The inherent robustness of constrained linear model predictive control

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Abstract

We show that a sufficient condition for the robust stability of constrained linear model predictive control is for the plant to be open-loop stable, for zero to be a feasible solution of the associated quadratic programme and for the input weighting be sufficiently high. The result can be applied equally to state feedback and output feedback controllers with arbitrary prediction horizon. If integral action is included a further condition on the steady state modelling error is required for stability. We illustrate the results with two forms of integral action commonly used with model predictive control.

1 Introduction

Model predictive control (MPC) is a popular control strategy widely used in industry for plants with constraints (Qin and Badgwell, 2003). We are concerned with demonstrating the robustness of linear MPC to plant uncertainty with stable plants. Linear MPC has a linear state space model, linear equality and inequality constraints and a quadratic cost function with weights on both predicted states and inputs.

It might seem intuitively obvious that with sufficiently high weighting on the control input such a controller would be both nominally and robustly stable. However there are remarkably few results in the literature concerning constrained linear MPC's robustness to model uncertainty. Zheng (1999) provides a sufficient condition for robust stability of state feedback MPC, while Zheng and Morari (1995) and Findeisen *et al.* (2003) provide sufficient conditions for nominal stability of output feedback MPC. More generally, the majority of the literature is devoted to the further augmentation of the MPC cost or constraints to guarantee stability: see (Mayne *et al.*, 2000) for a survey of methodologies for guaranteeing nominal state feedback stability and more recently (Kerrigan and Maciejowski, 2004), (Sakizlis *et al.*, 2004) and references therein for guaranteeing robustness. However we believe Zafiriou's critique of such approaches (Zafiriou, 1990) remains pertinent.

Heath *et al.* (2003) show that the multivariable circle criterion can be used to guarantee the closed-loop stability of certain MPC schemes, provided the constraints allow zero as a feasible solution to the associated constrained optimisation problem. This is always true (for example) if the only constraints are simple bounds on the inputs. In this paper we use the result to provide a sufficient condition for the robust stability of both state feedback and output feedback MPC. In particular, if there is no integral action, it is sufficient that both plant and model are stable and the input weighting is sufficiently high.

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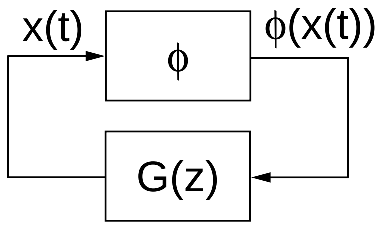

Figure 1: Feedback around the nonlinearity.

We also consider two popular forms of integral action which we will label velocity form and two-stage form respectively. The velocity form corresponds to the scheme of Prett and García (1988) where only input and output changes are weighted in the cost function. The two-stage form corresponds to the scheme of Muske and Rawlings (1993) where the input and state steady state values are computed via a separate optimization at each control stage. For both forms we require an additional condition for stability that the steady state behaviour of the plant and model are sufficiently close (in some sense).

Although the results are both conservative and limited to open-loop stable plants, we should note that the model and plant are not assumed to match, no terminal constraints are introduced and the results are independent of signal norms. Furthermore there is no requirement that the steady state should lie on the interior (as opposed to the boundary) of the constraint set.

The paper is structured as follows. In Section 2 we quote two sufficient conditions for closed-loop asymptotic stability. Each is derived from the discrete multivariable circle criterion. In Section 3 we introduce the MPC notation. Sections 4 to 6 contain the main contributions of the paper. In Section 4 we provide a stability analysis of linear MPC without integral action. In Section 5 we consider velocity form integral action while in Section 6 we consider two-stage form integral action. Proofs of the Lemmas may be found in Appendix A while a simulation example is presented in Appendix B.

2 Preliminaries: strongly positive real results

The discrete version of the multivariable circle criterion (Haddad and Bernstein, 1994) states that if ϕ is a continuous static map satisfying

$$\phi(x)^T\left(\phi(x)+x\right)\leq 0 \tag{1}$$

and if $I+G(z)$ is strongly positive real then the closed loop system $x(t)=G(z)\phi\left(x(t)\right)$ is stable (see Fig 1).

Simple multiplier theory (Khalil, 2002; Heath *et al.*, 2003) gives the following lemma as a corollary:

Lemma 1: Suppose ϕ is a continuous static map satisfying

$$\phi(x)^T H\phi(x)+\phi(x)^T x\leq 0 \tag{2}$$

If H is positive definite and $H+G(z)$ is strongly positive real then the closed-loop system $x(t)=G(z)\phi\left(x(t)\right)$ is stable. □

Heath *et al.* (2003) showed that certain quadratic programmes could be included in the class of such functions. Hence the further lemma:

Lemma 2: Suppose we have the closed-loop equations

$$\begin{array}{rcll} x(t) & = & G(z)\phi(x(t)) & \\ \phi(x(t)) & = & \arg\min\limits_{\nu} & \nu^T H\nu+2\nu^T x(t) \\ & & \text{s. t.} & A\nu\preceq b(t) \text{ and } C\nu=0 \end{array} \tag{3}$$

with H positive definite, $G(z)$ strictly proper and stable and $\nu=0$ always feasible. Then a sufficient condition for stability is that $H+G(z)$ be strongly positive real. □

3 MPC notation

3.1 MPC definition

Given a horizon N, let $J(X,U)$ describe the cost function

$$J(X,U) = ||x_N - x_{ss}||_P^2 + \sum_{k=1}^{N-1} ||x_k - x_{ss}||_Q^2 + \sum_{k=0}^{N-1} ||u_k - u_{ss}||_R^2 \tag{4}$$

Here X and U are sequences of predicted states and inputs

$$\begin{aligned} X &= (x_1, x_2, \ldots, x_N) \text{ with } x_k \in \mathbb{R}^{n_x} \\ U &= (u_0, u_1, \ldots, u_{N-1}) \text{ with } u_k \in \mathbb{R}^{n_u} \end{aligned} \tag{5}$$

Where convenient we will consider X and U to be stacked vectors $X \in \mathbb{R}^{Nn_x}$ and $U \in \mathbb{R}^{Nn_u}$ without change of notation. The terms x_{ss} and u_{ss} correspond to desired steady state values. The weighting matrices P and Q are positive semi-definite while R is positive definite.

We will consider two choices for the terminal cost weighting matrix P. One possibility is simply to choose $P = Q$. The other possibility, which we will term LQR tuning, is to choose P to satisfy the discrete algebraic Riccati equation (DARE)

$$A^T PA - P - A^T PB(R + B^T PB)^{-1} B^T PA + Q = 0 \tag{6}$$

With LQR tuning, unconstrained MPC is equivalent to unconstrained LQR control with an infinite cost horizon (Bitmead *et al.*, 1990). Furthermore the corresponding state-feedback constrained MPC with LQR tuning is nominally optimal for open-loop stable plants provided the horizon N is sufficiently large and the set-point is away from boundaries (Muske and Rawlings, 1993; Chmielewski and Manousiouthakis, 1996). Consequently LQR tuning with fixed N has been proposed by Muske and Rawlings (1993) for output feedback constrained MPC with integral action. Its successful industrial application has been reported, including by the current authors (Wills and Heath, 2004).

Given a state evolution model $x_{i+1} = Ax_i + Bu_i$ and state and input constraint sets $\mathbb{X}$ and $\mathbb{U}$ we may define the MPC law to be:

MPC: Set $u(t)$ to

$$u(t) = \bar{E}U^* \tag{7}$$

where

$$\bar{E} = \left[\begin{array}{cccc} I & 0 & \ldots & 0 \end{array}\right] \tag{8}$$

and

$$\begin{aligned} [X^*, U^*] = \arg \quad & \min_{X,U} \quad J(X,U) \\ & \text{s. t.} \quad x_{i+1} = Ax_i + Bu_i,\ x_{i+1} \in \mathbb{X} \text{ and } u_i \in \mathbb{U} \\ & \qquad\ \ \text{for } i = 0, \ldots, N-1 \end{aligned} \tag{9}$$

We will consider the cases with and without integral action (or "offset free" action) separately. With integral action we will only consider output feedback MPC.

- Without integral action, x_{ss} and u_{ss} are derived from external variables, and for stability analysis may be considered zero without loss of generality. In this case state feedback MPC defines a law

  $$u(t) = \kappa(x(t)) \text{ for some } \kappa \tag{10}$$

  with $x_0 = x(t)$ where $x(t)$ is the plant state (see e.g. Mayne *et al.*, 2000). Similarly output feedback MPC defines a law

  $$u(t) = \kappa(\hat{x}(t)) \tag{11}$$

  with $x_0 = \hat{x}(t)$ where $\hat{x}(t)$ is some observed state value.

