Fixed Points of EXIT Charts

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***Abstract*— Extrinsic Information Transfer (or EXIT) charts have provided a useful tool for analysing the convergence of iterative decoders. In this work, we abstract the EXIT chart as a feedback interconnection of two one-dimensional dynamical systems. For such feedback interconnections, we characterise the local stability properties of fixed points and demonstrate the existence of period two orbits and discuss their stability properties. Finally, we give a graphical procedure for finding the region of attraction for asymptotically stable fixed points or period two orbits.**

I. Introduction

The advent of turbo codes [3] and the rediscovery of low-density parity-check codes [4], [6] have made iterative decoding a subject of intense study over the past decade. Codes for iterative decoding are typically made up of two simple constituent codes, which are then decoded by individual decoders, with messages passed between the decoders at each iteration.

One of the more successful analysis and design tools for iteratively decoded codes has been the Extrinsic Information Transfer (EXIT) chart [11], which is a graphical tool enabling the mutual information exchanged between constituent decoders to be tracked.

An EXIT chart plots the average input-output mutual information from one constituent decoder against the other, where the EXIT functions corresponding to each decoder characterise the manner in which each decoder converts *a priori* and (possibly) channel log-likelihood ratio (LLR) messages into *a posteriori* LLR values.

In the case of asymptotically long block lengths, it has been observed that, for ideal decoding, it is necessary to have a "convergence tunnel" [2] such that the decoding trajectory can move from the origin to $(1,1)$. To date, the use of EXIT charts has focused heavily on the existence or absence of this convergence tunnel to describe the expected behaviour of the corresponding iterative decoder without the need for computationally intensive bit-error rate (BER) simulations.

In this paper we employ tools of dynamical systems theory to take a first step towards a more systematic understanding of EXIT charts. Our starting point is the observation that the exchange of messages between constituent decoders can be seen as the feedback interconnection of two discrete-time dynamical systems. To this end, we consider the following two-dimensional system

$$\begin{aligned} x_{k+1} &= f(y_k) \\ y_{k+1} &= g(x_k) \end{aligned} \tag{1}$$

where $f, g : [0,1] \to [0,1]$, and both $f(\cdot)$ and $g(\cdot)$ are monotonically increasing on $[0,1]$. We assume we are given initial conditions $x_0, y_0 \in [0,1]$.

An EXIT chart is then the plot of $x = f(y)$ and $y = g(x)$. Assuming the existence of an (at least local) inverse for the function f, fixed points of (1) consist of those points (x^*, y^*) satisfying

$$g(x^*) = f^{-1}(x^*), \quad y^* = g(x^*). \tag{2}$$

If we plot the two functions in the plane, we see that these points are the intersections of the graphs of $y = g(x)$ and $y = f^{-1}(x)$. If both f and g are continuous on the square, Brouwer's fixed point theorem guarantees the existence of at least one fixed point. An obvious question is: What can we say about the stability of these fixed points? Two further questions are: Can (1) give rise to periodic or chaotic solutions? What can we say about domains of attraction for fixed points and/or periodic orbits?

Fixed points for the turbo decoder have previously been studied in [1] and [8]. Whereas these references essentially examine the order $2n$ dynamical system described by the decoder (for block length n), we examine the second-order approximation given by the EXIT chart. While it was shown in [9] that the EXIT chart may not entirely capture the iterative decoding process, it has proven to be a useful approximation in practice. The analysis in the current work provides a precursor to a more thorough analysis to understand precisely how the EXIT chart approximates the decoding process.

In Section II we demonstrate a local condition on the graph that is sufficient to determine instability or asymptotic stability of fixed points of (1). As such, our assumptions on the nature of the functions (i.e., continuously differentiable with a continuously differentiable inverse) need only be satisfied near to the fixed points. In Section III we demonstrate the existence of period two orbits for (1) and give sufficient local conditions, again related to the graph, for instability or asymptotic stability of the period two orbits. Finally, in Section IV we take a more global view and demonstrate regions of attraction for asymptotically stable fixed points and period two orbits. Being a global result, we require the functions to be well-behaved over the square. In particular, we require them to be continuous and monotonically increasing.

