Robust cross-directional control of paper making machines with saturating actuators

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1 Introduction

Preface

In this report we discuss the design of cross-directional controllers which are guaranteed to be robustly stabilizing while incorporating a quadratic programme for steady state performance. The report is loosely based on the technical report of Akkermans (2003) who proposes using the multivariable circle criterion to guarantee closed-loop stability of cross-directional controllers in the presence of sector-bounded nonlinearities. Heath and Wills (2004) recommend implementing cross-directional controllers in modal form with a constrained internal model control (IMC) structure; nominal optimal steady state performance is guaranteed via a non-linear element that incorporates a quadratic programme. Finally Heath *et al.* (2003) observe that the quadratic programme suggested by Heath and Wills (2004) can be expressed as a continuous sector bounded nonlinearity together with two linear transformations.

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Extended Abstract

Cross-directional control has received considerable attention in the academic community—see for example (Featherstone *et al.*, 2000) and references therein, as well as more recently (Duncan, 2002; Dochain *et al.*, 2003) and associated contributions. There are two main schools of cross-directional control design:

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firstly unconstrained control (perhaps with limited anti-windup) based on robust control methodologies, and secondly constrained control achieved via MPC (model predictive control). In both cases it is advantageous to decompose the profile into modes (Duncan *et al.*, 1996), with control action reduced or zero at high modes. Briefly the first school *guarantees* robust closed-loop behavior, while if the actuator response is sufficiently well-known the second school offers considerably improved steady state behavior (Ma *et al.*, 2002; Wills and Heath, 2002). This in turn may have significant impact on the economic viability of the machine.

Recently Heath and Wills (2004) proposed incorporating a quadratic programme for optimal steady state behavior via a constrained IMC (internal model control) structure. The controller is straightforward to implement in real time, and allows robust control designs to be directly translated to the constrained control case. Furthermore Heath *et al.* (2003) showed that such quadratic programmes can be modelled as continuous sector bounded non-linearities together with two linear transformations. This opens the possibility of using the multivariable circle criterion (Khalil, 2002) to guarantee robust stability. In this paper we show how all these elements can be combined to design a cross-directional controller with optimal steady state performance and guaranteed robust stability.

Many authors (e.g. Stewart *et al.*, 2003) assume that if the actuator response is decomposed into modes, the plant uncertainty may also be decomposed in the same modes. This is reasonable for paper machines to the extent that the interaction matrix can be approximated as a circular symmetric matrix. In this case, the multivariable circle criterion may also be decomposed, and it suffices to ensure the dynamics of each mode satisfies a single variable circle criterion. This ensures a straightforward design methodology which we illustrate with a simulation example.

Report structure

The report is structured as follows. In Section 2 we introduce the plant model and the control problem. We briefly review the IMC (internal model control) structure with quadratic programme proposed by Heath and Wills (2004) for the cross-directional control problem. In Section 3 we discuss a simple robust unconstrained design and in Section 4 we discuss how this design can be used (and perhaps further modified) to give a robust constrained controller. As the plant dynamics are split into modes, the control is also performed mode by mode. In the Appendix we review a simple constrained SISO (single input single output) IMC design strategy. In Section 5 we discuss the implications of leakage between modes. In Section 6 we illustrate the results with a simulation.

2 Plant model and control structure

We will make the standard assumption (e.g. Heath, 1996) that the open loop behaviour of the output profile $y(t)$ may be well approximated by the model

$$y(t) = z^{-k}h(z)Bu(t) + d(t) \tag{1}$$

Here $y(t) \in \mathbb{R}^n$ represents the measured profile across the web and $u(t) \in \mathbb{R}^m$ represents the array of actuators. Typically $n > m$. We assume the whole profile $y(t)$ is available simultaneously. The dynamics are represented as a delay of k time samples and a biproper transfer function $h(z)$. Typically $h(z)$ is stable and low order. However it may be non-minimum phase and the delay k may be fairly large. We assume without loss of generality that $h(1) = 1$.

The $(n \times m)$ interaction matrix B describes the steady state response of the actuators on the profile. Finally $d(t) \in \mathbb{R}^n$ represents disturbances on the plant.

Let the singular value decomposition (SVD) of B be

$$B = \Phi \Sigma \Psi^T \tag{2}$$

with Φ and Ψ orthonormal and the non-zero block of Σ diagonal.

We can then write $y(t)$ and $u(t)$ in terms of basis functions which are the columns of Φ and Ψ respectively. We will assume the controller is designed to act only on $r \leq \min(n, m)$ modes, corresponding to the first r singular values. It will be useful to define a reduced interaction matrix $B_r \in \mathbb{R}^{(n,m)}$ which can be decomposed as

$$B_r = \Phi_r \Sigma_r \Psi_r^T \tag{3}$$

with $\Phi_r \in \mathbb{R}^{(n,r)}$ representing the modes we wish to control, $\Sigma_r \in \mathbb{R}^{(r,r)}$ and $\Psi_r \in \mathbb{R}^{(m,r)}$. Thus the diagonal entries of Σ_r are the first r diagonal entries of Σ, and the columns of Φ_r and Ψ_r are respectively the first r columns of Φ and Ψ.

Thus we will assume the plant is given by

$$y(t) = G(z)u(t) + d(t) \tag{4}$$

with

$$G(z) = z^{-k} h(z) \Phi_r \Sigma_r \Psi_r^T + \tilde{\Delta}(z) \tag{5}$$

The term $\tilde{\Delta}(z)$ takes into account unknown dynamics, neglected dynamics (for example we may choose to include partial delays in $\tilde{\Delta}(z)$) and the neglected higher order modes of B.