- For velocity form integral action an augmented model is used so that $\Delta u(t) = u(t) - u(t-1)$ is computed as a function of an augmented state $[\hat{w}(t)^T y(t)^T]^T$ where $w(t) = \Delta x(t)$. We will find it useful to think of output feedback MPC as defining a law

$$u(t) = \kappa(\hat{w}(t), y(t), u(t-1)) \text{ for some } \kappa \tag{12}$$

- For two-stage form integral action x_{ss} and u_{ss} depend on some disturbance term d_0. In this case output feedback MPC defines a law

$$u(t) = \kappa(\hat{x}(t), \hat{d}(t)) \text{ for some } \kappa \tag{13}$$

with $x_0 = \hat{x}(t)$ as before and $d_0 = \hat{d}(t)$ for some disturbance estimate $\hat{d}(t)$.

3.2 MPC in implicit form

It is standard to express MPC in implicit form by projecting onto the equality constraints defined by the model. Introduce the matrices

$$\begin{aligned}
\bar{P} &= \begin{bmatrix} Q & & & \\ & \ddots & & \\ & & Q & \\ & & & P \end{bmatrix} \text{ (with } \bar{P} = P \text{ when } N = 1\text{)} \\
\bar{R} &= \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix} \\
\Phi &= \begin{bmatrix} B & & & \\ AB & B & & \\ \vdots & \vdots & \ddots & \\ A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \\
\Lambda &= \begin{bmatrix} A \\ \vdots \\ A^N \end{bmatrix}
\end{aligned} \tag{14}$$

Also define

$$\begin{aligned}
\bar{H} &= \bar{R} + \Phi^T \bar{P} \Phi \\
\bar{L} &= \Phi^T \bar{P} \Lambda
\end{aligned} \tag{15}$$

and

$$\begin{aligned}
I_x &= \begin{bmatrix} I & \cdots & I \end{bmatrix}^T \text{ with } I_x \in \mathbb{R}^{n_x, Nn_x} \\
I_u &= \begin{bmatrix} I & \cdots & I \end{bmatrix}^T \text{ with } I_x \in \mathbb{R}^{n_x, Nn_x}
\end{aligned} \tag{16}$$

Define the implicit cost

$$J_I(U) = U^T \bar{H} U + 2U^T \left(\bar{L} x_0 - \Phi^T \bar{P} I_x x_{ss} - \bar{R} I_u u_{ss} \right) \tag{17}$$

We can then replace (9) in the MPC law by expressing U^* as

$$U^* = \arg \begin{array}{cc} \min\limits_{U} & J_I(U) \\ \text{s. t.} & U \in \bar{\mathbb{U}} \end{array} \tag{18}$$

where $\bar{\mathbb{U}}$ is the natural generalisation of $\mathbb{X}$ and $\mathbb{U}$ to U.

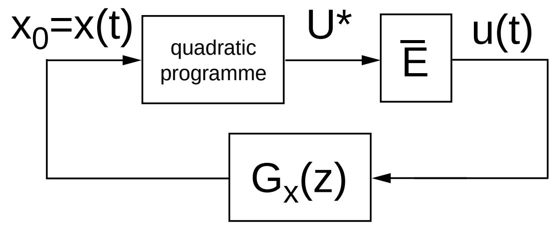

Figure 2: State feedback MPC.

4 Stability of MPC without integral action

4.1 State feedback

Consider the plant

$$x(t) = G_x(z)u(t) \tag{19}$$

with $G_x(z)$ stable and strictly proper. We will model the plant with some

$$\hat{G}_x(z) = (zI - A)^{-1}B \tag{20}$$

Note that we do *not* necessarily assume the plant $G_x(z)$ and model $\hat{G}_x(z)$ to be equal.

We wish to establish the stability of the state feedback system comprising $G_x(z)$ with the MPC control law $u(t) = \kappa(x(t))$. As stated above we assume x_{ss} and u_{ss} to be zero without loss of generality. We further assume the constraints $U \in \bar{\mathbb{U}}$ can be written as a set of (possibly time varying) linear inequalities and equalities

$$A_U U \preceq b_U \text{ and } C_U U = 0 \tag{21}$$

with $U = 0$ always feasible. Since the control law comprises a quadratic programme and linear multiplication (see Fig 2) we may apply Lemma 2 to prove stability. Specifically we may say:

Result 1. Consider the closed-loop feedback system comprising the plant $x(t) = G_x(z)u(t)$ and MPC controller $u(t) = \kappa(x(t))$ with horizon N and with P chosen either as $P = Q$ or as the solution of the DARE (6). If $G_x(z)$ is strictly proper and stable, if A has all eigenvalues in the unit circle, if the constraints on U can be written in the form (21) with $U = 0$ feasible and if R is sufficiently large then the system is stable.

Proof: From Lemma 2 and the implicit form of MPC, it is sufficient that

$$T(z) = \bar{H} + \bar{L}G_x(z)\bar{E} \tag{22}$$

be strongly positive real. Suppose we put $R = \rho R_0$ for some positive definite R_0 and $\rho > 0$. If P is chosen as the solution of the DARE (6) then for A stable $P_\infty = \lim_{\rho\to\infty} P$ exists (Kwakernaak and Sivan, 1972) and is the solution to the discrete Lyapunov equation

$$A^T P_\infty A - P_\infty + Q = 0 \tag{23}$$

Hence, for either choice of P,

$$\lim_{\rho\to\infty} \frac{1}{\rho} T(z) = \bar{R}_0 \text{ for } |z| = 1 \text{ and } z = 0. \tag{24}$$

where

$$\bar{R}_0 = \begin{bmatrix} R_0 & & \\ & \ddots & \\ & & R_0 \end{bmatrix} \tag{25}$$

Thus for sufficiently large ρ, $T(z)$ is strongly positive real and the closed-loop system is stable. $\square$

Result 1 is useful when the horizon N is small. But for large N it becomes somewhat unsatisfactory on two counts. Firstly the dimension of $T(z)$ increases with horizon N, and secondly we would like to find a ρ such that the closed-loop is guaranteed stable for any N. Following (Heath *et al.*, 2003) it is sufficient to examine the eigenvalues of

$$M(z) = \begin{bmatrix} \bar{E} \\ G_x(z)^H \bar{L}^T \end{bmatrix} \bar{H}^{-1} \begin{bmatrix} \bar{L}G_x(z) & \bar{E}^T \end{bmatrix} \tag{26}$$

We find $M(z) \in \mathbb{C}^{2n_u,2n_u}$ with dimension independent of horizon N.

In what follows we will consider only LQR tuning, where P is chosen as the solution of the DARE (6). Let $\mathrm{e}[X]$ denote the non-zero eigenvalues of matrix X. We have the following two lemmas:

Lemma 3: We have the identity

$$\mathrm{e}\left[M(z)\right] = \mathrm{e}\left[\bar{H}^{-\frac{1}{2}}(\bar{L}G_x(z)\bar{E} + \bar{E}^T G_x(z)^H \bar{L}^T)\bar{H}^{-\frac{1}{2}}\right] \tag{27}$$

Furthermore with LQR tuning we may express $M(z)$ as

$$M(z) = \begin{bmatrix} KG_x(z) & H^{-1} \\ G_x(z)^H \left(\sum_{i=1}^{N} (A^T)^i PBH^{-1}B^T PA^i\right) G_x(z) & G_x(z)^H K^T \end{bmatrix} \tag{28}$$

where

$$H = P + B^T RB \tag{29}$$

and K is the LQR gain

$$K = H^{-1}B^T PA \tag{30}$$

Proof: See Appendix A. $\square$

Lemma 4: For R sufficiently large and for all values of z on the unit circle,

$$2 + \min \mathrm{eig}\left[M(z)\right] > 0 \text{ for all } N \tag{31}$$

Proof: See Appendix A. $\square$

So me may say:

Result 2: Consider the closed-loop feedback system comprising the plant $x(t) = G_x(z)u(t)$ and MPC controller $u(t) = \kappa(x(t))$ with LQR tuning. If $G_x(z)$ is strictly proper and stable, if A has all eigenvalues in the unit circle, if the constraints on U can be written in the form (21) with $U = 0$ feasible and if R is sufficiently large then the system is stable for any horizon.

Proof: We require that $T(z)$ be strongly positive real. Given that $G_x(z)$ is both stable and strictly proper, it is sufficient to show for all values of z on the unit circle that

$$\min \mathrm{eig}\left[2\bar{H} + \bar{L}G_x(z)\bar{E} + \bar{E}^T G_x(z)^H \bar{L}^T\right] > 0 \tag{32}$$

Equivalently it is sufficient that for all values of z on the unit circle

$$2 + \min \mathrm{eig}\left[\bar{H}^{-\frac{1}{2}}(\bar{L}G_x(z)\bar{E} + \bar{E}^T G_x(z)^H \bar{L}^T)\bar{H}^{-\frac{1}{2}}\right] > 0 \tag{33}$$

Hence Lemmas 3 and 4 give the result. $\square$

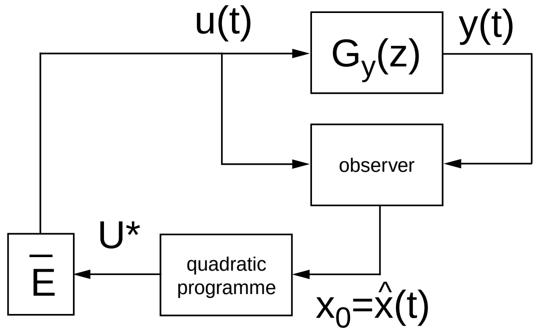

Figure 3: Output feedback MPC with an observer.