II. Local stability properties of fixed points

We first make precise our stability notions.

Definition 1: A fixed point x^* is said to be *stable* for $x_{k+1} = f(x_k)$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all initial states satisfying $|x_0 - x^*| < \delta$, solutions satisfy $|x_k - x^*| < \varepsilon$ for all $k \in \mathbb{Z}_{\geq 0}$. A fixed point that is not stable is said to be *unstable*.

Definition 2: A fixed point is said to be *asymptotically stable* if, in addition to being stable, $|x_k - x^*| \to 0$ as $k \to \infty$.

For nonlinear systems, such as those described by equation (1), we may clearly have more than one equilibrium point. Consequently, we can modify the above definitions to hold locally around a fixed point. In essence, this means that there exists a neighbourhood around the fixed point wherein we have asymptotic stability. We refer to this as *local asymptotic stability*.

A well-known result from stability theory is that the local stability of a fixed point for a nonlinear system can be determined from the linearisation of the system at the fixed point, so long as the fixed point is *hyperbolic*. A hyperbolic point is one such that the Jacobian of the system equations evaluated at that point has eigenvalues strictly within or strictly outside the unit circle. In the former case, the point is locally asymptotically stable, while in the latter case the point is unstable. (See, for example, [10, Theorem 1.3.7].)

We will make use of the following terminology:

Definition 3: We call (x^*, y^*) a *stable crossing* if for some $\varepsilon > 0$, the graphs intersect with $g(x) > f^{-1}(x)$ for $x \in (x^* - \varepsilon, x^*)$ (i.e., when approaching from the left) and $g(x) < f^{-1}(x)$ for $x \in (x^*, x^* + \varepsilon)$ (i.e., when moving away from the fixed point to the right). We call (x^*, y^*) an *unstable crossing* if the graphs intersect with $g(x) < f^{-1}(x)$ for $x \in (x^* - \varepsilon, x^*)$ and $g(x) > f^{-1}(x)$ for $x \in (x^*, x^* + \varepsilon)$

Theorem 1: Suppose $f(\cdot)$, $f^{-1}(\cdot)$, and $g(\cdot)$ are continuously differentiable in a neighbourhood of the fixed point (x^*, y^*). If, (x^*, y^*) is a stable crossing, then the fixed point is locally asymptotically stable. If, on the other hand, (x^*, y^*) is an unstable crossing, then the fixed point is unstable.

Fact 1: Consider the points $x^*, y^* \in [0, 1]$ satisfying $x^* = f(y^*)$ where $f(\cdot)$ is continuously differentiable and monotonically increasing. Then

$$\frac{d}{dy} f(y)\bigg|_{y=y^*} = \frac{1}{\frac{d}{dx} f^{-1}(x)}\bigg|_{x=x^*}.$$

Proof: The linearisation of (1) at a fixed point is given by

$$\begin{bmatrix} x_{k+1} - x^* \\ y_{k+1} - y^* \end{bmatrix} = \begin{bmatrix} 0 & f'(y^*) \\ g'(x^*) & 0 \end{bmatrix} \begin{bmatrix} x_k - x^* \\ y_k - y^* \end{bmatrix} \quad (3)$$

It is easy to see that the eigenvalues for the above system matrix are at $\lambda = \pm\sqrt{f'(y^*) g'(x^*)}$.

We first consider the case where $g(\cdot)$ is initially above $f^{-1}(\cdot)$. In order for the curves to cross, we see that, in some neighbourhood $\mathcal{B}$ of the fixed point, it is necessary that $\frac{d}{dx} f^{-1}(x) > \frac{d}{dx} g(x)$ for all $x \in \mathcal{B}$. Using Fact 1, we see that

$$g'(x^*) f'(y^*) < 1. \quad (4)$$

As a consequence, the eigenvalues of the linearisation are inside the unit circle, and hence the fixed point is locally asymptotically stable.

