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Research Article

Sensitivity Limitations for Multivariable Linear Filtering

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This paper examines fundamental limitations in performance which apply to linear filtering problems associated with multivariable systems having as many inputs as outputs. The results of this paper quantify unavoidable limitations in the sensitivity of state estimates to process and measurement disturbances, as represented by the maximum singular values of the relevant transfer matrices. These limitations result from interpolation constraints imposed by open right half-plane poles and zeros in the transfer matrices linking process noise and output noise with state estimates. Using the Poisson integral inequality, this paper shows how sensitivity limitations and tradeoffs in multivariable filtering problems are intimately related to the directionality properties of the open right half-plane poles and zeros in these transfer matrices.

1. INTRODUCTION

Estimating the state of a dynamical system from noise-corrupted measurements of the input and output is a problem of considerable engineering significance. While an enormous body of literature on the filtering problem has developed over the past 50 years, optimization procedures have played a dominant role addressing problems of the following form: given a plant model and some description of the noise characteristics, compute a state estimator which minimizes some measure of the state estimation error. The most well-known example of this approach is the Kalman filter, which minimizes the variance of the estimation error under the assumption that the initial state estimate and the state and measurement noise processes satisfy Gaussian probability distributions; see, for example, [1] and the references therein.

Questions of a different nature, and which have received somewhat less attention, are the following: what fundamental constraints, if any, are implied by the plant model? If the state estimation error is made insensitive to measurement (or process) noise over some frequency range, is the estimation error necessarily sensitive to this same noise over some other range of frequencies?

Questions of this form have been considered in the analysis of feedback control systems since at least the work of Bode [2], and more recently by Francis and Zames [3] and Freudenberg and Looze [4, 5] among others. In the context of feedback control, the requirement for internal stability of the feedback loop imposes interpolation constraints, (specified in terms of right half-plane poles and zeros of the open-loop transfer function) on closed-loop transfer functions mapping external inputs to all signals in the loop.

One of the key closed-loop operators is the sensitivity function $S(s)$, which in a feedback control system maps output disturbances and small parameter variations to the controlled output. The requirement that the sensitivity operator satisfy interpolation conditions at open-right half-plane (ORHP) plant zeros in turn forces integral constraints to hold, namely, that reduction of sensitivity over some frequency interval implies that sensitivity is increased over some other interval [4]. These are powerful results, and indicate bounds on achievable performance independent of the particular design method used.

The problem of understanding unavoidable constraints in the filtering context was first considered by Goodwin et al. [6]. This work shows the requirement that a state estimator being unbiased leads to the conclusion that the relative sensitivities of the state estimation error to input and output noise are complementary. That is, at any given frequency, the sensitivity of state estimates to input and output disturbances cannot be simultaneously reduced to zero. Seron and Goodwin [7] (see also [8]) extended this work, showing that for an even broader class of state estimators than unbiased estimators (the so-called *bounded error estimators*), the operators mapping process noise and measurement noise to state estimates are complementary. Furthermore, integral bounds were established on the peak modulus of the process and measurement noise sensitivity functions, providing bounds on the achievable performance of all linear, bounded error estimators, regardless of the particular design method used. For a comprehensive treatment of fundamental limitations in filtering and control, see [9].

The work [6–8] in restricts attention to filtering sensitivities of single-input, single-output (SISO) systems. The extension to multivariable systems was considered by Braslavsky et al. [10], pursuing an extension of the results of [7] along the lines of [11], see also [9, Chpater 9]. The multivariable extension in [10] leads to lower bounds on the infinity norm of the individual elements of the measurement noise sensitivity matrix.

In the present paper, we pursue a complementary route to multivariable filtering constraints, along the lines of Chen [12], wherein integral constraints on the maximum singular value of the sensitivity functions are obtained, see also [13] for a multivariable extension of the classical Poisson integral inequality which holds under slightly weaker conditions than in [12]. As in [10], the limits so obtained depend explicitly on the directionality properties of the plant poles and zeros, concepts which have no scalar counterpart.

The present paper is organized as follows. In the remainder of Section 1, we review the notion of directionality for zeros and poles of multivariable systems, and present some mathematical and notational preliminaries. In Section 2, we review the class of bounded error estimators introduced by Seron and Goodwin [7], see also [9, Section 7.3]. The process noise and measurement noise operators are introduced in Section 3, and multivariable interpolation conditions are developed in terms of these operators. The material in this section extends upon the work of Seron and Goodwin in [7, Section 5] by considering multivariable filtering constraints for square systems. In Section 4, the interpolation conditions, together with the Poisson integral inequality [12, 13], are used to develop lower bounds on suitably weighted integrals of the maximum singular value of the measurement noise operator. In Section 5, we show how the integral constraints of Section 4, together with a mild design requirement on the measurement noise operator, imply an unavoidable increase in measurement noise sensitivity over some frequency range whenever the plant is unstable and there is at least one nonminimum phase zero in the transfer function from process noise to system output. Conclusions are drawn in Section 6.