4.2 Output feedback

A similar result for output feedback MPC follows immediately. Specifically, suppose the plant is given by

$$y(t) = G_y(z)u(t) \tag{34}$$

and have an observer for the state

$$\hat{x}(t) = J_u(z)u(t) + J_y(z)y(t) \tag{35}$$

for some strictly proper stable transfer function matrix $J_u(z)$ and some stable transfer function matrix $J_y(z)$. Then we can combine the observer with the MPC law $u(t) = \kappa(\hat{x}(t))$; see Fig 3. We may say:

Result 3: Consider the closed-loop feedback system comprising the plant $y(t) = G_y(z)u(t)$, the observer $\hat{x}(t) = J_u(z)u(t) + J_y(z)y(t)$ and MPC controller $u(t) = \kappa(\hat{x}(t))$ with either $P = Q$ or LQR tuning. If $G_y(z)$ is strictly proper and stable, if A has all eigenvalues in the unit circle, if $J_u(z)$ and $J_y(z)$ are stable (with $J_u(z)$ strictly proper), if the constraints on U can be written in the form (21) with $U = 0$ feasible and if R is sufficiently large then for given horizon N the system is stable. If furthermore we have LQR tuning and if R is sufficiently large then the system is stable for any horizon.

Proof: The is exactly the same form as the previous case if we write

$$G_x(z) = J_u(z) + J_y(z)G_y(z) \tag{36}$$

Since there is no requirement for the plant $G_x(z)$ to match the model $(Iz - A)^{-1}B$, the result follows immediately from Results 1 and 2. □

5 Velocity form integral action for output feedback MPC

In the previous section we gave a sufficient condition for closed-loop stability when the controller does not incorporate integral action. However most practical applications of MPC require (when feasible) the rejection of constant disturbances. In this section we consider one well-known scheme for achieving this (Prett and García, 1988; Maciejowski, 2002), which we term velocity form integral action. It is similar in spirit to the integral action used in GPC (Clarke *et al.*, 1987).

5.1 Control structure

Suppose we have a cost function of the form

$$J = \sum_{i=1}^{N} ||y_i - r||_Q^2 + \sum_{i=0}^{N-1} ||\Delta u_i||_R^2 \tag{37}$$

where $\Delta u_i = u_i - u_{i-1}$. Integral action can then be incorporated by including a disturbance in the model. For example we can express an output disturbance model

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i \\ y_i &= Cx_i + d \end{aligned} \tag{38}$$

as

$$\begin{aligned} \begin{bmatrix} w_{i+1} \\ y_{i+1} \end{bmatrix} &= A_a \begin{bmatrix} w_i \\ y_i \end{bmatrix} + B_a \Delta u_i \\ y_i &= C_a \begin{bmatrix} w_i \\ y_i \end{bmatrix} \end{aligned} \tag{39}$$

with

$$\begin{aligned} w_i &= \Delta x_i \\ A_a &= \begin{bmatrix} A & 0 \\ CA & I \end{bmatrix} \\ B_a &= \begin{bmatrix} B \\ CB \end{bmatrix} \\ C_a &= \begin{bmatrix} 0 & I \end{bmatrix} \end{aligned} \tag{40}$$

The cost can then be expressed as

$$J = \left|\left| \begin{bmatrix} w_N \\ y_N - r \end{bmatrix} \right|\right|_P^2 + \sum_{i=1}^{N-1} \left|\left| \begin{bmatrix} w_i \\ y_i - r \end{bmatrix} \right|\right|_{Q_a}^2 + \sum_{i=0}^{N-1} ||\Delta u_i||_R^2 \tag{41}$$

with

$$P = Q_a = C_a^T Q C_a \tag{42}$$

We might choose to modify the controller by adopting LQR tuning. In this case we would let P in (41) satisfy the DARE

$$A_a^T P A_a - P - A_a^T P B_a (R + B_a^T P B_a)^{-1} B_a^T P A_a + Q_a = 0 \tag{43}$$

Note that in this case the control cost no longer has the structure of (37). In both cases for the output feedback case we set w_0 equal to some observer $\hat{w}(t)$, $y_0 = y(t)$ and $\Delta u_0 = u(t) - u(t-1)$ so that MPC is a feedback law

$$u(t) = \kappa\left(\hat{w}(t), y(t), u(t-1)\right) \text{ for some } \kappa \tag{44}$$

See Fig 4. We will find it useful to define the sequence

$$U_a = (u_0, \Delta u_1, \ldots, \Delta u_{N-1}) \tag{45}$$

The implicit cost is

$$J_{IV}(U_a) = [U_a - \bar{E}^T u_{-1}]^T \bar{H}_a [U_a - \bar{E}^T u_{-1}] + 2[U_a - \bar{E}^T u_{-1}]^T \left(\bar{L}_a w_0 - \Phi_a^T \bar{P}_a I_x w_{ss}\right) \tag{46}$$

The implicit velocity form MPC law can be expressed as

Velocity form MPC: Set $u(t)$ to

$$u(t) = \bar{E} U_a^* \tag{47}$$

where

$$U_a^* = \arg \begin{array}{ll} \min\limits_{U_a} & J_{IV}(U_a) \\ \text{s. t.} & U_a \in \bar{\mathbb{U}}_a \end{array} \tag{48}$$

with $w_0 = \hat{w}(t)$ and $u_{-1} = u(t-1)$.

We will assume the observer is given by

$$\hat{w}(t) = J_u(z)(1 - z^{-1}) u(t) + J_y(z)(1 - z^{-1}) y(t) \tag{49}$$

for some stable $J_u(z)$ and $J_y(z)$.

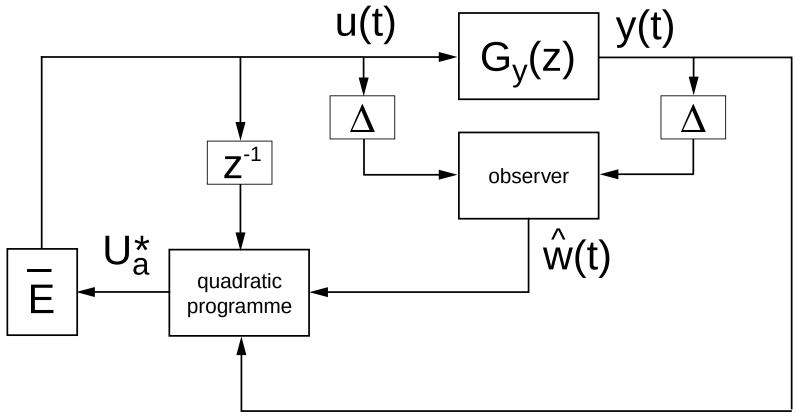

Figure 4: Output feedback MPC with velocity form integral action.

5.2 Stability analysis

It follows from Lemma 2 that the closed-loop system is stable if and only if

$$T_a(z) = \bar{H}_a - \bar{H}_a \bar{E}^T z^{-1} \bar{E} + \bar{L}_a \left[\begin{array}{c} J_u(z)(1-z^{-1}) + J_y(z)(1-z^{-1})G_y(z) \\ G_y(z) \end{array} \right] \bar{E} \tag{50}$$

is strongly positive real.

Lemma 5: Let $T_{1,1} + T_{1,1}^T$ be positive definite with

$$T_{1,1} = B^T \sum_{i=0}^{N-2} (N-1-i)(A^T)^i C^T Q G_y(1) + B_a^T (A_a^T)^{N-1} P_a \left[\begin{array}{c} 0 \\ G_y(1) \end{array} \right] \tag{51}$$

Then if R is chosen sufficiently large, $T_a(z)$ is strongly positive real.