Design choice 1. The choice of nominal model $z^{-k} h(z) \Phi_r \Sigma_r \Psi_r^T$ together with (implicitly) the number of modes constitute the first design choice. Furthermore we have chosen SVD, so Σ_r is diagonal.

We will put

$$\Delta(z) = \Phi_r^T \tilde{\Delta}(z) \Psi_r \tag{6}$$

Heath and Wills (2004) proposed the following control structure (see Fig 5):

$$\begin{aligned}
\hat{\delta}(t) &= \Phi_r^T y(t) - z^{-k} h(z) \Sigma_r \mu(t) \\
\hat{\delta}_q(t) &= Q_f(z) \hat{\delta}(t) - Q_b(z) \Sigma_r \mu(t) \\
\mu(t) &= \arg\min_{\mu} \left\| \Sigma_r \mu + \hat{\delta}_q(t) \right\|_2^2 \text{ such that } \Psi_r \mu \in \mathbb{U} \\
u(t) &= \Psi_r \mu(t)
\end{aligned} \tag{7}$$

Both $Q_f(z)$ and $Q_b(z)$ are chosen as diagonal transfer function matrices. This is the natural generalization of the single-input single-output control scheme depicted in Fig 3 (see Appendix), but with the saturation replaced by an optimizing anti-windup non-linear element, and implemented in modal form.

The actuator signal $u(t)$ is constrained to lie in some space $u(t) \in \mathbb{U}$. This will include both saturation constraints and bending constraints, as is usual for cross-directional controllers. Thus, for example, we may require

$$\begin{aligned}
|u_i(t)| &\leq U_m \\
|u_{i-1}(t) - 2u_i(t) + u_{i+1}(t)| &\leq U_b
\end{aligned} \tag{8}$$

for some U_m and U_b, and for each actuator position i.

3 Unconstrained design

As with the single-input single-output case (see Appendix), we begin our design by considering the unconstrained dynamics. When no constraints are active we have

$$Q_\delta(z) = \left[I - Q_b(z)\right]^{-1} Q_f(z) \tag{9}$$

Thus we will begin by designing a diagonal transfer function matrix $Q_\delta(z)$ and only later choose $Q_f(z)$ and $Q_b(z)$.

Design choice 2. Choose $Q_\delta(z)$. Specifically we will choose $Q_\delta(z)$ to be diagonal, and tune each loop according to standard IMC design laws. For simple plants we have two design parameters per loop: (i) speed of closed-loop response and (ii) incorporation of integral action.

When constraints are not active, the controller appears as

$$\begin{aligned} \hat{\delta}(t) &= \Phi_r^T y(t) - z^{-k} h(z) \Sigma_r \mu(t) \\ \mu(t) &= -\Sigma_r^{-1} Q_\delta(z) \hat{\delta}(t) \\ u(t) &= \Psi_r \mu(t) \end{aligned} \tag{10}$$

See Fig 6. If we substitute for $y(t)$ this gives in closed-loop

$$\hat{\delta}(t) = \Delta(z)\mu(t) + \Phi_r^T d(t) \tag{11}$$

and hence

$$\begin{aligned} u(t) &= -\Psi_r \left[\Sigma_r + Q_\delta(z)\Delta(z)\right]^{-1} Q_\delta(z) \Phi_r^T d(t) \\ y(t) &= \left\{ I - \left[z^{-k} h(z) \Phi_r \Sigma_r + \tilde{\Delta}(z) \Psi_r^T \right] \left[\Sigma_r + Q_\delta(z)\Delta(z)\right]^{-1} Q_\delta(z) \Phi_r^T \right\} d(t) \\ \Phi_r^T y(t) &= \left\{ I - \left[z^{-k} h(z) \Sigma_r + \Delta(z) \right] \left[\Sigma_r + Q_\delta(z)\Delta(z)\right]^{-1} Q_\delta(z) \right\} \Phi_r^T d(t) \end{aligned} \tag{12}$$

In particular, putting $\tilde{\Delta}(z) = 0$ gives the nominal sensitivity $S(z)$ and complementary sensitivity $T(z)$ as

$$\begin{aligned} S(z) &= I - z^{-k} h(z) \Phi_r Q_\delta(z) \Phi_r^T \\ T(z) &= z^{-k} h(z) \Phi_r Q_\delta(z) \Phi_r^T \end{aligned} \tag{13}$$

Such expressions are quite standard for multivariable IMC (Morari and Zafirou, 1989); however the modular approach allows considerable simplification (Featherstone *et al.*, 2000). For example, we find

$$\Phi_r^T S(z) \Phi_r = I - z^{-k} h(z) Q_\delta(z) \tag{14}$$

so design is straightforward provided we choose $Q_\delta(z)$ to be diagonal.

With a fixed disturbance d_{ss} the steady state output becomes:

$$\begin{aligned} y_{ss} &= \left\{ I - \left[\Phi_r \Sigma_r + \tilde{\Delta}(1) \Psi_r^T \right] \left[\Sigma_r + Q_\delta(1)\Delta(1)\right]^{-1} Q_\delta(1) \Phi_r^T \right\} d_{ss} \\ \Phi_r^T y_{ss} &= \left\{ I - \left[\Sigma_r + \Delta(1)\right] \left[\Sigma_r + Q_\delta(1)\Delta(1)\right]^{-1} Q_\delta(1) \right\} \Phi_r^T d_{ss} \end{aligned} \tag{15}$$

We see immediately the standard result that if $Q_\delta(1)$ is set to

$$Q_\delta(1) = I \tag{16}$$

then

$$\Phi_r^T y_{ss} = 0 \tag{17}$$

Note that this corresponds to integral action, and Stewart *et al.* (2003) recommend replacing such action with high gain proportional action. Wills and Heath (2002) discuss steady state behaviour in more detail.