Consider now the case where $g(\cdot)$ is initially below $f^{-1}(\cdot)$. Then, in some neighbourhood $\mathcal{B}$ of the fixed point, it is necessary that $\frac{d}{dx} f^{-1}(x) < \frac{d}{dx} g(x)$ for all $x \in \mathcal{B}$. Using Fact 1 again, we see that the eigenvalues of the linearisation must lie outside the unit circle. Therefore the fixed point is unstable. ■

Remark 1: Note that, if the curves intersect, but do not cross, then the eigenvalues of the linearisation in (3) are at ± 1. This follows because, at such a point the derivatives

$$\frac{d}{dx} f^{-1}(x^*) = \frac{d}{dx} g(x^*) \quad \Rightarrow \quad f'(x^*) g'(x^*) = 1.$$

As a consequence, this approach does not allow us to determine the stability of the fixed point. In Section IV we use a different approach to show that such points are saddle points. ■

Remark 2: We observe that, so long as f and g are continuous in a neighourhood of $(1, 1)$, then the rightmost fixed point must be either a stable crossing or at $(1, 1)$. This follows from the fact that f and g must be defined over the entirety of $[0, 1]$. ■

Example 1: In [2], Ashikhmin *et al.*presented an EXIT chart for a $(2, 4)$-regular LDPC code over a binary erasure channel with erasure probability q. They derive the EXIT function for the check nodes as $I_{Ec} = (I_{Ac})^3$ and the EXIT function for the variable nodes is $I_{Ev} = 1 - q\,(1 - I_{Av})$. In the above notation, this yields the system

$$\begin{aligned} x_{k+1} &= y_k^3 =: f(y_k) \\ y_{k+1} &= 1 - q(1 - x_k) =: g(x_k). \end{aligned}$$

The linearization about a fixed point (x^*, y^*) is easily calculated as

$$\begin{bmatrix} \hat{x}_{k+1} \\ \hat{y}_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & 3(y^*)^2 \\ q & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_k \\ \hat{y}_k \end{bmatrix} \quad (5)$$

The eigenvalues of the system matrix are then at $\lambda = \pm\sqrt{3(y^*)^2 q}$.

We examine the same values $q = 0.3, 0.5$ as in [2]. When $q = 0.3$, there is only one crossing and it is at $(1, 1)$. We see in this case that the eigenvalues of the above Jacobian are at $\lambda = \pm\sqrt{9/10}$ and the fixed point is locally asymptotically stable as expected. (In fact, it is globally asymptotically stable, but that does not follow from Theorem 1. Rather, see Lemma 2.) On the other hand, when $q = 0.5$, we see that there are two crossings; one at $(1, 1)$ and the other at about $(0.2361, 0.6180)$. For the crossing at $(1, 1)$, we see that the eigenvalues are at $\lambda = \pm\sqrt{3/2}$. Therefore, the fixed point at $(1, 1)$ is unstable. For the crossing at $(0.2361, 0.6180)$, the

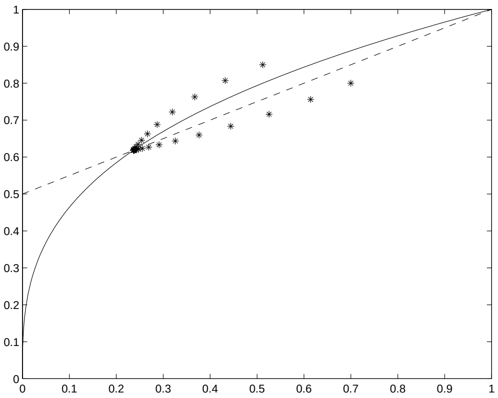

Fig. 1. Dashed line: $y = g(x)$, Solid line: $y = f^{-1}(x)$, $*$: Solution from $(0.7, 0.8)$

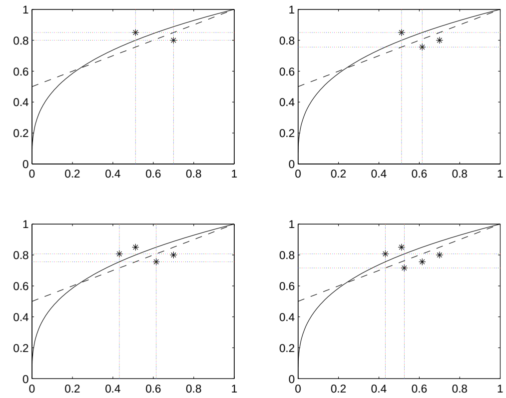

Fig. 2. First four iterations for Example 1. Dashed line: $y = g(x)$, Solid line: $y = f^{-1}(x)$, $*$: Solution from $(0.7, 0.8)$

eigenvalues are at $\lambda = \pm\sqrt{0.5729}$. Consequently, the fixed point is locally asymptotically stable, as expected.