Proof: See Appendix A. □

The following result follows immediately:

Result 4: Consider the closed-loop feedback system comprising the plant $y(t) = G_y(z)u(t)$, the observer $\hat{w}(t) = J_u(z)\Delta u(t) + J_y(z)\Delta y(t)$ and MPC controller with velocity form integral action $u(t) = \kappa(\hat{w}(t), y(t), u(t-1))$ with horizon N and either $P = Q$ or LQR tuning. If $G_y(z)$ is strictly proper and stable, if A has all eigenvalues in the unit circle, if $J_u(z)$ and $J_y(z)$ are stable (with $J_u(z)$ strictly proper), if the constraints on U_a can be written in the form (21) with $U_a = 0$ feasible, if R is sufficiently large and if $T_{1,1} + T_{1,1}^T$ is positive definite then the system is stable in closed-loop. □

Note that if $P = Q_a$ then

$$T_{1,1} = B^T \sum_{i=0}^{N-1} (N-i)(A^T)^i C^T Q G_y(1) \tag{52}$$

5.3 Computation

As before, rather than examine $T_a(z)$, it is sufficient to check that $2 + \min \mathrm{eig}[M_a(z)] > 0$ with

$$M_a(z) = \left[\begin{array}{c} \bar{E} \\ M_1(z)^H \end{array} \right] \bar{H}_a^{-1} \left[\begin{array}{cc} M_1(z) & \bar{E}^T \end{array} \right] \tag{53}$$

where

$$M_1(z) = \bar{L}_a \left[\begin{array}{c} J_u(z)(1-z^{-1}) + J_y(z)(1-z^{-1})G_y(z) \\ G_y(z) \end{array} \right] - \bar{H}_a \bar{E}^T z^{-1} \tag{54}$$

Once again, $M_a(z) \in \mathbb{R}^{2n_u, 2n_u}$ has dimension independent of horizon length N.

5.4 Note on steady state conditions

Suppose the closed-loop system reaches steady state with input and output values u_{ss} and y_{ss}, and furthermore the plant is such that y_{ss} may be expressed as some continuous function $y_{ss} = y_{ss}(u_{ss})$. Then u_{ss} satisfies

$$\begin{aligned} u_{ss} = \arg \quad & \min_u \quad ||y_{ss}(u) - r||^2_{[C_a P C_a^T + (N-1)Q]} \\ & \text{s. t.} \quad u \in \mathbb{U}, \\ & \qquad (I-A)^{-1}Bu \in \mathbb{X} \end{aligned} \tag{55}$$

In this sense, it is more straightforward to put $P = C_a^T Q C_a$ so that u_{ss} satisfies

$$\begin{aligned} u_{ss} = \arg \quad & \min_u \quad ||y_{ss}(u) - r||^2_Q \\ & \text{s. t.} \quad u \in \mathbb{U}, \\ & \qquad (I-A)^{-1}Bu \in \mathbb{X} \end{aligned} \tag{56}$$

For both cases the stability result confirms that such a steady state is achieved.

6 Two stage form integral action for output feedback MPC

Muske and Rawlings (1993) recommend an alternate form of integration. Specifically they recommend a two-stage MPC for both regulator and servo problems (see also Pannocchia and Rawlings, 2003, for a recent discussion).

6.1 Controller structure

We will consider output feedback MPC for the plant

$$y(t) = G_y(z)u(t) \tag{57}$$

For integral action we let x_{ss} and u_{ss} be dependent on some disturbance estimate $\hat{d} = \hat{d}(t)$ so that the MPC law may be expressed as $u(t) = \kappa(\hat{x}(t), \hat{d}(t))$ for some κ. Specifically, given an output disturbance model (the idea can be straightforwardly generalised to an input disturbance)

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i \\ y_i &= Cx_i + d_i \end{aligned} \tag{58}$$

we put

$$\begin{aligned} u_{ss} &= \arg \min_u \left|\left| C(I-A)^{-1}Bu + \hat{d} - r \right|\right|^2_{Q_{ss}} \\ & \qquad \text{s. t.} \quad u \in \mathbb{U} \text{ and } (I-A)^{-1}Bu \in \mathbb{X} \\ x_{ss} &= (I-A)^{-1}Bu_{ss} \end{aligned} \tag{59}$$

Here r is the external set-point. We assume the weighting matrix Q_{ss} to be positive definite.

Given plant input $u(t)$ and output $y(t)$ the state and disturbance estimates are given by

$$\begin{aligned} \hat{x}(t) &= J_u(z)u(t) + J_y(z)\left(y(t) - \hat{d}(t)\right) \\ \hat{d}(t) &= J_d(z)\left(y(t) - C\hat{x}(t)\right) \end{aligned} \tag{60}$$

with $J_u(z)$ stable and strictly proper, $J_y(z)$ stable and $J_d(z)$ stable with $J_d(1) = I$. See Fig 5.

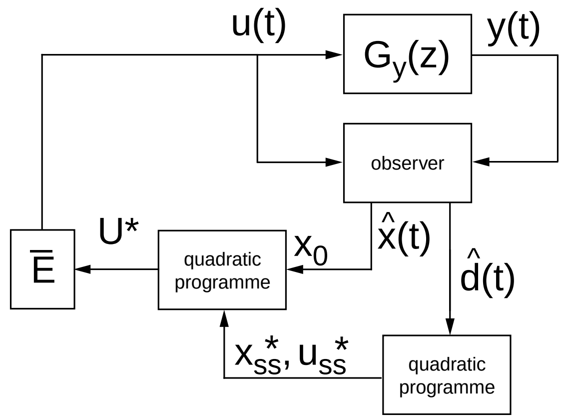

Figure 5: Output feedback MPC with two-stage form integral action.

6.2 Sector bound result

We now have two quadratic programmes in the closed-loop system, so can no longer apply Lemma 2 for stability analysis. Instead we will show that the mapping from a linear combination of $\hat{x}(t)$ and $\hat{d}(t)$ to a linear combination of U^* and u_{ss} takes the form of ϕ in Lemma 1. We will assume the conditions $U \in \bar{\mathbb{U}}$, $u \in \mathbb{U}$ and $(I - A)^{-1}Bu \in \mathbb{X}$ can be written as the (possibly time varying) linear inequality and equality constraints (21) with $U = 0$ feasible (and hence $u = 0$ also feasible). We will also define

$$
\begin{aligned}
\bar{F} &= -\frac{1}{2}\left(\Phi^T \bar{P} I_x (I-A)^{-1}B + \bar{R} I_u\right) \\
F_{ss} &= B^T (I-A)^{-T} C^T Q_{ss} \\
H_{ss} &= B^T (I-A)^{-T} C^T Q_{ss} C (I-A)^{-1} B \\
\bar{\bar{H}} &= \begin{bmatrix} \bar{H} & \bar{F} \\ \bar{F}^T & \mu H_{ss} \end{bmatrix}
\end{aligned} \tag{61}
$$

Then we may say:

Lemma 6: Let ϕ define the map

$$
\begin{bmatrix} U^* \\ u_{ss} \end{bmatrix} = \phi\left(\begin{bmatrix} \bar{L}x_0 \\ \mu F_{ss}(\hat{d} - r) \end{bmatrix}\right) \tag{62}
$$

For any $\mu > 0$ we find $\phi(.)$ is a continuous function satisfying

$$
\phi(x)^T \bar{\bar{H}} \phi(x) + \phi(x)^T x \leq 0 \tag{63}
$$

Also $\bar{\bar{H}}$ is positive definite provided $\mu > 0$ is chosen sufficiently big.

Proof: See Appendix A. □

6.3 Stability analysis

If we put $U^* = U^*(t)$ and $u_{ss} = u_{ss}(t)$ we have the dynamic relationship

$$
\begin{bmatrix} \hat{x}(t) \\ \hat{d}(t) \end{bmatrix} = \begin{bmatrix} I & J_y(z) \\ J_d(z)C & I \end{bmatrix}^{-1} \begin{bmatrix} G_x(z) \\ J_d(z)G_y(z) \end{bmatrix} \begin{bmatrix} \bar{E} & 0 \end{bmatrix} \begin{bmatrix} U^*(t) \\ u_{ss}(t) \end{bmatrix} \tag{64}
$$

where, as before

$$
G_x(z) = J_u(z) + J_y(z)G_y(z) \tag{65}
$$

It follows from Lemmas 1 and 5 that the system is closed-loop stable provided $T_\mu(z)$ is strongly positive real with

$$T_\mu(z) = \begin{bmatrix} \bar{H} & \bar{F} \\ \bar{F}^T & \mu H_{ss} \end{bmatrix} + \begin{bmatrix} \bar{L} & 0 \\ 0 & \mu F_{ss} \end{bmatrix} \begin{bmatrix} I & J_y(z) \\ J_d(z)C & I \end{bmatrix}^{-1} \begin{bmatrix} G_x(z) \\ J_d(z)G_y(z) \end{bmatrix} \begin{bmatrix} \bar{E} & 0 \end{bmatrix} \tag{66}$$

Define the model and model error as

$$\begin{aligned} \hat{G}_y(z) &= C(zI - A)^{-1}B \\ \Delta G_y(z) &= G_y(z) - \hat{G}_y(z) \end{aligned} \tag{67}$$

Furthermore put

$$\hat{G}_x(z) = J_u(z) + J_y(z)\hat{G}_y(z) \tag{68}$$

We will assume $J_u(z)$ and $J_y(z)$ take the form

$$\begin{aligned} J_u(z) &= (zI - A + LC)^{-1}B \\ J_y(z) &= (zI - A + LC)^{-1}L \end{aligned} \tag{69}$$

so that (68) is consistent with (20).