For the analysis of stability robustness, it suffices to consider the feedback loop equations

$$\begin{aligned} \hat{\delta}(t) &= \Delta(z)\mu(t) + \Phi_r^T d(t) \\ \mu(t) &= -\Sigma_r^{-1} Q_\delta(z)\hat{\delta}(t) \end{aligned} \tag{18}$$

See Fig 7. Thus a standard robustness requirement might be that we choose Q_δ such that

$$\left|\left|-\Sigma_r^{-1} Q_\delta(.)\Delta(.)\right|\right|_\infty < 1 \tag{19}$$

Such robust control design is discussed by Featherstone *et al.* (2000) and Stewart *et al.* (2003). We only differ in that we recommend constraints are *not* taken into account at this stage, but rather are taken explicitly into account when designing $Q_f(z)$ and $Q_b(z)$ (see below).

A standard assumption is that $\Delta(z)$ is diagonal. When this is the case each loop is decoupled and the design problem is reduced to a series of single-input single-output loops. That is to say, we choose each diagonal element $[Q_\delta(z)]_i$ such that

$$\left|\left| - [\Sigma_r]_i^{-1} [Q_\delta(.)]_i [\Delta(.)]_i \right|\right|_\infty < 1 \tag{20}$$

One possibility (c.f. the Appendix) is to put

$$[Q_\delta(z)]_i = k_i \frac{1 - b_i}{1 - b_i z^{-1}} \tilde{h}(z) \tag{21}$$

where $\tilde{h}(z)$ is a fixed stable transfer function satisfying $\tilde{h}(1) = 1$. Usually $\tilde{h}(z)$ would be chosen to approximate the inverse of $h(z)$. The design parameters for each mode are then k_i and b_i, each taking values between 0 and 1. Choosing b_i to be small corresponds to a fast closed loop response. Choosing $k_i = 1$ corresponds to the inclusion of integral action.

4 Constrained design

Design choice 3. Choose $Q_f(z)$ and $Q_b(z)$. We require $Q_f(z)$ and $Q_b(z)$ to be diagonal transfer function matrices, with $Q_f(z)$ as fast as possible, given certain steady state requirements, and given that the multivariable circle criterion is satisfied.

When constraints are active we have the following steady state behaviour (Heath and Wills, 2004):

$$y_{ss} = \Phi_r \Sigma_r \Psi_r^T u_{ss} + \tilde{\Delta}(1) u_{ss} + d_{ss} \tag{22}$$

and

$$\begin{aligned}
\hat{\delta}_{ss} &= \Phi_r^T y_{ss} - \Sigma_r \mu_{ss} \\
\mu_{ss} &= \arg\min_{\mu} \left|\left| [I - Q_b(1)]\, \Sigma_r \mu + Q_f(1) \hat{\delta}_{ss} \right|\right|_2^2 \text{ such that } \Psi_r \mu \in \mathbb{U} \\
u_{ss} &= \Psi_r \mu_{ss}
\end{aligned} \tag{23}$$

The constrained optimization may be written

$$\mu_{ss} = \arg\min_{\mu} \left|\left| Q_f(1) Q_\delta(1) \Sigma_r \mu + Q_f(1) \hat{\delta}_{ss} \right|\right|_2^2 \text{ such that } \Psi_r \mu \in \mathbb{U} \tag{24}$$

Thus the value of $Q_f(1)$ determines how much each mode is weighted in the cost function. For this discussion we will assume the requirement

$$Q_f(1) = Q_\delta(1) \tag{25}$$

In particular, when $Q_\delta(1) = I$ (corresponding to integral action on all modes), then each mode is weighted equally. But we also require $Q_b(z)$ to be strictly proper—i.e. $Q_b(\infty) = 0$. This in turn corresponds to

$$Q_f(\infty) = Q_\delta(\infty) \tag{26}$$

There are many ways of satisfying (25) and (26) simultaneously. We will choose

$$Q_f(z) = \Lambda Q_\delta(z) + (I - \Lambda) Q_\delta(\infty)(I - M z^{-1}) \tag{27}$$

with Λ a diagonal weighting matrix with terms between 0 and 1, and M a diagonal matrix satisfying

$$M = I - Q_\delta^{-1}(\infty) Q_\delta(1) \tag{28}$$

If we wish to satisfy (9) then we must choose

$$Q_b(z) = I - Q_f(z) Q_\delta^{-1}(z) \tag{29}$$

It remains to choose the diagonal elements of Λ. As with the SISO case, we would like to choose the elements as close as possible to zero (thus ensuring a fast dynamic response with constraints active), whilst guaranteeing robust stability. Such a choice is discussed by Zheng *et al.* (1994); note that choosing the element to be zero does not necessarily correspond to the optimal, even in the absence of uncertainty (Heath and Wills, 2004). The guarantee may be provided by applying the discrete multivariable circle criterion (Haddad and Bernstein, 1994). Specifically, we may observe the implemented quadratic programme takes the form

$$\mu(t) = \arg\min_{\mu} \frac{1}{2} \mu^T \Sigma_r^T \Sigma_r \mu + \mu^T \Sigma_r^T \hat{\delta}_q(t) \text{ such that } \Psi_r \mu \in \mathbb{U} \tag{30}$$