Finally, in [2], they note that the threshold for convergence to $(1, 1)$ is at $q = 1/3$; i.e., the decoder converges for $q < 1/3$. If we examine the location of the eigenvalues given above, we see that $\lambda = 1$ precisely when $q = 1/3$, and $|\lambda| < 1$ when $q < 1/3$, thus guaranteeing asymptotic stability of the fixed point. ■

Remark 3: We note that the trajectories shown in Figure 1 and 4 do not lie between the EXIT functions as in common EXIT chart plots in the literature. This comes from our having taken initial conditions other than the origin and examining how the system evolves. To see that trajectories outside the EXIT function "envelope" are indeed correct for these initial conditions, consider specifically Figure 1. From the initial condition $(0.7, 0.8)$, draw a vertical line until intersecting $y = g(x)$ (the dashed line). This is the y-value of the next point. Consequently, the next point must lie on a horizontal line drawn from this intersection. Now draw a horizontal line from $(0.7, 0.8)$ until intersecting the curve $x = f(y)$ (the solid line). This gives the x-value of the next point. So we can draw a vertical line through this point. The next iterate lies on the intersection of the secondary lines. In Figure 2, we show this procedure for the first four iterations.

Our choice of initial conditions was made to illustrate the wider range of behaviour possible for systems such as those described by (1). This would correspond to the decoders having access to non-zero *a priori* knowledge (see, e.g., [12, Section III.A]). In particular, we assume we can initialise both decoders with independent values, and we then map the iterates; i.e., the points (x_0, y_0), (x_1, y_1), and so on.

By way of comparison, the common plots with points lying on the EXIT functions come about by iterating only one state at a time while holding the other state constant. That is, the common staircase plot can be obtained by plotting (x_0, y_0), (x_1, y_0), (x_1, y_1), (x_2, y_1), and so on. Note that, if we start from an initial condition on one of the EXIT curves,

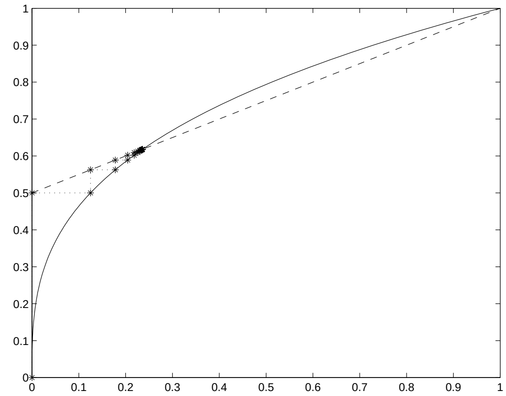

Fig. 3. Dashed line: $y = g(x)$, Solid line: $y = f^{-1}(x)$, $*$: Solution from $(0, 0)$

the plots of the iterates are the same regardless of which approach is taken. For example, consider Figure 3 where we take as initial condition $(0, y_1)$, where $y_1 = g(0) = 0.5$. We see that the iterates evolve as $(0, y_1)$, $(f(y_1), g(0)) = (f(y_1), y_1)$, $(f(y_1), g(f(y_1)))$. In other words, each coordinate only changes every other iteration. ■

Rather than consider the two-dimensional system described in (1), which is a natural and direct abstraction of the EXIT chart, we may consider the evolution of the composition mappings $f \circ g(\cdot)$ and $g \circ f(\cdot)$.