Preliminaries

Given a proper rational transfer matrix $P(s)$ with a minimal realization (A, B, C, D), a point $z \in \mathbf{C}$ is called a *transmission zero* of $P(s)$ if there exist vectors ζ and η such that the relation

$$\begin{bmatrix} zI - A & -B \\ -C & -D \end{bmatrix} \begin{bmatrix} \zeta \\ \eta \end{bmatrix} = 0 \tag{1}$$

holds, where $\eta^H \eta = 1$, and η is called the *input zero direction* associated with z. Similarly, a transmission zero z of $P(s)$ satisfies the relation

$$\begin{bmatrix} x^H & w^H \end{bmatrix} \begin{bmatrix} zI - A & -B \\ -C & -D \end{bmatrix} = 0, \tag{2}$$

where x and w are some vectors with w, called the *output zero direction* associated with z, satisfying the condition $w^H w = 1$.

By definition, the poles of $P(s)$ are the eigenvalues of A. The system $P(s)$ is said to be *unstable* if any eigenvalue of A lies in the open right half-plane, and stable otherwise. For D nonsingular, we will call the input and output directions of $P^{-1}(s)$ the input and output pole directions of $P(s)$, respectively, using the well-known fact that $p \in \mathbf{C}$ is a pole of $P(s)$ if and only if it is a zero of $P^{-1}(s)$ [14].

A *nonminimum phase* zero is a transmission zero that lies in the closed right half-plane. A transfer matrix with at least one nonminimum phase zero is said to be nonminimum phase, otherwise it is said to be minimum phase. It is well known that the right half-plane zeros of a nonminimum phase transfer matrix may be collected into a stable all-pass factor. One procedure for performing this factorization amounts to sequentially extracting nonminimum phase zeros by repeated use of a formula developed in [15].

Let $\mathbf{C}_+ \triangleq \{s : \mathrm{Re}(s) > 0\}$ denote the open right half-plane (ORHP), and let $\overline{\mathbf{C}}_+$ denote its closure. Let $z_i \in \mathbf{C}_+, i = 1, \ldots, k$ be the nonminimum phase zeros of $P(s)$. The transfer matrix $P(s)$ can be factorized as $P(s) = P^{(1)}(s)\mathcal{B}_1(s)$, where $\mathcal{B}_1(s) = I - (2\mathrm{Re}(z_1)/(s+\overline{z}_1))\eta_1\eta_1^H$, and η_1 is the input direction associated with z_1. Note that after this factorization, z_1 is no longer a zero of $P^{(1)}(s)$. This procedure can be continued to obtain $P^{(i-1)}(s) = P^{(i)}(s)\mathcal{B}_i(s)$, where

$$\mathcal{B}_i(s) = I - \frac{2\mathrm{Re}(z_i)}{s + \overline{z}_i}\eta_i\eta_i^H, \tag{3}$$

and η_i, satisfying $\eta_i^H\eta_i = 1$, is obtained as if it were the direction of z_i but is computed from $P^{(i-1)}(s)$ rather than $P(s)$. By repeating this procedure, we obtain a factorization of the form

$$P(s) = P_{\mathrm{m}}(s)\prod_{i=1}^{k}\mathcal{B}_{k+1-i}(s), \tag{4}$$

where $P_{\mathrm{m}}(s)$ is the minimum phase factor of $P(s)$ and the $\mathcal{B}_i(s)$'s obtained above are the all-pass factors associated with z_i.

Given a unitary vector $u \in \mathbf{C}^n$, the one-dimensional subspace spanned by u is termed the *direction* of u. The angle between the directions of two unitary vectors $u, v \in \mathbf{C}$ is defined as the *principal angle* [16] between the two vectors, and denoted by $\angle(u, v)$:

$$\cos\angle(u, v) \triangleq |u^H v|. \tag{5}$$

2. A GENERAL CLASS OF ESTIMATORS

Consider the following linear, time-invariant system having measured input u, unmeasured disturbance inputs v and w, and measured output y:

$$\dot{x} = Fx + G_1 u + G_2 v, \quad x(0) = x_0, \tag{6}$$

$$y = Hx + w \tag{7}$$

where $x \in \mathbf{R}^n$, $v \in \mathbf{R}^q$, $u \in \mathbf{R}^m$, and $y, w \in \mathbf{R}^p$, and F, G_1, G_2, and H are real-valued matrices of appropriate dimensions.

We assume that (6)-(7) is detectable but not necessarily stabilizable from either u or v. No restrictions are placed on the disturbance inputs v and w.

By taking the Laplace transform of (6)-(7), the system state x and output y can be expressed in terms of the initial state x_0, and the measured input u and disturbances v and w as

$$x = T_0(s)x_0 + T_0G_1u + T_0(s)G_2v, \tag{8}$$

$$y = HT_0(s)x_0 + HT_0(s)G_1u + HT_0(s)G_2v + w, \tag{9}$$

where

$$T_0(s) \triangleq (sI - F)^{-1} \tag{10}$$

is not necessarily assumed to be stable. Here and in the sequel, we will use the same symbols to denote both the time domain functions and their corresponding transforms.