A function $f(\cdot)$ is said to lie in the sector $[0, I]$ if (Khalil, 2002)

$$f(x)^T f(x) - f^T(x) x \leq 0 \text{ for all } x \tag{31}$$

Following the analysis of Heath *et al.* (2003), we may express $\mu(t)$ as

$$\mu(t) = -\Sigma_r^{-1} f\left(\hat{\delta}_q(t)\right) \tag{32}$$

with $f(\cdot)$ a continuous static nonlinearity that lies in the sector $[0, I]$ provided $\mu(t) = 0$ is feasible. We find for typical saturation and bending constraints (8) that $u(t) = 0$ is feasible, in turn corresponding to $\mu(t) = 0$ feasible.

The remaining dynamic elements reduce to

$$\hat{\delta}_q(t) = Q_f(z)\Phi_r^T d(t) + [Q_f(z)\Delta(z) - Q_b(z)\Sigma_r]\,\mu(t) \tag{33}$$

See Fig 8. Hence, from the discrete multivariable circle criterion (Haddad and Bernstein, 1994) we require $P(z)$ to be strongly positive real with

$$P(z) = I + Q_f(z)\Delta(z)\Sigma_r^{-1} - Q_b(z) \tag{34}$$

As with the single-input single-output case, if $\Lambda = I$ the unconstrained robustness condition (19) implies this is automatically satisfied. For other values of Λ, if $\Delta(z)$ is diagonal then $P(z)$ itself is diagonal, and it suffices to check that each element of $P(z)$ is itself strongly positive real. That is to say each diagonal element $P_{i,i}(z)$ satisfies

1. $P_{i,i}(z)$ is stable.
2. $\text{Real}[P_{i,i}(e^{-j\omega})] > 0$ for $0 \leq \omega \leq \pi$.
3. $P_{i,i}(\infty) > 0$.

5 The case where $\Delta(z)$ is non-diagonal

The transformed uncertainty $\Delta(z)$ will be diagonal when the following assumptions hold:

1. The plant dynamics $G(z)$ are truly separable between spatial and dynamic modes. I.e. we may write
$$G(z) = Bh(z) \tag{35}$$
for some (not necessarily known) interaction matrix B and SISO transfer function $h(z)$.
2. The matrix $\Phi_r^T B \Psi_r^T$ is diagonal.

The second condition is equivalent to saying the set of basis functions corresponding to the singular value decomposition of B_r is a subset of those corresponding to the singular value decomposition of B. It is satisfied for circulant matrices where the basis functions correspond to Fourier terms (Stewart *et al.*, 2003).

But for most cross-directional problems edge effects violate the second assumption (Wills and Heath, 2002; Mijanovic *et al.*, 2002). Thus $\Delta(z)$ is only approximately diagonal. Characterisation of $\Delta(z)$ in such circumstances is beyond the scope of this discussion.

If the non-diagonal elements of $\Delta(z)$ are sufficiently small, it would be possible to synthesise diagonal $Q_f(z)$ and $Q_\delta(z)$ such that $P(z)$ is guaranteed strongly positive real via an appeal to diagonal dominance (see for example Maciejowski, 1989).

More generally we observe that it is sufficient to choose $Q_f(z)$ and $Q_b(z)$ such that $\left|\left|Q_f \Delta \Sigma_r^{-1} - Q_b\right|\right|_\infty < 1$ since if a transfer function matrix G is strictly proper, stable and satisfies $||G||_\infty < 1$, then $I+G$ is strongly positive real. This follows from the inequalities (e.g. Golub and Van Loan, 1996)

$$\max \left|\text{eig}\left(G(e^{j\omega})\right)\right| < \bar{\sigma}\left(G(e^{j\omega})\right) < 1 \tag{36}$$

Hence for all $x \in \mathbb{C}$

$$\text{Re}\left[x^H G\left(G(e^{j\omega})\right)x\right] > -1 \tag{37}$$

and so for all $x \in \mathbb{C}$

$$\text{Re}\left[x^H(I + G\left(G(e^{j\omega})\right) + I + G\left(G(e^{j\omega})\right)^H)x\right] > 0 \tag{38}$$

Note that the transfer function matrices $Q_f(z)$, $Q_b(z)$ and $\Delta(z)$ are all assumed stable. Both $Q_b(z)$ and $\Delta(z)$ are assumed strictly proper. It follows immediately that $Q_f \Delta \Sigma_r^{-1} - Q_b$ is both stable and strictly proper.

Furthermore, we observe once again that if we choose $Q_b(z) = 0$ and $Q_f(z) = Q_\delta(z)$ then unconstrained robust stability ensures constrained robust stability. A similar observation is made by Turner *et al.* (2004) in the context of anti-windup with saturation functions for continuous plants.

6 Simulation example

A simulation was performed with uncertainty both in the width of actuator response and in the delay. The plant was assumed to have 101 actuators and 501 measurement positions. The nominal dynamics were given by

$$h(z) = \frac{1 - e^{0.2}}{1 - e^{0.2}z^{-1}} \tag{39}$$

with a delay of 10. The uncertainty in the delay was assumed to be ± 1 sample, while the nominal and allowed range of actuator response (for one actuator) is shown in Fig 9. The actuators were subject to the constraints (8) with $U_m = 1$ and $U_b = 0.1$.