Consider the iterative mapping

$$z_{k+1} = f \circ g(z_k). \tag{6}$$

with initial condition $z_0 \in [0, 1]$. We observe that this comes directly from equation (1) where we have simply taken z_k as a subsequence of the sequence of x_k. We note that a fixed point of $f \circ g$, say z^*, must correspond to an x^* and y^* that are fixed points of (1). In particular, $z^* = x^*$ and $y^* = g(z^*)$. Taking the Taylor series expansion of $f \circ g(\cdot)$ around a fixed point z^* gives us the following difference equation

$$\begin{aligned} z_{k+1} &\approx f \circ g(z^*) + f'(g(z^*))g'(x^*)(z_k - z^*) \\ &= z^* + f'(y^*)g'(x^*)(z_k - z^*). \end{aligned}$$

Therefore, near the fixed point (x^*, y^*), we have

$$z_{k+1} - x^* \approx f'(y^*)g'(x^*)(z_k - x^*) \tag{7}$$

This then gives an alternate proof of Theorem 1 since a similar equation can be derived for $g \circ f$. By using the same relationship as in equation (4), we see that sequences will tend to contract to the fixed point x^*. If, on the other hand, we have that

$$\frac{d}{dx} f^{-1}(x^*) < \frac{d}{dx} g(x^*) \quad \Rightarrow \quad f'(y^*)g'(x^*) > 1$$

then sequences will move away from the fixed point x^*. Note that, if this approach is taken, one must carefully account for the initial conditions. We will return to this point in the following section.

III. Period two orbits

Suppose we have two isolated fixed points, (x_1^*, y_1^*) and (x_2^*, y_2^*). Take as an initial condition (x_1^*, y_2^*). The solution from this point is then

$$\left(\begin{array}{c} x_1^* \\ y_2^* \end{array}\right), \left(\begin{array}{c} f(y_2^*) = x_2^* \\ g(x_1^*) = y_1^* \end{array}\right), \left(\begin{array}{c} f(y_1^*) = x_1^* \\ g(x_2^*) = y_2^* \end{array}\right), \cdots \tag{8}$$

which is a period two orbit.

We see, then, that an easy way to locate the period two orbits of (1) is to draw a horizontal and vertical line through each fixed point in the plane. Each intersection is then one point of a period two orbit.

Lemma 1: If n is the total number of fixed points in the square, then there are $\frac{n(n-1)}{2}$ period two orbits.

Proof: Suppose we have n fixed points which correspond to $\phi(n)$ period two orbits. We observe that adding another fixed point will result in n new period two orbits. We also observe that, in the case of a single fixed point, there are no period two orbits. Consequently, we have the difference equation

$$\phi(n+1) = \phi(n) + n, \quad \phi(1) = 0$$

the solution of which is $\frac{n(n-1)}{2}$. ■

When discussing period two orbits, it is useful to consider the mapping obtained over two time steps:

$$\begin{aligned} x_{k+1} &= f \circ g(x_{k-1}) \\ y_{k+1} &= g \circ f(y_{k-1}) \end{aligned} \tag{9}$$

Definition 4: A period two orbit is stable (asymptotically stable, unstable, hyperbolic) if each periodic point on the orbit is stable (asymptotically stable, unstable, hyperbolic) as a fixed point of the two time step mapping.

The following theorem states that it is only necessary to examine the stability properties of one point on an orbit (rather than both) [10, Theorem 1.3.9]:

Theorem 2: Let $f : \Re^n \to \Re^n$ be continuously differentiable and let $x^* \in \Re^n$ be a point on an orbit of period q. Then x^* is hyperbolic and stable (hyperbolic and unstable) as a fixed point of $f^{(q)}$ if and only if the period q orbit is hyperbolic and stable (hyperbolic and unstable).

The above theorem simplifies the proof of:

Theorem 3: Suppose (x_1^*, y_1^*) and (x_2^*, y_2^*) are stable crossings. Then the period two orbit consisting of (x_1^*, y_2^*) and (x_2^*, y_1^*) is locally asymptotically stable. Suppose, on the other hand, that (x_1^*, y_1^*) and (x_2^*, y_2^*) are unstable crossings. Then the period two orbit is unstable.