Now, consider the general class of linear, time-invariant, stable state estimators having the form

$$\dot{\xi} = \hat{F}\xi + K_1u + K_2y, \quad \xi(0) = \xi_0, \tag{11}$$

$$\hat{x} = N\xi, \tag{12}$$

where $\xi \in \mathbf{R}^l$, $\hat{x} \in \mathbf{R}^n$, u and y are as in (6)-(7), and $\hat{F}, K_1, K_2$, and N are real-valued matrices of appropriate dimension, with $\hat{F}$ a stability matrix. Note that there is not necessarily a connection between F in (6) and $\hat{F}$ in (11). In particular, the order of the estimator (11)-(12) need not be the same as that of the system (6)-(7).

By taking Laplace transforms of (11)-(12), the state estimate $\hat{x}$ can be expressed in terms of the initial state estimate ξ_0 and the measurable system input and output as

$$\hat{x} = N\hat{T}_0(s)\xi_0 + N\hat{T}_0(s)K_1u + N\hat{T}_0(s)K_2y, \tag{13}$$

where

$$\hat{T}_0(s) \triangleq (sI - \hat{F})^{-1} \tag{14}$$

is a stable, proper transfer matrix.

From (8), (9), and (13), the state estimation error $\tilde{x} \triangleq x - \hat{x}$ is clearly a function of the signals u, v, and w, together with the initial states x_0 and ξ_0. The following lemma makes this connection specific by developing an expression for the state estimation error which will be of key importance in Section 3.

Lemma 1 ([7]). *For the system* (6)-(7), *and the general class of estimators* (11), (12), *the estimation error* $\tilde{x}$ *satisfies*

$$\begin{aligned}\tilde{x} = {} & (I - N\hat{T}_0(s)K_2H)T_0(s)x_0 - N\hat{T}_0(s)\xi_0 \\ & + T_{\tilde{x}u}(s)u + T_{\tilde{x}v}(s)v + T_{\tilde{x}w}(s)w,\end{aligned} \tag{15}$$

where

$$T_{\tilde{x}u}(s) \triangleq T_{xu}(s) - T_{\hat{x}y}(s)T_{yu}(s) - T_{\hat{x}u}(s) \tag{16}$$

$$T_{\tilde{x}v}(s) \triangleq T_{xv}(s) - T_{\hat{x}v}(s) \triangleq T_{xv}(s) - T_{\hat{x}y}(s)T_{yv}(s), \tag{17}$$

$$T_{\tilde{x}w}(s) \triangleq -T_{\hat{x}w}(s) \triangleq -T_{\hat{x}y}(s), \tag{18}$$

and where in (16)–(18) *we have used the following:*

$$T_{xu}(s) \triangleq T_0(s)G_1, \qquad T_{xv}(s) \triangleq T_0(s)G_2, \tag{19}$$

$$T_{yu}(s) \triangleq HT_0(s)G_1, \qquad T_{yv}(s) \triangleq HT_0(s)G_2, \tag{20}$$

$$T_{\hat{x}u}(s) \triangleq N\hat{T}_0(s)K_1, \qquad T_{\hat{x}y}(s) \triangleq N\hat{T}_0(s)K_{2.} \tag{21}$$

Proof. Using (9) to eliminate y from the expression for the estimator output $\hat{x}$, we have

$$\begin{aligned}\hat{x} = {} & N\hat{T}_0(s)\xi_0 + N\hat{T}_0(s)K_1u \\ & + N\hat{T}_0(s)K_2(HT_0(s)x_0 + HT_0(s)G_1u + HT_0(s)G_2v + w),\end{aligned} \tag{22}$$

which, together with (8), implies

$$\begin{aligned}x - \hat{x} = {} & (I - N\hat{T}_0(s)K_2H)T_0(s)x_0 - N\hat{T}_0(s)\xi_0 \\ & + ((I - N\hat{T}_0(s)K_2H)T_0(s)G - N\hat{T}_0(s)K_1)u \\ & + ((I - N\hat{T}_0(s)K_2H)T_0(s)G_2)v - N\hat{T}_0(s)K_2w.\end{aligned} \tag{23}$$

The result now follows from definitions (16)–(21). □

Of the general class of estimators described in Section 2, we will restrict attention to those estimators which, in addition to being stable, produce a bounded estimation error when all signals entering the system are bounded. A vector signal $x(t) = [x_1(t), \dots, x_i(t), \dots, x_n(t)]^T$ is said to be *bounded* if every component of the vector is bounded, that is, $\|x_i\|_\infty < \infty$, $i = 1, \dots, n$, where $\|x_i\|_\infty \triangleq \sup_{t \geq 0}|x_i(t)|$. The following definition, together with Lemma 2, is due to Seron [8]; see also [7, Section 7.3].

Definition 1 (Bounded error estimator). A stable state estimator $\hat{x}$ is said to be a bounded error estimator (BEE) if, for all finite initial states $x(0)$, $\hat{x}(0)$, the estimation error $\tilde{x} = x - \hat{x}$ is bounded whenever the system inputs u, v, and w are bounded.