Then we may express $T_\mu(z)$ as:

Lemma 7:

$$\begin{aligned} T_\mu(z) &= \begin{bmatrix} \bar{H} + \bar{L}\hat{G}_x(z)\bar{E} & \bar{F} \\ \bar{F}^T & \mu H_{ss} \end{bmatrix} \\ &+ \begin{bmatrix} \bar{L}J_y(z)\left[I - J_d(z)CJ_y(z)\right]^{-1}\left[I - J_d(z)\right]\Delta G_y(z)\bar{E} & 0 \\ \mu F_{ss}\left[I - J_d(z)CJ_y(z)\right]^{-1} J_d(z)\left[I - CJ_y(z)\right]\Delta G_y(z)\bar{E} & 0 \end{bmatrix} \end{aligned} \tag{70}$$

Proof: See Appendix A. □

The following results for special cases follow immediately:

- When $J_d(z) = 0$, we have the relation

$$T_\mu(z) = \begin{bmatrix} \bar{H} + \bar{L}G_x(z)\bar{E} & \bar{F} \\ \bar{F}^T & \mu H_{ss} \end{bmatrix} \tag{71}$$

  Thus $T_\mu(z)$ is strongly positive real when $J_d(z) = 0$ and μ is sufficiently big.

- If we put $J_d(z) = I$ we have

$$T_\mu(z) = \begin{bmatrix} \bar{H} + \bar{L}\hat{G}_x(z)\bar{E} & \bar{F} \\ \bar{F}^T & \mu H_{ss} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \mu F_{ss}\Delta G_y(z)\bar{E} & 0 \end{bmatrix} \tag{72}$$

  Thus $T_\mu(z)$ is positive definite provided the model is sufficiently close to the plant and provided μ is sufficiently large. Note that we always have, at steady state, $J_d(1) = I$.

Thus, if there is sufficiently small uncertainty at low frequency, stability can be guaranteed by ensuring R is sufficiently large and $J_d(z)$ has sufficiently low bandwidth. To be specific:

Result 5: Consider the feedback system comprising the plant $y(t) = G_y(z)u(t)$, the state and disturbance observers (60) satisfying (69) and MPC controller with two stage form integral action $u(t) = \kappa(\hat{x}(t), \hat{d}(t))$ with horizon N. The weighting matrix P is chosen either as $P = Q$ or via LQR tuning. If $G_y(z)$ is strictly proper and stable, if A has all eigenvalues in the unit circle, if $J_u(z)$ and $J_y(z)$ are stable (with $J_u(z)$ strictly proper), if the constraints on U can be written in the form (21) with $U = 0$ feasible, if R is sufficiently large, if $J_d(z)$ has sufficiently low bandwidth, and if a μ can be found such that both $T_\mu(1) + T_\mu(1)^T$ is positive definite and $T_\mu(z)$ evaluated with $J_d(z) = 0$ is strongly positive real then the system is stable in closed-loop. □

We also have the following useful special cases:

- If $R = \rho R_0$ as before, then we have the relation

$$\lim_{\rho\to\infty}\frac{1}{\rho}T_\mu(z) = \begin{bmatrix} \bar{R}_0 & -\frac{1}{2}\bar{R}_0 I_u \\ -\frac{1}{2}I_u^T\bar{R}_0^T & \lim_{\rho\to\infty}\frac{\mu}{\rho}H_{ss}\end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \lim_{\rho\to\infty}\frac{\mu}{\rho}F_{ss}\left[I - J_d(z)CJ_y(z)\right]^{-1}J_d(z)[I - CJ_y(z)]\Delta G_y(z)\bar{E} & 0\end{bmatrix} \quad (73)$$

- If we put the observer gain $L = 0$ then

$$T_\mu(z) = \begin{bmatrix} \bar{H} + \bar{L}G_x(z)\bar{E} & \bar{F} \\ \bar{F}^T + \mu F_{ss}J_d(z)\Delta G_y(z)\bar{E} & \mu H_{ss}\end{bmatrix} \quad (74)$$

- One possibility (see Maciejowski, 2002, p59) is to choose $L = 0$ and $J_d(z) = I$. This yields

$$T_\mu(z) = \begin{bmatrix} \bar{H} + \bar{L}G_x(z)\bar{E} & \bar{F} \\ \bar{F}^T + \mu F_{ss}\Delta G_y(z)\bar{E} & \mu H_{ss}\end{bmatrix} \quad (75)$$

6.4 Computation

If we put

$$\begin{aligned} W_1(z) &= \hat{G}_x(z) + J_y(z)\left[I - J_d(z)CJ_y(z)\right]^{-1}\left[I - J_d(z)\right]\Delta G_y(z) \\ W_2(z) &= \left[I - J_d(z)CJ_y(z)\right]^{-1}J_d(z)\left[I - CJ_y(z)\right]\Delta G_y(z) \end{aligned} \quad (76)$$

then we may write

$$T_\mu(z) = \begin{bmatrix} \bar{H} & \bar{F} \\ \bar{F}^T & \mu H_{ss}\end{bmatrix} + \begin{bmatrix} \bar{L}W_1(z) \\ \mu F_{ss}W_2(z)\end{bmatrix}\begin{bmatrix} \bar{E} & 0\end{bmatrix} \quad (77)$$

In a similar manner to before, it is sufficient to check $2 + \min\text{eig}[(M_{ts}(z)] > 0$ with

$$M_{ts}(z) = \begin{bmatrix} \bar{E} & 0 \\ W_1(z)^H\bar{L}^T & \mu W_2(z)^H F_{ss}\end{bmatrix}\begin{bmatrix} \bar{H} & \bar{F} \\ \bar{F}^T & \mu H_{ss}\end{bmatrix}^{-1}\begin{bmatrix} \bar{L}W_1(z) & \bar{E}^T \\ \mu F_{ss}W_2(z) & 0\end{bmatrix} \quad (78)$$

7 Conclusion

We have demonstrated the closed-loop asymptotic stability of constrained linear MPC for stable plants. Without integral action we simply require the input weighting to be sufficiently high. With integral action a further condition on the accuracy of the steady state model is required. We have illustrated such a requirement for both velocity form and two-stage form integral action. The results are equally applicable to state feedback and output feedback MPC schemes.

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Appendix A: Proof of the Lemmas

Proof of Lemma 3: We will exploit the relation, for arbitrary matrices A and B,

$$\begin{aligned} e\left(AB + (AB)^H\right) &= e\left(\begin{bmatrix} A & B^H \end{bmatrix}\begin{bmatrix} B \\ A^H \end{bmatrix}\right) \\ &= e\left(\begin{bmatrix} B \\ A^H \end{bmatrix}\begin{bmatrix} A & B^H \end{bmatrix}\right) \end{aligned} \tag{79}$$

Hence

$$\begin{aligned} &\mathrm{e}\left(\bar{H}^{-\frac{1}{2}}(\bar{L}G_x(z)\bar{E} + \bar{E}^T G_x(z)^H \bar{L}^T)\bar{H}^{-\frac{1}{2}}\right) \\ &= \mathrm{e}\left(\begin{bmatrix} \bar{E} \\ G_x(z)^H \bar{L}^T \end{bmatrix} \bar{H}^{-1} \begin{bmatrix} \bar{L}G_x(z) & \bar{E}^T \end{bmatrix}\right) \\ &= \mathrm{e}\left(\begin{bmatrix} \bar{E}\bar{H}^{-1}\bar{L}G_x(z) & \bar{E}\bar{H}^{-1}\bar{E}^T \\ G_x(z)^H \bar{L}^T \bar{H}^{-1}\bar{L}G_x(z) & G_x(z)^H \bar{L}^T \bar{H}^{-1}\bar{E}^T \end{bmatrix}\right) \end{aligned}$$

Substituting for Q via the DARE (6) we can write

$$\bar{H} = \bar{S}^T \begin{bmatrix} H & & & & \\ & H & & & \\ & & \ddots & & \\ & & & H & \\ & & & & H \end{bmatrix} \bar{S} \tag{80}$$

with

$$\bar{S} = \begin{bmatrix} I & & & & \\ KB & I & & & \\ \vdots & \ddots & \ddots & & \\ KA^{N-3}B & & \ddots & I & \\ KA^{N-2}B & \cdots & & KB & I \end{bmatrix} \tag{81}$$

and

$$\bar{L} = \begin{bmatrix} B^T PA + \sum_{k=1}^{N-1} B^T (A^T)^k PBKA^k \\ B^T PA^2 + \sum_{k=1}^{N-2} B^T (A^T)^k PBKA^{k+1} \\ \vdots \\ B^T PA^N \end{bmatrix} \tag{82}$$