Fig 10 shows that the maximum and minimum possible gain at each mode, taking into account both spatial and dynamic uncertainty. Spillage values (corresponding to non-diagonal terms in $\Delta(z)$ are also shown, although no account of these are taken in the following design. Above mode 48 there is uncertainty in the sign of the gain, so a maximum of 48 modes was chosen. The range of possible dynamic responses for mode 10 is shown in Fig 12. Values of b_i and k_i for each mode were chosen according to the following criterion:

1. For each mode we should have $b_i \leq 0.9$.
2. For each consecutive mode we should have $b_{i+1} \geq b_i$.
3. k should be chosen as near to 1 as possible.
4. The maximum possible achieved sensitivity should be less than 2.
5. The robustness criterion (47) should be satisfied.

Values for b and k are shown in Fig **??**. Note that k is less than 1 (no integral action) only for the last few modes.

This determines $Q_\delta(z)$, with

$$[Q_\delta]_i(z) = \left(\frac{1-b_i}{1-a}\right)\left(\frac{z-a}{z-b_i}\right) \tag{40}$$

It was found that with these values the circle criterion was satisfied for all modes with $\Lambda = 0$. For example, the values of $[P(z)]_{10}$ are shown in Fig 13 for $0 \leq \omega \leq \pi$; it is clear that $[P(z)]_{10}$ is strongly positive real for any permissible value of $\Delta(z)$. Hence from (28) and (29) this leads to

$$\begin{aligned}
Q_f(z) &= Q_\delta(\infty)(I - Mz^{-1}) \\
[M]_i &= \left(\frac{a-b_i}{1-b_i}\right) \\
[Q_f(z)]_i &= k\left(\frac{1-b_i}{1-a}\right) - k\left(\frac{a-b_i}{1-a}\right)z^{-1} \\
[Q_b(z)]_i &= \left(\frac{a-b_i}{1-b_i}\right)\left(\frac{z-1}{z-a}\right)b_i z^{-1}
\end{aligned} \tag{41}$$

Final output profile and actuator positions are shown for a simulation with worst case mismatch in Figs 14 and 15. These results were obtained with a stochastic disturbance. Fig 16 shows the profile evolving in response to an output disturbance which appears as a step in the machine direction (but stochastic in the cross direction). Fig 17 shows the corresponding mode evolutions.

7 Conclusions

The incorporation of a quadratic programme into a cross-directional controller has been proposed by many authors, and originally by Boyle (1977). It can be shown to have significant advantage provided the actuator response is sufficiently well-known (Ma *et al.*, 2002; Wills and Heath, 2002), in particular with respect to steady state performance. In this paper we have shown that if the control scheme suggested by Heath and Wills (2004) is used, then the multivariable circle criterion can be used to guarantee robust stability whilst reaping the benefits of a quadratic programme. We have also suggested a simple design methodology for the case where the uncertainty may be assumed to be diagonal (in the appropriate modal space).

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Appendix: review of constrained SISO IMC design

We design our controller mode by mode, so it will be useful to review some salient features of constrained SISO (single-input single-output) IMC (internal model control). The following treatment is quite standard (Kosut, 1983; Morari and Zafirou, 1989; Campo and Morari, 1990; Zheng *et al.*, 1994), save for some of the notation which becomes useful when generalized to the multivariable case.

Unconstrained design

Consider the set-up shown in Fig 1. We assume the plant is given by

$$\begin{aligned} y(t) &= g(z)u(t) + d(t) \\ g(z) &= z^{-k}\sigma h(z) + \Delta(z) \end{aligned} \tag{42}$$

with $h(z)$ stable. If $g(1) \neq 0$ we may assume without loss of generality $h(1) = 1$ and $h(\infty) \neq 0$. The controller is given by

$$\begin{aligned} u(t) &= -q_\delta(z)\sigma^{-1}\hat{\delta}(t) \\ \hat{\delta}(t) &= y(t) - z^{-k}\sigma h(z)u(t) \end{aligned} \tag{43}$$

We will put

$$q_\delta(z) = \left(\frac{1-b}{1-bz^{-1}}\right)\tilde{h}(z)k \tag{44}$$

where $\tilde{h}(z)$ is stable and an approximate inverse of $h(z)$. In particular we will assume $\tilde{h}(1) = 1$. This gives us two design parameters b and k. The nominal sensitivity is

$$s(z) = 1 - z^{-k}\left(\frac{1-b}{1-bz^{-1}}\right)h(z)\tilde{h}(z)k \tag{45}$$

Note that $s(z)$ is only zero at steady state ($z = 1$) if $k = 1$, corresponding to integral action.