Lemma 2. *A stable estimator given by* (11)–(14) *for the system* (6)-(7) *is a BEE if and only if the transfer matrix* $(I - N\hat{T}_0(s)K_2H)T_0(s)$ *is stable.*

Proof. From the expression for the estimation error given in (23), $\hat{x}$ is bounded whenever inputs u, v, and w are bounded if and only if both $\hat{T}_0(s)$ and $(I - N\hat{T}_0(s)K_2H)T_0(s)$ are stable. From (14), $\hat{T}_0(s)$ is stable by assumption, and the result follows. □

The class of BEE's for the system (6)-(7) is very broad, and includes all stable unbiased estimators [17] as a subclass.

3. INTERPOLATION CONDITIONS FOR MULTIVARIABLE FILTERING

Given measurements of the input u and output y of the system (6)-(7), two factors conspire to introduce errors into the state estimates $\hat{x}$ obtained by any bounded error estimator, unmeasured process disturbances v and measurement disturbances w. While it is clearly desirable to minimize the effect of both types of disturbance on state estimates, Seron and Goodwin [7] have shown that at any given frequency, it is impossible for any bounded error estimator to simultaneously eliminate the effect of both types of disturbances. In this section, we review the process noise and measurement noise sensitivity operators introduced by Seron and Goodwin [7], and show how their complementary form leads to interpolation constraints expressed in terms of the ORHP poles and zeros of these operators.

Consider the following pair of operators, expressed in terms of various transfer matrices defined in Lemma 1:

$$\mathcal{P}(s) \triangleq T_{\tilde{x}v}(s)T_{xv}^{-1}(s), \quad (24)$$

$$\mathcal{M}(s) \triangleq -T_{\tilde{x}w}(s)T_{yv}(s)T_{xv}^{-1}(s). \quad (25)$$

The operator $\mathcal{P}(s)$represents the relative effect of process disturbances v on the estimation error $\hat{x}$, and is thus termed the process noise sensitivity. Likewise, $\mathcal{M}(s)$ represents the effect of the measurement noise w on $\hat{x}$, weighted by the ratio between the measured output y, and the state vector x. That the interpretation of $\mathcal{M}(s)$ is not quite so straightforward as that given to $\mathcal{P}(s)$ reflects the somewhat more indirect route from measurement noise to state estimates as compared with process noise to state estimates.

By appropriate state estimator design, it is possible to shape $\mathcal{P}(s)$ in such a way that this operator is made small (in some appropriate sense, typically that the maximum singular value is made small) over the range of frequencies where process noise is large. Similarly, $\mathcal{M}(s)$ can be shaped to reduce the effect of measurement noise on state estimates. At any given frequency, however, it is impossible to make both $\mathcal{P}(s)$ and $\mathcal{M}(s)$ small, as the following key result makes clear.

Theorem 3.1. *The process noise sensitivity operator $\mathcal{P}(s)$ and the measurement noise sensitivity operator $\mathcal{M}(s)$ satisfy*

$$\mathcal{P}(s) + \mathcal{M}(s) = I, \quad (26)$$

at any complex frequency $s \in \mathbf{C}$ that is not a pole of either $\mathcal{P}(s)$ or $\mathcal{M}(s)$.

Proof. See [7, Theorem 3.1]. □

Suppose now that we wish to estimate not simply the state vector x of the system (6)-(7), but some linear combination of states given by

$$z = Jx. \quad (27)$$

Since attention is restricted to linear, finite-dimensional, time-invariant and stable state estimators of the form (11)-(12), it follows from (13) that the estimate $\hat{z}$ is given by

$$\hat{z} = JN\hat{T}_0(s)\xi_0 + JN\hat{T}_0(s)K_1u + JN\hat{T}_0(s)K_2y. \quad (28)$$

With the estimation error defined as $\tilde{z} = z - \hat{z}$, the process and measurement noise sensitivity operators in the complementarity constraint (26) become

$$\mathcal{P}(s) = T_{\tilde{z}v}(s)T_{zv}^{-1}(s), \quad (29)$$

$$\mathcal{M}(s) = -T_{\tilde{z}w}(s)T_{yv}(s)T_{zv}^{-1}(s). \quad (30)$$

Our aim is to show how Theorem 3.1 leads to interpolation constraints expressed in terms of the ORHP poles and zeros of $\mathcal{P}(s)$ and $\mathcal{M}(s)$. To do so, we will express both $\mathcal{P}(s)$ and $\mathcal{M}(s)$ in terms of right coprime matrix-fraction descriptions (MFDs) of the transfer matrices in (29)-(30).

There are two reasons for choosing MFDs, as opposed to state-space descriptions, for representing the transfer matrices of interest. First, since $T_{zv}(s)$ is strictly proper, the indicated inverse in (29) and (30) will be improper, and keeping track of its transmission zeros is simpler using MFDs than state-space representations, since no particular significance is placed on properness of MFDs. Second, for clarity of exposition in Section 4, the use of MFDs makes it simpler to keep track of the plant-dependent and estimator-dependent terms in the sensitivity operators. Note, however, that the lower bounds on peak sensitivity developed in Section 5 (see Theorem 5.1) can be computed entirely in a state-space framework, thereby simplifying the necessary calculations.