Hence

$$\bar{S}^{-T}\bar{E}^T = \bar{E}^T \tag{83}$$

and

$$\bar{S}^{-T}\bar{L} = \begin{bmatrix} B^T PA \\ B^T PA^2 \\ \vdots \\ B^T PA^N \end{bmatrix} \tag{84}$$

Thus we have the identities

$$\begin{aligned} \bar{E}\bar{H}^{-1}\bar{L} &= K \\ \bar{E}\bar{H}^{-1}\bar{E}^T &= H^{-1} \\ \bar{L}^T\bar{H}^{-1}\bar{L} &= \sum_{i=1}^{N}(A^T)^i PBH^{-1}B^T PA^i \\ &= Q - P + \Lambda^T \bar{P}\Lambda \end{aligned} \tag{85}$$

Note that the first of these is well-known, and usually shown via a dynamic programming argument (Bitmead *et al.*, 1990). Hence the result. □

Proof of Lemma 4: We find

$$M(z) = M_1(z) + \begin{bmatrix} 0 & 0 \\ G_x(z)^H \sum_{i=2}^{N} (A^T)^i PBH^{-1}B^T PA^i G_x(z) & 0 \end{bmatrix} \tag{86}$$

From Lemma 3

$$\mathrm{e}\left[M_1(z)\right] = \mathrm{e}\left(H^{-\frac{1}{2}}\left(B^T PAG_x(z) + G_x(z)^H A^T PB\right) H^{-\frac{1}{2}}\right) \tag{87}$$

So if we let $R = R(\rho)$ as before we find

$$\lim_{\rho\to\infty} \min \mathrm{e}\left(M_1(z)\right) = 0 \tag{88}$$

But

$$\min \mathrm{e}\left(M(z)\right) > \min \mathrm{e}\left(M_1(z)\right) - \max \left| \mathrm{e}\left(G_x(z)^H \sum_{i=2}^{N} (A^T)^i PBH^{-1}B^T PA^i G_x(z)\right)\right| \tag{89}$$

Suppose

$$\Pi = \sum_{i=2}^{N} (A^T)^i PBH^{-1}B^T PA^i = \sum_{i=1}^{N} (A^T)^i K^T HKA^i \tag{90}$$

Then for fixed ρ, Π_N is bounded. Specifically (Anderson and Moore, 1979) $\Pi_\infty = \lim_{N\to\infty} \Pi_N$ exists and is the solution to the Lyapunov equation

$$\Pi_\infty - A^T \Pi_\infty A = A^T K^T HKA \tag{91}$$

Thus for ρ sufficiently large, $2 + \min \mathrm{e}(M) > 0$ for all N. □

Proof of Lemma 5:

It will be useful to consider three cases separately: (i) where $z = 1$, (ii) $|z| = 1$, $z \neq 1$ and (iii) $z = 0$. In each case we need to show $T_a(z) + T_a(z)^H$ is positive definite for sufficiently large R. Equivalently, we will put $Q = \varepsilon Q_0$ for some positive definite Q_0, and allow $\varepsilon > 0$ to be sufficiently small. Note that whether P is chosen as $P = Q_a$ or via LQR tuning, $\lim_{\varepsilon\to 0} P = 0$.

(i) Let $z = 1$. We have

$$T_a(1) = \bar{H}_a - \bar{H}_a \bar{E}^T \bar{E} + \bar{L}_a \begin{bmatrix} 0 \\ G_y(1) \end{bmatrix} \bar{E} \tag{92}$$

Thus with $N = 1$ we find

$$T_a(1) = B_a^T P_a A_a \begin{bmatrix} 0 \\ G_y(1) \end{bmatrix} = \begin{bmatrix} B \\ CB \end{bmatrix}^T P_a \begin{bmatrix} 0 \\ G_y(1) \end{bmatrix} = T_{1,1} \tag{93}$$

For $N > 1$, we can partition

$$T_a(1) = \begin{bmatrix} T_{1,1} & T_{1,2} \\ T_{2,1} & T_{2,2} \end{bmatrix} \tag{94}$$

with $T_{1,1} \in \mathbb{R}^{n_u \times n_u}$ given by

$$\begin{aligned} T_{1,1} &= B_a^T \sum_{i=1}^{N-1} (A_a^T)^{i-1} \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix} A_a^i \begin{bmatrix} 0 \\ G_y(1) \end{bmatrix} + B_a^T (A_a^T)^{N-1} P_a A_a^N \begin{bmatrix} 0 \\ G_y(1) \end{bmatrix} \\ &= B^T \sum_{i=0}^{N-2} (N-1-i)(A^T)^i C^T Q G_y(1) + B_a^T (A_a^T)^{N-1} P_a \begin{bmatrix} 0 \\ G_y(1) \end{bmatrix} \end{aligned} \tag{95}$$

Taking Schur complements, for $T_a(1) + T_a(1)^T$ to be positive definite we require

(a) $T_{1,1} + T_{1,1}^T > 0$

(b) $T_{2,2} + T_{2,2}^T - (T_{2,1} + T_{1,2}^T)(T_{1,1} + T_{1,1}^T)^{-1}(T_{1,2} + T_{2,1}^T) > 0$

Furthermore

$$\lim_{\varepsilon \to 0} \left[T_{2,2} + T_{2,2}^T - (T_{2,1} + T_{1,2}^T)(T_{1,1} + T_{1,1}^T)^{-1}(T_{1,2} + T_{2,1}^T) > 0 \right] = 2 \begin{bmatrix} R & & \\ & \ddots & \\ & & R \end{bmatrix} \tag{96}$$

Hence for $\varepsilon > 0$ sufficiently small $T_a(1) + T_a(1)^T$ is positive definite.

1. Suppose $|z| = 1$ with $z \neq 0$. Then

$$\lim_{\varepsilon \to 0} T_a(z) = \begin{bmatrix} R(1 - z^{-1}) & & & \\ & R & & \\ & & \ddots & \\ & & & R \end{bmatrix} \tag{97}$$

and hence $\lim_{\varepsilon \to 0}(T_a(z) + T_a(z)^H)$ is positive definite.

2. Finally with $z = 0$ we find

$$\lim_{\varepsilon \to 0} T_a(0) = \bar{R} \tag{98}$$

□

Proof of Lemma 6: Continuity follows since each quadratic programme is continuous (Fiacco, 1983). The KKT conditions for U^* and u_{ss} can be written

$$\bar{H}U^* + C_U \zeta_U + A_U^T \lambda_U + \bar{L}x_0 + 2\bar{F}u_{ss} = 0 \tag{99}$$

with

$$\begin{aligned} C_U U^* &= 0 \\ A_U U^* + s_U &= b_U \\ s_U^T \lambda_U &= 0 \\ s_U &\succeq 0 \\ \lambda_U &\succeq 0 \end{aligned} \tag{100}$$

and

$$H_{ss} u_{ss} + C_u \zeta_u + A_u^T \lambda_u + F_{ss}(\hat{d} - r) = 0 \tag{101}$$

with

$$\begin{aligned} C_u u_{ss} &= 0 \\ A_u u_{ss} + s_u &= b_u \\ s_u^T \lambda_u &= 0 \\ s_u &\succeq 0 \\ \lambda_u &\succeq 0 \end{aligned} \tag{102}$$

Pre-multiplying (99) by ${U^*}^T$, (101) by ${u_{ss}}^T$ and substitution yields

$$ {U^*}^T \bar{H} U^* + b_U \lambda_U + {U^*}^T \bar{L} x_0 + 2{U^*}^T \bar{F} u_{ss} = 0 \tag{103}$$

and

$$ {u_{ss}}^T H_{ss} u_{ss} + b_u^T \lambda_u + {u_{ss}}^T F_{ss}(\hat{d} - r) = 0 \tag{104}$$

Adding the two equations together (with an arbitrary scaling constant μ to the second) yields

$$\begin{aligned} \begin{bmatrix} U^* \\ u_{ss} \end{bmatrix}^T \begin{bmatrix} \bar{H} & \bar{F} \\ \bar{F}^T & \mu H_{ss} \end{bmatrix} \begin{bmatrix} U^* \\ u_{ss} \end{bmatrix} + \begin{bmatrix} U^* \\ u_{ss} \end{bmatrix}^T \begin{bmatrix} \bar{L}x_0 \\ \mu F_{ss}(\hat{d}-r) \end{bmatrix} &= -b_U^T\lambda_U - \mu b_u^T \lambda_u \\ &\leq 0 \end{aligned} \tag{105}$$

for all $\mu > 0$.