Stability of the loop can be assessed by observing that

$$\hat{\delta}(t) = d(t) + \Delta(z)u(t) \tag{46}$$

and hence that stability is guaranteed provided negative feedback stabilizes the forward transfer function $\Delta(z)q_\delta(z)\sigma^{-1}$ (see Fig 2). Thus we might impose the condition

$$\left|\left|\Delta(.)q_\delta(.)\sigma^{-1}\right|\right|_\infty < 1 \tag{47}$$

Constrained design

Now consider the constrained system shown in Fig 3. The control input $u(t)$ is given by

$$u(t) = \begin{cases} u_{\max} & \text{for} \quad \sigma u_{\max} < -\hat{\delta}_q(t) \\ -\sigma^{-1}\hat{\delta}_q(t) & \text{for} \quad \sigma u_{\min} \leq -\hat{\delta}_q(t) \leq \sigma u_{\max} \\ u_{\min} & \text{for} \quad -\hat{\delta}_q(t) < \sigma u_{\min} \end{cases}$$

The unconstrained behavior is the same as previously provided

$$[1 - q_b(z)]^{-1} q_f(z) = q_\delta(z) \tag{48}$$

We will satisfy this by choosing

$$\begin{aligned} q_f(z) &= \lambda q_\delta(z) + (1-\lambda) q_\delta(\infty) \\ q_b(z) &= 1 - q_f(z) q_\delta^{-1}(z) \end{aligned} \tag{49}$$

for some λ. Choosing λ nearer to zero generally (but not always) gives a faster response. The closed-loop equations can be expressed as

$$\begin{aligned} \hat{\delta}(t) &= d(t) + \Delta(z) u(t) \\ u(t) &= -\sigma^{-1} \text{sat}[\hat{\delta}_q(t)] \\ \hat{\delta}_q(t) &= q_f(z)\hat{\delta}(t) - \sigma q_b(z) u(t) \end{aligned} \tag{50}$$

Eliminating $\hat{\delta}(t)$ gives

$$\begin{aligned} u(t) &= -\sigma^{-1} \text{sat}[\hat{\delta}_q(t)] \\ \hat{\delta}_q(t) &= q_f(z) d(t) + [q_f(z)\Delta(z) - \sigma q_b(z)]\, u(t) \end{aligned} \tag{51}$$

See Fig 4. Hence by the circle criterion (Haddad and Bernstein, 1994), stability is guaranteed provided $p(z)$ is strongly positive real with

$$p(z) = 1 + \sigma^{-1} q_f(z)\Delta(z) - q_b(z) \tag{52}$$

That is to say we require $p(z)$ to be stable and

$$\begin{aligned} \text{Real}[p(e^{-j\omega})] &> 0 \text{ for } 0 \leq \omega \leq \pi \\ p(\infty) &> 0 \end{aligned} \tag{53}$$

Note that we may write

$$\begin{aligned} p(z) &= 1 + \sigma^{-1} q_f(z)\Delta(z) - q_b(z) \\ &= \sigma^{-1} q_f(z)\Delta(z) + q_f(z) q_\delta^{-1}(z) \\ &= \lambda + (1-\lambda) q_\delta(\infty) q_\delta^{-1}(z) + \sigma^{-1}\lambda q_\delta(z)\Delta(z) + \sigma^{-1}(1-\lambda) q_\delta(\infty)\Delta(z) \end{aligned} \tag{54}$$

It follows that

$$\begin{aligned} p(\infty) &= 1 + \sigma^{-1} q_\delta(\infty)\Delta(\infty) \\ &= 1 \end{aligned} \tag{55}$$

provided $\Delta(\infty) = 0$. Furthermore, if we choose $\lambda = 1$, we find

$$\begin{aligned} \text{Real}[p(e^{-j\omega})] &= \text{Real}[1 + \sigma^{-1} q_\delta(e^{-j\omega})\Delta(e^{-j\omega})] \\ &> 0 \end{aligned} \tag{56}$$

provided (47) holds.

Hence we have a natural tuning procedure:

1. Find simple model and corresponding $\tilde{h}(z)$. Evaluate the uncertainty $\Delta(z)$.
2. Tune the unconstrained response by choice of b and k. We want good (nominal) disturbance response and to satisfy (47). Let $k = 1$ if possible.
3. Choose λ to give satisfactory constrained response and to satisfy $p(z)$ strongly positive real.