From Lemma 1, (27) and (28), it follows that

$$T_{zv}(s) = JT_0(s)G_2 = C(s)A^{-1}(s), \quad (31)$$

$$T_{yv}(s) = HT_0(s)G_2 = B(s)A^{-1}(s), \quad (32)$$

$$T_{\hat{z}v}(s) = JN\hat{T}_0(s)K_2 = L(s)E^{-1}(s), \quad (33)$$

$$\begin{aligned} T_{\tilde{z}v}(s) &= T_{zv}(s) - T_{\hat{z}y}(s)T_{yv}(s) \\ &= C(s)A^{-1}(s) - L(s)E^{-1}(s)B(s)A^{-1}(s), \end{aligned} \quad (34)$$

$$T_{\tilde{z}w}(s) = -T_{\hat{z}y}(s) = -L(s)E^{-1}(s), \quad (35)$$

where $A(s) = sI - F$ and $E(s) = sI - \hat{F}$. Thus from (29)–(35) the process noise and measurement noise sensitivity operators associated with any bounded error estimator are given by

$$\begin{aligned} \mathcal{P}(s) &= T_{\tilde{z}v}(s)T_{zv}^{-1}(s) \\ &= \left(C(s)A^{-1}(s) - L(s)E^{-1}(s)B(s)A^{-1}(s)\right)\left(C(s)A^{-1}(s)\right)^{-1} \\ &= I - L(s)E^{-1}(s)B(s)C^{-1}(s), \end{aligned} \quad (36)$$

$$\begin{aligned} \mathcal{M}(s) &= -T_{\tilde{z}w}(s)T_{yv}(s)T_{zv}^{-1}(s) \\ &= -\left(L(s)E^{-1}(s)\right)\left(B(s)A^{-1}(s)\right)\left(C(s)A^{-1}(s)\right)^{-1} \\ &= L(s)E^{-1}(s)B(s)C^{-1}(s). \end{aligned} \quad (37)$$

From (37),

$$\det \mathcal{M}(s) = \frac{\det L(s) \cdot \det B(s)}{\det E(s) \cdot \det C(s)}, \quad (38)$$

and since $\det E(s)$ is Hurwitz for any stable estimator, it follows that the poles and zeros of $\mathcal{M}(s)$ are given by $\mathcal{Z}_+(C(s))$ and $\mathcal{Z}_+(L(s)B(s))$, respectively, where $\mathcal{Z}_+(X(s))$ denotes the set of ORHP zeros of the polynomial $\det X(s)$.

Since $\mathcal{Z}_+(C(s))$ is not necessarily empty (i.e., the transfer matrix $T_{zv}(s)$ may be nonminimum phase), the possibility exists for unstable pole-zero cancellations to occur in (38). For simplicity in notation, we will assume in this paper that no such cancellations occur; see Seron [9] for a thorough treatment of the more general situation in the SISO case. For our purposes here, it is sufficient to note that any results in the sequel using elements of $\mathcal{Z}_+(B(s))$ are restricted to entries distinct from $\mathcal{Z}_+(C(s))$.

In the following section, we will develop integral relations satisfied by the measurement noise sensitivity $\mathcal{M}(s)$. Similar results can be obtained for $\mathcal{P}(s)$ using the complementarity constraint (26). The interpolation constraints are expressed in terms of the ORHP zeros of $\mathcal{P}(s)$ and $\mathcal{M}(s)$, for which we introduce the following definitions:

$$\begin{aligned} \mathcal{Z}_P &\triangleq \{\text{ORHP zeros of } \mathcal{P}(s)\}, \\ \mathcal{Z}_M &\triangleq \{\text{ORHP zeros of } \mathcal{M}(s)\} = \mathcal{Z}_+(L(s)B(s)). \end{aligned} \tag{39}$$

Of particular importance for this paper is the fact that all unstable plant poles are contained within the set $\mathcal{Z}_P$. To see this, note that the transfer matrix $T_{\tilde{z}v}(s)$must be stable for a bounded error estimator. But from (34),

$$\det T_{\tilde{z}v}(s) = \frac{\det\left(C(s) - L(s)E^{-1}(s)B(s)\right)}{\det A(s)}, \tag{40}$$

so that $\det(C(s) - L(s)E^{-1}(s)B(s)) = \det M(s) \det A_{\mathrm{u}}(s)$ for some polynomial matrix $M(s)$ of appropriate degree, and $A_{\mathrm{u}}(s)$ contains all ORHP plant poles. Now from (36),

$$\begin{aligned} \det \mathcal{P}(s) &= \det\left(I - L(s)E^{-1}(s)B(s)C^{-1}(s)\right) \\ &= \frac{\det\left(C(s) - L(s)E^{-1}(s)B(s)\right)}{\det C(s)} \\ &= \frac{\det M(s) \det A_{\mathrm{u}}(s)}{\det C(s)}, \end{aligned} \tag{41}$$

and the assertion is proved.