Taking Schur complements, the matrix $\bar{\bar{H}}$ is positive definite if and only if both $\bar{H}$ and $\mu H_{ss} - \bar{F}^T\bar{H}^{-1}\bar{F}$ are positive definite. This latter condition is guaranteed for sufficiently large $\mu > 0$. □

Proof of Lemma 7: We may write

$$\begin{bmatrix} I & J_y(z) \\ J_d(z)C & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -J_y(z) \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & [I - J_d(z)CJ_y(z)]^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -J_d(z)C & I \end{bmatrix} \tag{106}$$

and

$$\begin{aligned} G_x(z) &= J_u(z) + J_y(z)G_y(z) \\ &= J_u(z) + J_y(z)\hat{G}_y(z) + J_y(z)\Delta G_y(z) \\ &= \hat{G}_x(z) + J_y(z)\Delta G_y(z) \end{aligned} \tag{107}$$

Hence

$$\begin{aligned} &\begin{bmatrix} I & J_y(z) \\ J_d(z)C & I \end{bmatrix}^{-1} \begin{bmatrix} G_x(z) \\ J_d(z)G_y(z) \end{bmatrix} \\ &= \begin{bmatrix} I & -J_y(z) \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & [I - J_d(z)CJ_y(z)]^{-1} \end{bmatrix} \begin{bmatrix} G_x(z) \\ J_d(z)\left[G_y(z) - CG_x(z)\right] \end{bmatrix} \\ &= \begin{bmatrix} I & -J_y(z) \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & [I - J_d(z)CJ_y(z)]^{-1} \end{bmatrix} \begin{bmatrix} \hat{G}_x(z) + J_y(z)\Delta G_y(z) \\ J_d(z)\left[I - CJ_y(z)\right]\Delta G_y(z) \end{bmatrix} \\ &= \begin{bmatrix} \hat{G}_x(z) \\ 0 \end{bmatrix} + \begin{bmatrix} J_y(z)\Delta G_y(z) - J_y(z)\left[I - J_d(z)CJ_y(z)\right]^{-1} J_d(z)\left[I - CJ_y(z)\right]\Delta G_y(z) \\ \left[I - J_d(z)CJ_y(z)\right]^{-1} J_d(z)\left[I - CJ_y(z)\right]\Delta G_y(z) \end{bmatrix} \\ &= \begin{bmatrix} \hat{G}_x(z) \\ 0 \end{bmatrix} + \begin{bmatrix} J_y(z)\left[I - J_d(z)CJ_y(z)\right]^{-1}\left[I - J_d(z)\right]\Delta G_y(z) \\ \left[I - J_d(z)CJ_y(z)\right]^{-1} J_d(z)\left[I - CJ_y(z)\right]\Delta G_y(z) \end{bmatrix} \end{aligned} \tag{108}$$

The result then follows. □

Appendix B: Simulation example

To illustrate these results, consider the two-input two-output discrete plant given by transfer function matrix

$$G_y(z) = \begin{bmatrix} \dfrac{2.8z - 2.2}{z^3 - 2.2z^2 + 1.79z - 0.57} & \dfrac{-0.9z + 0.78}{z^3 - 1.1z^2 + 0.81z - 0.522} \\ \dfrac{z^2 - 0.5z - 0.34}{z^3 - 1.7z^2 + 1.26z - 0.432} & \dfrac{-z + 1.55}{z^3 - 1.4z^2 + 0.59z - 0.093} \end{bmatrix} \tag{109}$$

The plant $G_y(z)$ is stable with two non-minimum phase transmission zeros. We model it with the reduced order plant model

$$\hat{G}_y(z) = \begin{bmatrix} \dfrac{2.527}{z - 0.9029} & \dfrac{-0.4555}{z - 0.3139} \\ \dfrac{1.236}{z - 0.5404} & \dfrac{0.3467}{z - 0.9469} \end{bmatrix} \tag{110}$$

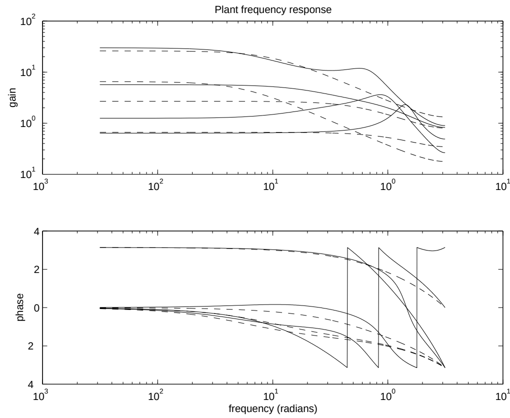

Figure 6: Frequency responses of plant $G_y(z)$ (solid lines) and model $\hat{G}_y(z)$ (dashed lines).

The model $\hat{G}_y(z)$ is also stable, but has no non-minimum phase transmission zeros. The frequency responses of both plant and model are shown in Figure 6.

In what follows we make no attempt to find a "good" design. We simply show that it is possible to find tuning parameters such that the various stability criteria are satisfied.

7.1 No integral action

The model $\hat{G}_y(z)$ can be represented in state-space as

$$y(t) = C(zI - A)^{-1}Bu(t) \tag{111}$$

with

$$\begin{aligned} A &= \begin{bmatrix} 0.9029 & 0 & 0 & 0 \\ 0 & 0.3139 & 0 & 0 \\ 0 & 0 & 0.5404 & 0 \\ 0 & 0 & 0 & 0.9469 \end{bmatrix} \\ B &= \begin{bmatrix} -1.5896 & 0 \\ 0 & 0.6749 \\ -1.1117 & 0 \\ 0 & 0.5888 \end{bmatrix} \\ C &= \begin{bmatrix} -1.5896 & -0.6749 & 0 & 0 \\ 0 & 0 & -1.1117 & 0.5888 \end{bmatrix} \end{aligned} \tag{112}$$

An observer was chosen with gain

$$L = \begin{bmatrix} -0.4034 & -0.2152 \\ -0.0395 & 0.0744 \\ -0.1595 & -0.0935 \\ -0.1755 & 0.3699 \end{bmatrix} \tag{113}$$

Figure 7 illustrates the eigenvalues of $T(z)$ with a horizon $N = 1$ with Q the identity matrix and R set to $R = \rho I$ with $\rho = 8$. With an infinite horizon such a choice of ρ fails the criterion. Figure 8 shows the eigenvalues of $M(z)$ (with 2 added) with ρ increased to 64. Figure 9 shows the values of the minimum eigenvalues of $M(z)$ (offset by 2) as ρ increases with $N \to \infty$. Note that they are below zero for small ρ. By contrast Figure 10 shows the values of the minimum eigenvalues of $M(z)$ (with 2 added) as N increases with $\rho = 64$. Note that they are above zero.

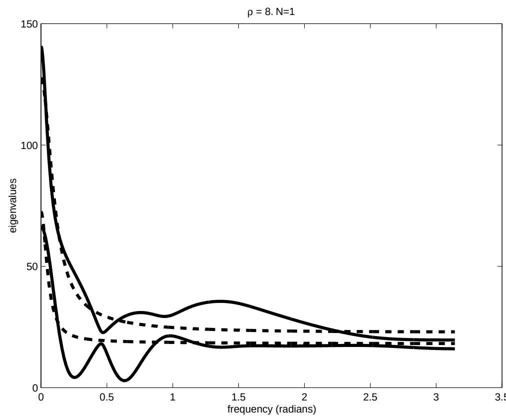

Figure 7: Eigenvalues of $T(z)$ with horizon $N = 1$ and weighting $\rho = 8$. Values using the model $\hat{G}_y(z)$ are also shown (dashed)

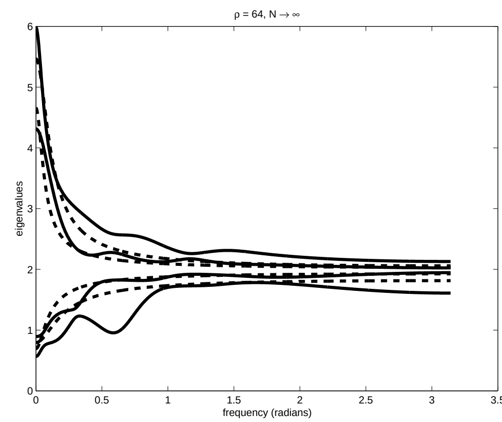

Figure 8: Eigenvalues of $M(z)$ (offset by 2: i.e. eig$[M(z)] + 2$ is shown) with horizon $N \to \infty$ and weighting $\rho = 64$. Values using the model $\hat{G}_y(z)$ are also shown (dashed)

7.2 Velocity form integral action

To illustrate the velocity form integration, we will set $P = Q_a$, with $Q = \varepsilon I$ and $R = I$. The same observer gain was used as in the previous case (note the observer estimates $w(t)$ in this case). We then have a condition on the steady state response of the plant and model (52). The eigenvalues of $T_{1,1}$ for various values of N are illustrated in Fig 11). We arbitrarily chose a horizon of N. Figure 12 illustrates the eigenvalues of $M_a(z)$ (offset by 2) for this example (same controller as before, but in velocity form) with $\varepsilon = 10^{-4}$. The constrained controller is thus guaranteed stable with these (rather cautious) tuning parameters. Figures 13) and 14) illustrate the output and input time responses respectively. Both set point changes and disturbances were added to the system. The constraints were that the first input should lie between ± 0.04 while the second input should lie between ± 0.15. Note that the latter constraint prevents offset-free performance on occasion. The corresponding unconstrained responses are also shown.