Figures

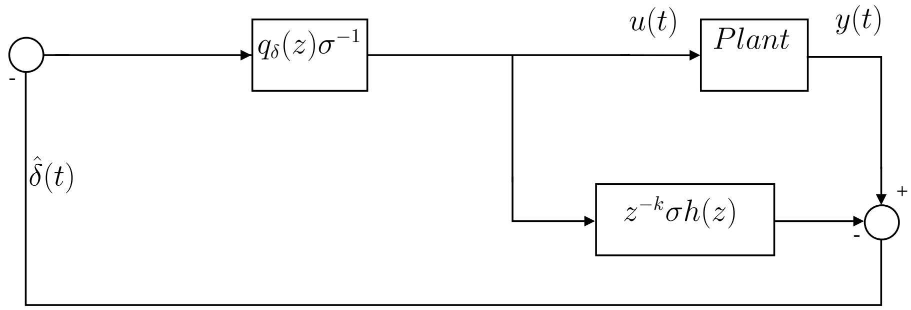

Figure 1: *Unconstrained SISO IMC.*

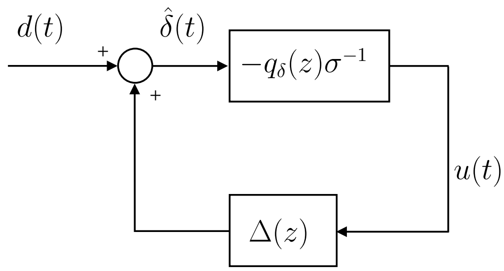

Figure 2: *Loop with uncertainty for SISO unconstrained control.*

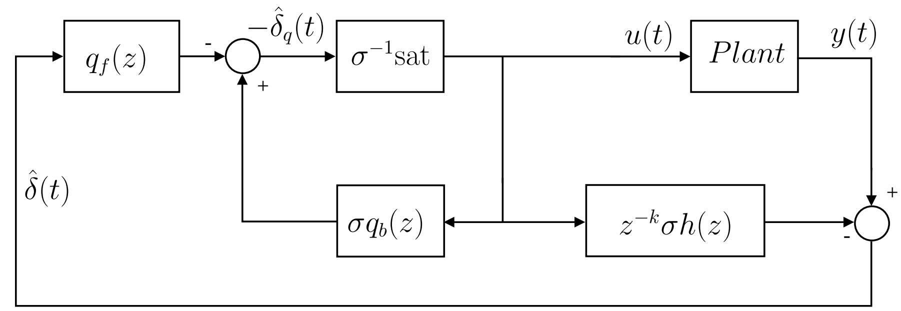

Figure 3: *SISO IMC with feedback around the saturation.*

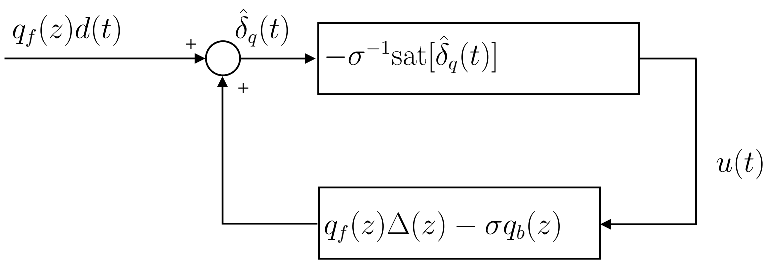

Figure 4: *Loop with uncertainty for constrained control.*

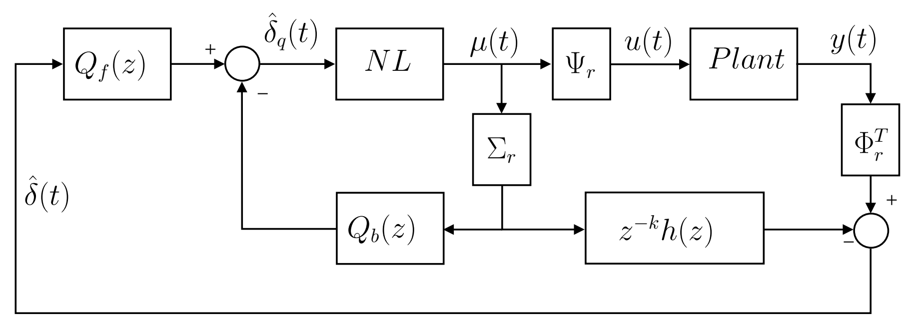

Figure 5: *IMC in modal form with feedback around the non-linear element.*

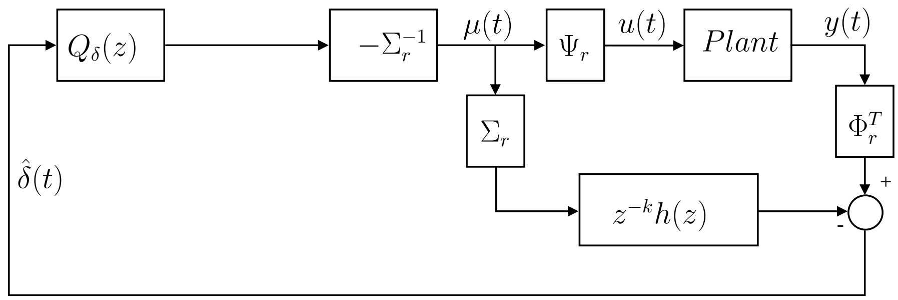

Figure 6: *IMC in modal form without constraints.*

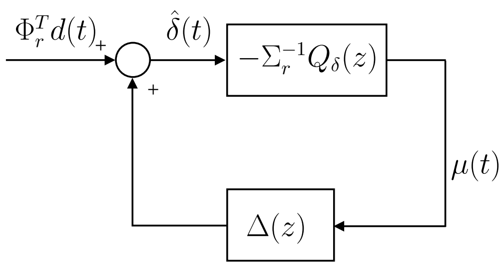

Figure 7: *Loop with uncertainty for unconstrained control.*

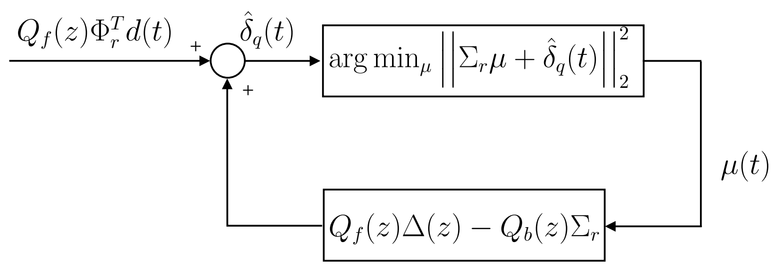

Figure 8: *Loop with uncertainty for constrained control.*

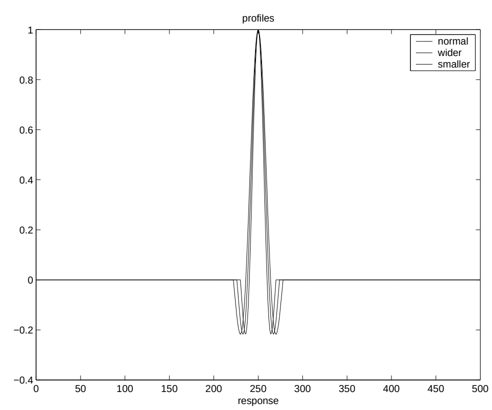

Figure 9: *Range of actuator response we consider. This shows three possible bump test responses on the profile for one actuator.*