We then have the following interpolation result, the scalar version of which is due to Seron and Goodwin [7, Section 5]. The proof follows immediately from the definitions of $\mathcal{Z}_P$ and $\mathcal{Z}_M$, together with the key complementarity constraint (26).

Theorem 3.2 (Interpolation constraints on $\mathcal{P}(s)$ and $\mathcal{M}(s)$). *One has the following:*

(i) $\mathcal{Z}_M \bigcap \mathcal{Z}_P = \varnothing$;
(ii) $\mathcal{P}(q) = 0$ *and* $\mathcal{M}(q) = I$, *if* $q \in \mathcal{Z}_P$;
(iii) $\mathcal{P}(p) = I$ *and* $\mathcal{M}(p) = 0$, *if* $p \in \mathcal{Z}_M$.

4. INTEGRAL CONSTRAINTS FOR MULTIVARIABLE FILTERING

In this section, we develop integral constraints on the admissible behavior of the measurement noise sensitivity $\mathcal{M}(s)$. The key tool used to develop these constraints is the Poisson integral inequality [12, 13], and the basic theme of the constraints is that suitably weighted integrals of the largest singular value of $\mathcal{M}(s)$ can be bounded below by values which depend only on the plant (6)-(7), and are thus independent of any particular estimator design procedure.

To begin, we note that the minimum phase/all-pass factorization detailed in Section 1 can be used to factorize the measurement noise sensitivity as

$$\mathcal{M}(s) = \overline{\mathcal{M}}(s)\mathcal{B}_L(s)\mathcal{B}_B(s), \tag{42}$$

where $\overline{\mathcal{M}}(s)$ is minimum phase, and $\mathcal{B}_L(s)$ and $\mathcal{B}_B(s)$ are, respectively, the all-pass factors corresponding to the ORHP zeros of the polynomial matrices $L(s)$ and $B(s)$, that is, the numerators in the MFDs (32) and (33). Since $\det C(s)$ is not necessarily Hurwitz, it follows from (38) that the minimum phase factor of $\mathcal{M}(s)$ (namely, $\overline{\mathcal{M}}(s)$) is not necessarily stable. To extract a stable minimum phase factor $\widetilde{\mathcal{M}}(s)$ from $\mathcal{M}(s)$, it is necessary to postmultiply $\overline{\mathcal{M}}(s)$ by $\mathcal{B}_C(s)$, the all-pass factor corresponding to the ORHP zeros of the polynomial matrix $C(s)$. It follows that we can write

$$\mathcal{M}(s) = \widetilde{\mathcal{M}}(s)\mathcal{B}(s)_C^{-1}(s)\mathcal{B}_L(s)\mathcal{B}_B(s), \tag{43}$$

where $\overline{\mathcal{M}}(s)$ is both stable and minimum phase. Letting $\overline{\sigma}(X)(s)$ denote the maximum singular value of X, we can now state the following.

Theorem 4.1. *Consider any bounded error estimator such that $\mathcal{M}(s)$ is proper, let $q = x_1 + jx_2$ be any element of $\mathcal{Z}_P$ with associated output direction w, and define*

$$\mathcal{I}_M \triangleq \int_{-\infty}^{\infty} \log \overline{\sigma}(\mathcal{M}(j\omega)) \frac{x_1}{x_1^2 + (x_2 - \omega)^2} d\omega. \tag{44}$$

Then

$$\mathcal{I}_M \geq \pi \log \overline{\sigma}\left(\mathcal{B}_B^{-1}(q)\mathcal{B}_L^{-1}(q)\mathcal{B}_C(q)\right) \tag{45}$$

$$\geq \pi \log \left\| w^H \mathcal{B}_B^{-1}(q)\mathcal{B}_L^{-1}(q)\mathcal{B}_C(q) \right\|, \tag{46}$$

where $\|\cdot\|$ denotes the spectral norm.

Proof. By construction, $\widetilde{\mathcal{M}}(s) = \mathcal{M}(s)\mathcal{B}_B^{-1}(s)\mathcal{B}_L^{-1}(s)\mathcal{B}_C(s)$ is analytic in the ORHP. If the multiplicity of $\overline{\sigma}(\widetilde{\mathcal{M}}(s))$ is constant over $\overline{\mathbf{C}}_+$ then, since $\widetilde{\mathcal{M}}(s)$ is proper by assumption, the Poisson integral inequality (see, e.g., [12, 13, 18]) can be applied to $\log \overline{\sigma}(\widetilde{\mathcal{M}}(j\omega))$, which yields

$$\frac{1}{\pi} \int_{\infty}^{\infty} \log \overline{\sigma}(\widetilde{\mathcal{M}}(j\omega)) \frac{x_1}{x_1^2 + (x_2 - \omega)^2} d\omega \geq \log \overline{\sigma}(\widetilde{\mathcal{M}}(q)). \tag{47}$$

Now from the interpolation constraint in Theorem 3.2, $\mathcal{M}(q) = I$, from which

$$\widetilde{\mathcal{M}}(q) = \mathcal{B}_B^{-1}(q)\mathcal{B}_L^{-1}(q)\mathcal{B}_C(q). \tag{48}$$

Even if the multiplicity of $\overline{\sigma}(\widetilde{\mathcal{M}}(s))$ is *not* constant over $\overline{\mathbf{C}}_+$, inequality (47)—and hence equality (48)—still hold, by the line of reasoning in the proof of [13, Theorem 3.1].