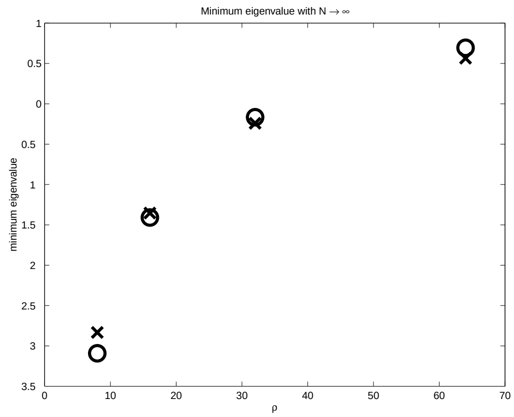

Figure 9: Minimum eigenvalues of $M(z)$ (offset by 2) with horizon $N \to \infty$ for different values of ρ. Values using the model $\hat{G}_y(z)$ are also shown ('o's)

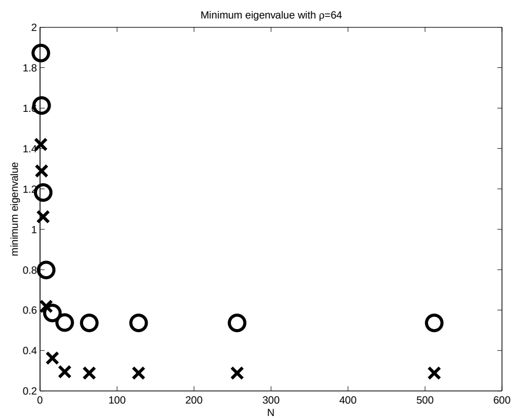

Figure 10: Minimum eigenvalues of $M(z)$ (offset by 2) with $\rho = 64$ and horizon N varying. Values using the model $\hat{G}_y(z)$ are also shown ('o's)

7.3 Two-stage form integral action

To illustrate the two-stage form integral action, we will set $Q_{ss} = I$, $Q = I$ and $R = \rho I$. The same state observer gain was used once again. Figure 15) illustrates the eigenvalues of $M_{ts}(z)$ (with two added) with $N = 10$, $\mu = 32$, $\rho = 155$ and $J_d(z)$ given by

$$J_d(z) = \frac{0.1z}{z - 0.9} I \tag{114}$$

Figures 16) and 17) illustrate the output and input time responses respectively. The simulation was run with the same set point changes, disturbances and constraints as the simulation with velocity form integration. The corresponding unconstrained responses are also shown.

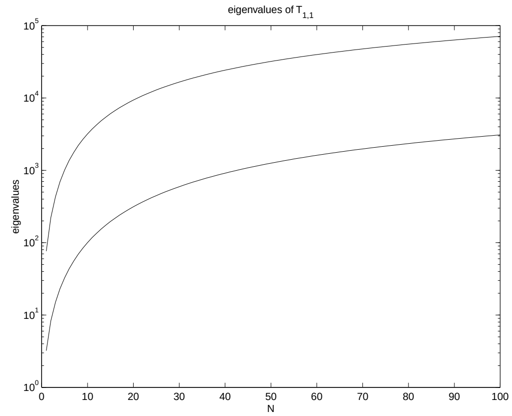

Figure 11: Eigenvalues of $T_{1,1}$. In this example they are positive for all N between 1 and 100.

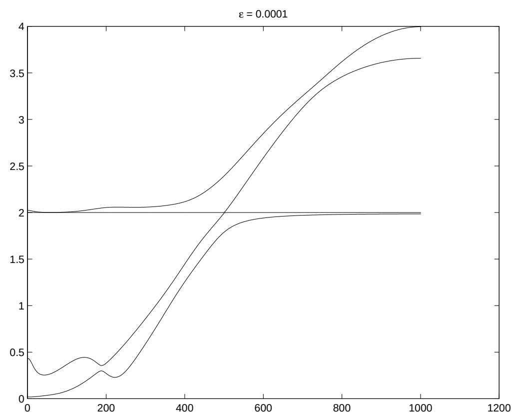

Figure 12: Eigenvalues of $M_a(z)$ (offset by 2) with horizon $N = 10$ and weighting $\varepsilon = 10^{-4}$.

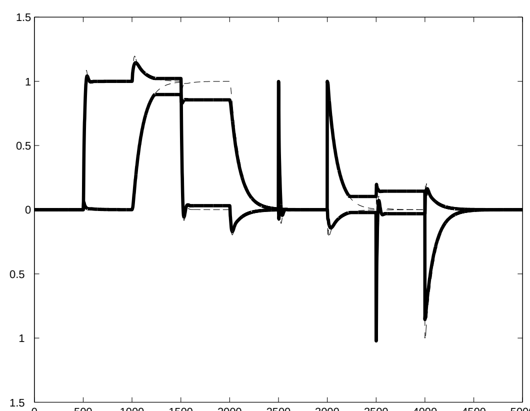

Figure 13: Output time response for velocity form simulation. The unconstrained response is also shown (dashed).

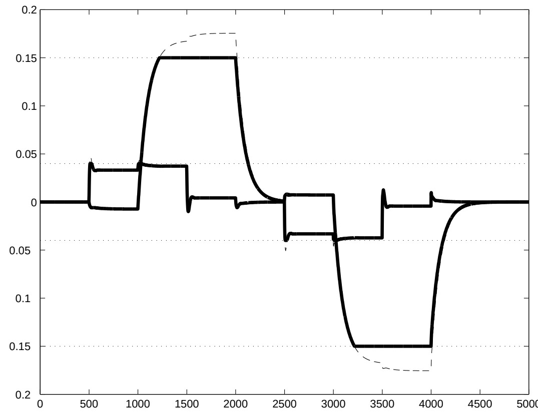

Figure 14: Input time response for velocity form simulation. The unconstrained response is also shown (dashed).

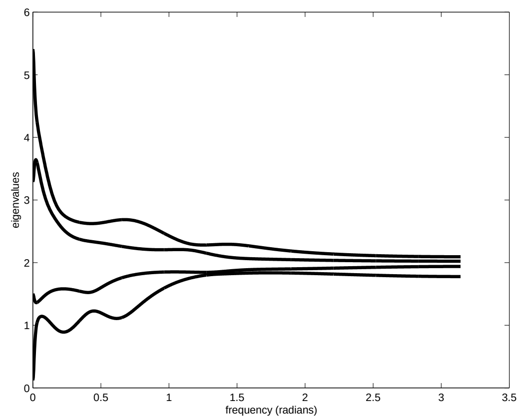

Figure 15: Eigenvalues of $M_{ts}(z)$ (offset by 2) with horizon $N = 10$ and weighting $\rho = 155$.

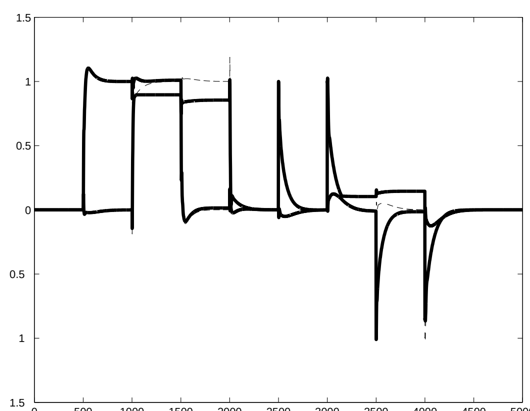

Figure 16: Output time response for two-stage form simulation. The unconstrained response is also shown (dashed).

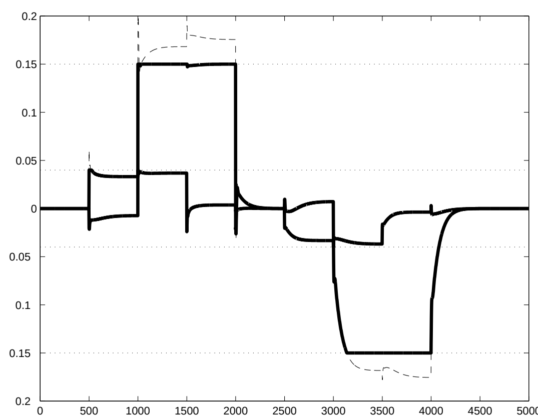

Figure 17: Input time response for two-stage form simulation. The unconstrained response is also shown (dashed).