Proof: Theorem 2 states that we need only examine the linearisation of the two step mapping at one point of a period two orbit which, for the orbit above, is

$$\left[\begin{array}{cc} f'(y_2^*)g'(x_2^*) & 0 \\ 0 & f'(y_1^*)g'(x_1^*) \end{array}\right]. \tag{10}$$

As the linearisation is diagonal, the eigenvalues are simply the diagonal entries. As in the proof of Theorem 1, we observe that if (x_1^*, y_1^*) and (x_2^*, y_2^*) are stable crossings, then both of $f'(y_2^*)g'(x_2^*)$ and $f'(y_1^*)g'(x_1^*)$ are less than unity. Therefore, both eigenvalues of the linearisation are within the unit circle and the period two orbit is locally asymptotically stable. On the other hand, if both points are unstable crossings, then both $f'(y_2^*)g'(x_2^*)$ and $f'(y_1^*)g'(x_1^*)$ will be greater than unity and, consequently, the period two orbit will be unstable. ■

Remark 4: We note that, as in the case of saddle points, the above approach gives us no information about the stability of period two orbits when one of the intersecting lines corresponds to a saddle point since the period two orbit will not be hyperbolic. ■

Example 2: We consider the sixth order polynomial given by

$$g(x) = \frac{31}{16}x - \frac{247}{32}x^2 + \frac{675}{32}x^3 - \frac{365}{16}x^4 + \frac{15}{2}x^5 + x^6$$

and take $f(y) = g(y)$. We see that there are crossings at $(0,0)$, $\left(\frac{1}{4}, \frac{1}{4}\right)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{3}{4}, \frac{3}{4}\right)$, and $(1,1)$. Looking at Figure 4, we quickly see that $(0,0)$, $\left(\frac{1}{2}, \frac{1}{2}\right)$, and $(1,1)$ are unstable crossings, yielding unstable fixed points from Theorem 1. Stable crossings are at $\left(\frac{1}{4}, \frac{1}{4}\right)$ and $\left(\frac{3}{4}, \frac{3}{4}\right)$.

Since there are five fixed points, Lemma 1 gives that there are ten period two orbits. From Theorem 3 we see that there is one asymptotically stable period two orbit consisting of $\left\{\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right\}$. We also have three unstable period two orbits at

$$\left\{\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right)\right\}, \left\{\left(\frac{1}{2}, 1\right), \left(1, \frac{1}{2}\right)\right\}, \{(0,1), (1,0)\}.$$

In Figure 4, we plot the curves $y = g(x)$ and $x = f(y)$. The stars show iterations of the system converging to the asymptotically stable period two orbit. Figure 5 shows the evolution of x_k and y_k as a function of time k. ■

Looking at (9), we see that we have effectively decoupled the equations from each other. However, we note that, were we to propagate these mappings forward, we would need two initial conditions for each equation. We are given initial conditions x_0 and y_0. These then define our second conditions as $x_1 = f(y_0)$ and $y_1 = g(x_0)$.

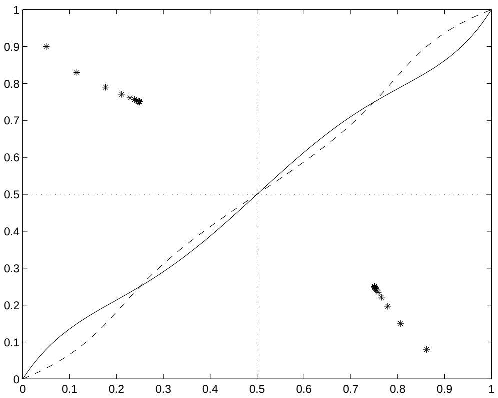

Fig. 4. Solid line: $y = g(x)$, Dashed line: $y = f^{-1}(x)$, $*$: Solution from $(0.05, 0.9)$