Inequality (45) follows from the fact that $\overline{\sigma}(\widetilde{\mathcal{M}}(j\omega)) = \overline{\sigma}(\mathcal{M}(j\omega))$, and on substituting (48) into the right-hand side of (47).

To establish inequality (46), we proceed as in the proof of Theorem 4.1 of Chen [12]. Since by assumption q is an ORHP zero of $\mathcal{P}$ with output direction w, we have that $w^H(q) = 0$, from which $w^H(I + \mathcal{P}(q)) = w^H$, and thus $w^H = w^H\mathcal{M}(q)$. Inequality (46) now follows from (47) and the fact that

$$\begin{aligned}\overline{\sigma}(\widetilde{\mathcal{M}}(q)) &= \overline{\sigma}(\mathcal{M}(q)\mathcal{B}_B^{-1}(q)\mathcal{B}_L^{-1}(q)\mathcal{B}_C(q)) \\ &\geq \|w^H\mathcal{M}(q)\mathcal{B}_B^{-1}(q)\mathcal{B}_L^{-1}(q)\mathcal{B}_C(q)\| \\ &= \|w^H\mathcal{B}_B^{-1}(q)\mathcal{B}_L^{-1}(q)\mathcal{B}_C(q)\|.\end{aligned} \quad (49)$$

□

The results in Theorem 4.1 are estimator-dependent, as evidenced by the presence of $\mathcal{B}_L^{-1}(q)$ on the right-hand side of inequalities (45) and (46). The following corollary provides a constraint independent of the particular state estimator chosen.

Corollary 1. *Consider any bounded error estimator such that $\mathcal{M}(s)$ is proper, and let $q = x_1 + jx_2$ be any element of $\mathcal{Z}_P$ with associated output direction w. Then with $\mathcal{I}_M$ as defined in* (44),

$$\mathcal{I}_M \geq \pi\log\overline{\sigma}(\mathcal{B}_B^{-1}(q)\mathcal{B}_C(q)) \quad (50)$$

$$\geq \pi\log\|w^H\mathcal{B}_B^{-1}(q)\mathcal{B}_C(q)\|. \quad (51)$$

Proof. Both inequalities follow immediately from Theorem 4.1 and the fact that $\log\overline{\sigma}(\mathcal{B}_L^{-1}(s)) \geq 0$ for any polynomial matrix $L(s)$. □

Inequalities (46) and (51) strongly suggest that in the multivariable case, the sensitivity properties of $\mathcal{M}(s)$ are closely related to the directionality of the elements of $\mathcal{Z}_P$. The following lemma makes this statement more precise, by explicitly linking the output directions of the elements of $\mathcal{Z}_P$ (the unstable system poles) and the input directions of elements of $\mathcal{Z}_M$ (the nonminimum phase zeros of the transfer matrix $T_{yv}(s)$).

Lemma 3. *Consider any bounded error estimator such that $\mathcal{M}(s)$ is proper. Also, let $\mathcal{M}(s)$ be factorized as in* (43), *and assume that the multiplicity of $\overline{\sigma}(\widetilde{M}(s))$ is constant on $\overline{\mathbf{C}}_+$. Let $q = x_1 + jx_2$ be any element of $\mathcal{Z}_P$ with associated output direction w and suppose $p_1, \ldots, p_k$ are the ORHP zeros of* $\det B(s)$, *having input directions $\eta_1, \ldots, \eta_k$, respectively. Then*

$$\mathcal{I}_M \geq \frac{\pi}{2}\max_{1\leq i\leq k}\log\left[\cos^2\angle(w,\eta_i)\left|\frac{q+\overline{p}_i}{q-p_i}\right|^2 + \sin^2\angle(w,\eta_i)\right]. \quad (52)$$

Proof. The proof mirrors the proof of [12, Corollary 4.1], and is only sketched here. Suppose, without loss of generality, that the maximum in (52) corresponds to $i = 1$. Then

$$\|w^H\mathcal{B}_1^{-1}(q)\|^2 = 1 + \frac{4\text{Re}(p_1)\text{Re}(q)}{|q-p_1|^2}|w^H\eta_1|^2, \quad (53)$$

where $\mathcal{B}_1$ is the all-pass factor corresponding to p_1. But since

$$\begin{aligned}\overline{\sigma}(\mathcal{B}_i^{-1}(s)) &= \left|\frac{s+\overline{p}_i}{s-p_i}\right|, \\ \overline{\sigma}^2(\mathcal{B}_i^{-1}(s)) &= 1 + \frac{4\text{Re}(s)\text{Re}(p_i)}{|s-p_i|^2},\end{aligned} \quad (54)$$

it follows that

$$\frac{4\text{Re}(p_1)\text{Re}(q)}{|q-p_1|^2} = \left|\frac{q+\overline{p}_1}{q-p_1}\right|^2 - 1. \quad (55)$$