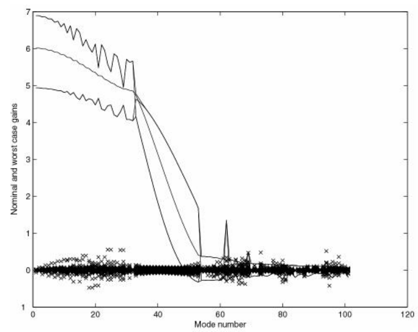

Figure 10: *Corresponding range in the modes. This shows the singular values of the nominal response, together with the maximum and minimum possible values in this space. Spillage values can also be seen.*

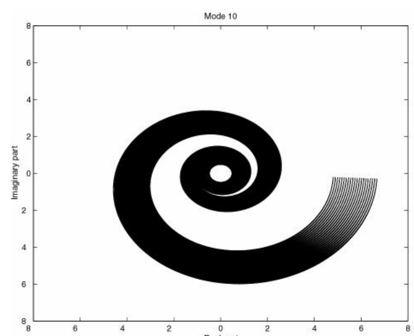

Figure 11: *Possible range of values for* $[G(z)]_{10}$. *The nominal response is first order with delay, while* $\Delta(z)$ *accounts for both spatial and dynamic uncertainty.*

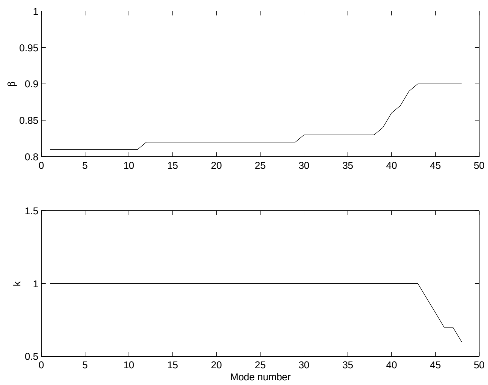

Figure 12: *Values of b_i and k_i at each mode.*

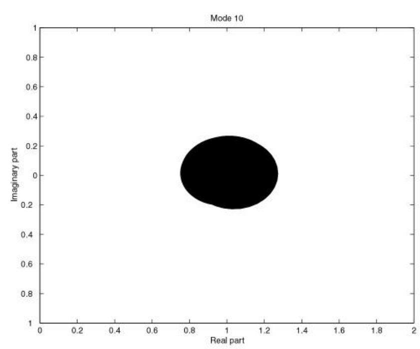

Figure 13: *Possible values of $[P(z)]_{10}$ with $[\Lambda]_{10} = 0$. We see the robustness criterion for this mode is satisfied.*

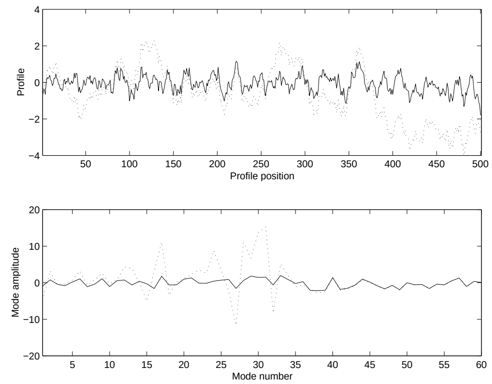

Figure 14: *Final profile value (with and without control) in both profile and mode space.*

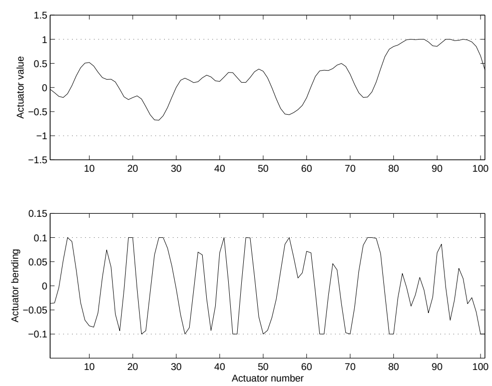

Figure 15: *Corresponding actuator position, and second moment of actuators.*

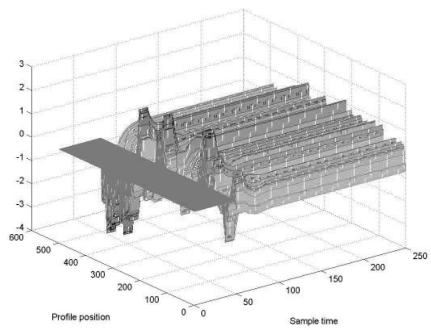

Figure 16: *Profile response to step disturbance change with model mismatch (illustration of stability).*

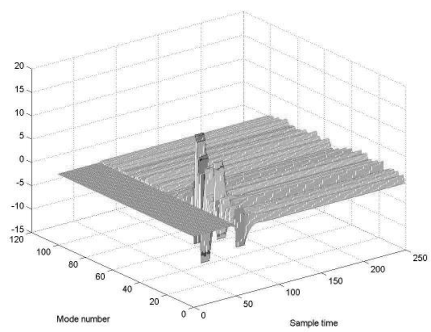

Figure 17: *Corresponding mode evolution (illustration of stability).*