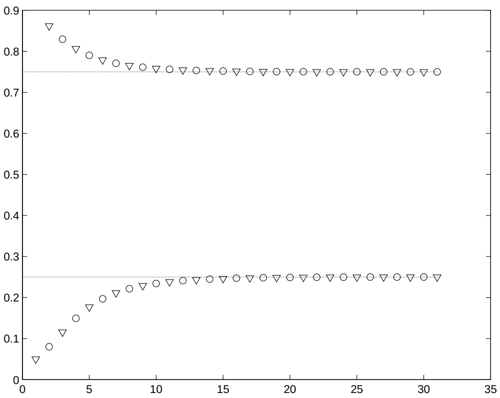

Fig. 5. ∇: x_k, $\circ$: y_k

We saw in the previous section that we can use the Taylor series expansion to determine local stability about a fixed point for the mappings $f \circ g(\cdot)$ and $g \circ f(\cdot)$. Suppose the two fixed points (x_1^*, y_1^*) and (x_2^*, y_2^*) are locally asymptotically stable. If we consider an initial condition (x_0, y_0) close to (x_1^*, y_2^*), we see that even elements of the sequence x_k converge to x_1^*, while odd elements of the sequence converge to x_2^*. Similarly, even or odd elements of the sequence y_k will converge to y_2^* or y_1^*, respectively. We therefore have a period two orbit with the orbit defined by the points (x_1^*, y_2^*) and (x_2^*, y_1^*).

IV. Domains of Attraction

In the previous two sections, all of our results were local to the fixed points or to the period two orbits. In this section, we take a more global view. Firstly, we can show that if there is only one fixed point, then it is globally asymptotically stable; by which we mean that all initial conditions in the square eventually converge to the fixed point.

Lemma 2: Suppose f and g are both continuous. If $f \circ g(\cdot)$ has a unique fixed point x^*, then x^* is globally asymptotically stable for the difference equation

$$x_{k+1} = f \circ g(x_k). \tag{11}$$

Proof: We first note that, if there is only one fixed point, then it is a stable crossing. This follows since both mappings must be defined over their entire domain and continuous. In other words, g must (continuously) connect the left and right sides of the unit square, while f^{-1} must (continuously) connect the bottom and top sides of the unit square. As a consequence, f^{-1} must initially be below g. If there is only a single intersection, then it is clearly a stable crossing. The exception to this is if the sole intersection is at the origin, in which case $f^{-1}(x) > g(x)$ for those $x > x^*$.

Suppose that the fixed point is neither at 0 nor 1 and consider any initial condition $x_0 \in [0, x^*)$. Then, since x^* is unique, we know that $f \circ g(x) > x$ for all $x \in [0, x^*)$. Consequently, we have

$$x_{k+1} = f \circ g(x_k) > x_k = f \circ g(x_{k-1}) > x_{k-1} > \cdots > x_0.$$

Since $f \circ g(\cdot)$ is continuous, monotone, and $f \circ g(x^*) = x^*$, we see that the x_k form a bounded, monotonically increasing sequence. Therefore the sequence converges to x^*. A similar argument holds for all $x \in (x^*, 1]$, except that any sequence will be monotonically decreasing as a consequence of $f \circ g(x) < x$ for all $x \in (x^*, 1]$. Therefore, for every initial condition, the solution to (11) will asymptotically converge to x^*.

If the fixed point is at 0, we see that $g(x) > f^{-1}(x)$ for all $x \in (0, 1]$. Consequently, any initial condition will generate a sequence converging to 0. Similarly, if the fixed point is at 1, then $g(x) > f^{-1}(x)$ for all $x \in [0, 1)$ and all solutions will converge to 1. ■

Suppose we have two fixed points (x_1^*, y_1^*) and (x_2^*, y_2^*) ordered such that $x_1^* < x_2^*$ and $y_1^* < y_2^*$ and such that no fixed point lies between them. Further suppose that, on the interval (x_1^*, x_2^*) we have that $f^{-1}(x) > g(x)$. Then, on the interval (x_1^*, x_2^*), $x > f \circ g(x)$. Consequently, iterating $x_{k+1} = f \circ g(x_k)$ leads to a monotonically decreasing sequence bounded below by x_1^*. Alternately, if $f^{-1}(x) < g(x)$ for all $x \in (x_1^*, x_2^*)$, then iterating $x_{k+1} = f \circ g(x_k)$ generates a monotonically increasing sequence bounded above by x_2^*.

