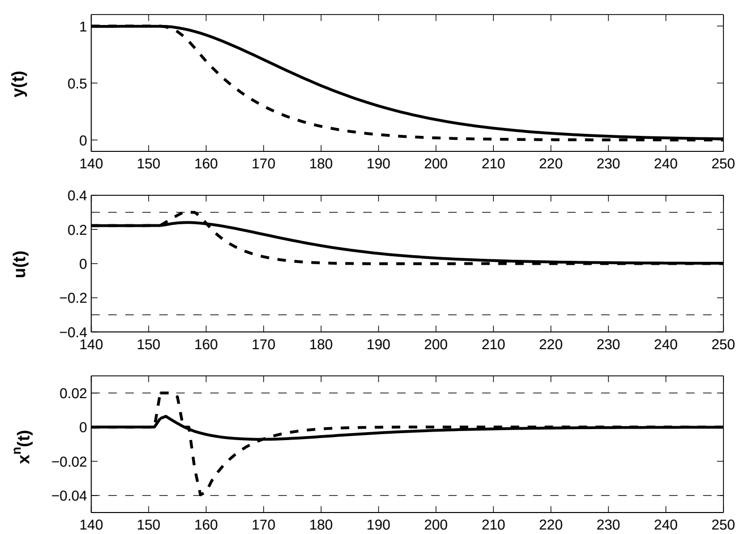

Figure 4: Simulation example with asymmetric rate constraints. There are two cases: (i) $\mu = 10^{-6}$ shown as a dashed line and (ii) $\mu = 1$ shown as a solid line. The barriers are now recentred.

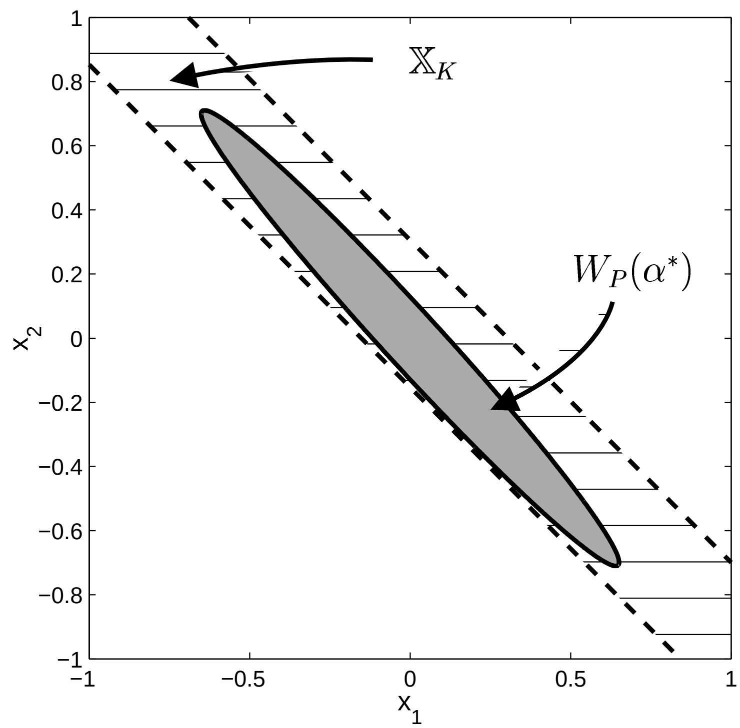

Figure 5: The two sets $\mathbb{X}_K$ and $W_P(\alpha^*)$ corresponding to $\beta = \sqrt{0.99}$ for for the simulation example in Section 7.1.

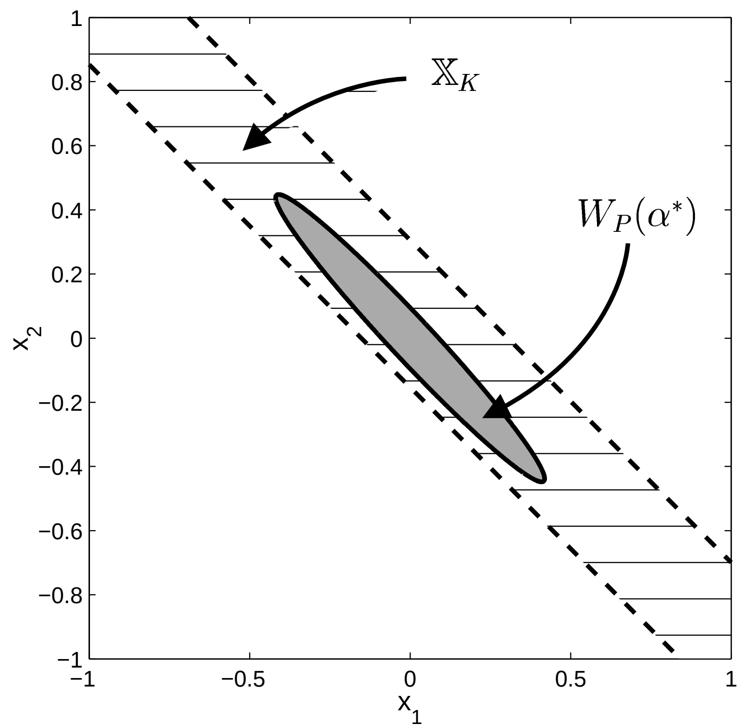

Figure 6: The two sets $\mathbb{X}_K$ and $W_P(\alpha^*)$ corresponding to $\beta = \sqrt{0.5}$ for the simulation example in Section 7.1.