We note that, if $f^{-1}(x) < g(x)$, this implies that $g^{-1}(y) < f(y)$, or that $y < g \circ f(y)$. Therefore, for an initial point $y_0 \in (y_1^*, y_2^*)$, we see that

$$\begin{aligned} y_{2(k+1)} &= g \circ f(y_{2k}) > y_{2k} = g \circ f(y_{2(k-1)}) > y_{2(k-1)} \\ &> \cdots > y_0 > y_1^*. \end{aligned}$$

Consequently, the sequence y_{2k} will approach y_2^*. In a similar fashion, $f^{-1}(x) > g(x)$ implies $g^{-1}(y) > f(y)$ or, in other words, $y > g \circ f(y)$. In this case, then, the sequence y_{2k} will approach y_1^*.

Finally, we observe that the above statements hold for the case when a fixed point is the leftmost or rightmost fixed point in the square. This allows us to define a simple partition of the square which quickly gives the regions of attractions for the fixed points, as well as the regions of attraction for the period two orbits. For each unstable or saddle point, draw both a vertical and a horizontal line through the point. A locally asymptotically stable fixed point attracts all points in its (open) partition. Note that a saddle point will be at a corner. If $f^{-1}(x) \geq g(x)$ around a saddle point, then it will lie on the bottom left corner of the partition containing its domain of attraction. If $f^{-1}(x) \leq g(x)$ around a saddle point, then the saddle point will be on the top right corner of the partition containing its domain of attraction.

Example 3: Looking again at the system in Example 2, we see that the square is partitioned into four sections. The domain of attraction for $\left(\frac{1}{4}, \frac{1}{4}\right)$ is $\left\{(x, y) \in \left(0, \frac{1}{2}\right) \times \left(0, \frac{1}{2}\right)\right\}$ while that for $\left(\frac{3}{4}, \frac{3}{4}\right)$ is $\left\{(x, y) \in \left(\frac{1}{2}, 1\right) \times \left(\frac{1}{2}, 1\right)\right\}$. We see that the domain of attraction for the period two orbit $\left\{\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{3}{4}, \frac{1}{4}\right)\right\}$ is

$$\left\{(x, y) \in \left(\left(\frac{1}{2}, 1\right) \times \left(0, \frac{1}{2}\right)\right) \bigcup \left(\left(0, \frac{1}{2}\right) \times \left(\frac{1}{2}, 1\right)\right)\right\}.$$

Finally, we observe that the lines through the unstable fixed points are invariants. ■

V. Conclusions

The EXIT chart has proved to be one of the more successful design tools available for iterative decoding. Furthermore, it potentially has important connections to fundamental quantities in coding theory [7] and has been shown to be useful in other parts of the physical layer receiver chain [5].

We have presented a first step towards understanding how EXIT charts capture the key elements of the iterative decoding process, particularly with non-zero *a priori* initial conditions. We have presented both a local analysis of stability properties of fixed points and shown the existence of period two orbits. Note that, away from the fixed points, this analysis does not require that the EXIT functions be continuous or monotonic. Future extensions of this work will include removing the differentiability assumption and perhaps even the continuity assumption at the fixed points.

We also demonstrated that regions of attraction can be found over the square when the EXIT functions are monotonic and continuous. It should be possible to at least get conservative estimates on regions of attraction for more general classes of functions.

In the case where we start the iterations with zero *a priori* knowledge, this implies we start at the origin. In this case, the results of Section IV guarantee that the trajectory converges to the first fixed point to the right of, or at, the origin (so long as the origin is not an unstable crossing). However, our results demonstrate that the ability to bias the decoders or make use of some *a priori* knowledge will not always guarantee convergence to $(1, 1)$.

We note that, with a stable crossing yielding a locally asymptotically stable fixed point, a small perturbation away from the fixed point in the direction of $(1, 1)$ will not cause the decoder to converge. Rather, the trajectory will simply return to the fixed point. If, on the other hand, the curves merely intersect but do not cross, a perturbation in the direction of $(1, 1)$ will, in fact, allow the iterations to continue to the next fixed point.

Finally, we note that these results have the interesting implication that it is actually possible to *reduce* the information content via iterative decoding. As the trajectory plotted in Figure 1 shows, it is possible to start close to $(1, 1)$ and yet to ultimately move quite far away. In fact, since $(1, 1)$ is unstable and $(0.2361, 0.6180)$ attracts every initial condition except $(1, 1)$, we can start arbitrarily close to "perfect information" and yet iterate to a distant point.

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