Hence

$$\begin{aligned}\|w^H\mathcal{B}_1^{-1}(q)\|^2 &= 1 + \left|\frac{q+\overline{p}_1}{q-p_1}\right|^2|w^H\eta_1|^2 - |w^H\eta_1|^2 \\ &= \cos^2\angle(w,\eta_1)\left|\frac{q+\overline{p}_1}{q-p_1}\right|^2 + \sin^2\angle(w,\eta_1).\end{aligned} \quad (56)$$

The result then follows from (51) and the fact that

$$\|w^H\mathcal{B}_B^{-1}(q)\mathcal{B}_C(q)\| \geq \|w^H\mathcal{B}_B^{-1}(q)\| \geq \|w^H\mathcal{B}_1^{-1}(q)\|. \quad (57)$$

□

5. DESIGN TRADEOFFS

The results of the previous section imply that arbitrary shaping of the measurement noise sensitivity function is, in general, not possible, Thus the reduction of measurement noise sensitivity in one frequency range is typically accompanied by an increased sensitivity over some other frequency range. As in the scalar case [7], the presence of ORHP poles and zeros in the system transfer functions $T_{zv}(s)$ and $T_{yv}(s)$ determine the nature and severity of these tradeoffs. In the multivariable case, however, the directionality properties of these ORHP poles and zeros play a role which has no scalar counterpart.

While the integral constraints of the previous section allow us to draw general conclusions regarding the overall behavior of the measurement and process noise sensitivity transfer functions, more quantitative conclusions are possible if we impose further constraints on the admissible forms of the sensitivity operators.

From the definition of $\mathcal{M}(s)$, the effect of measurement noise on the quality of state estimates will be minimal if $\overline{\sigma}(\mathcal{M}(j\omega))$ is small over the range of frequencies where measurement noise power is significant. Since measurement noise tends to dominate at high frequencies as compared with process noise, a realistic design goal is to impose an upper bound on $\overline{\sigma}(\mathcal{M}(j\omega))$ at high frequencies. A reasonable question then is; if $\overline{\sigma}(\mathcal{M}(j\omega))$ is kept small over some specified range of (high) frequencies, does the peak sensitivity over the low frequency range become unacceptably large? (Or, at least, can a lower bound on the peak sensitivity be computed?)

For the purposes of illustration, we will consider only the tradeoffs associated with the largest singular value of the measurement noise sensitivity. Suppose that a bounded error estimator has been designed such that the following level of sensitivity reduction is achieved:

$$\overline{\sigma}(\mathcal{M}(j\omega)) \leq \alpha_m < 1, \quad \text{for } \omega \in [\omega_m, \infty). \tag{58}$$

Let $q = x_1 + jx_2$ be any complex number, and define

$$\begin{aligned} \theta(q) &\triangleq \int_{-\omega_m}^{\omega_m} \frac{x_1}{x_1^2 + (\omega - x_2)^2} d\omega \\ &= \arctan \frac{\omega_m - x_2}{x_1} + \arctan \frac{\omega_m + x_2}{x_1} \leq \pi. \end{aligned} \tag{59}$$

We then have the following

Theorem 5.1. *Consider any bounded error estimator such that $\mathcal{M}(s)$ is proper, and suppose that condition* (58) *is satisfied. Then for each element $q = x_1 + jx_2$ of $\mathcal{Z}_P$ with output direction w,*

$$||\mathcal{M}(s)||_\infty \geq \left(\frac{1}{\alpha_m}\right)^{\theta(q)/(\pi-\theta(q))} \left(\overline{\sigma}(\mathcal{B}_B^{-1}(q)\mathcal{B}_C(q))\right)^{\pi/\theta(q)}, \tag{60}$$

$$||\mathcal{M}(s)||_\infty \geq \left(\frac{1}{\alpha_m}\right)^{\theta(q)/(\pi-\theta(q))} ||w^H \mathcal{B}_B^{-1}(q)\mathcal{B}_C(q)||^{\pi/\theta(q)}, \tag{61}$$

where $\|\mathcal{M}(s)\|_\infty \triangleq \sup_{\omega\in\mathbf{R}} \overline{\sigma}(\mathcal{M}(j\omega))$ denotes the H_∞ norm of $\mathcal{M}(s)$.

Proof. From (50),

$$\begin{aligned} &\pi \log \overline{\sigma}(\mathcal{B}_B^{-1}(q)\mathcal{B}_C(q)) \\ &\quad \leq \int_{-\infty}^{\infty} \log \overline{\sigma}(\mathcal{M}(j\omega)) \frac{x_1}{x_1^2 + (x_2 - \omega)^2} d\omega. \end{aligned} \tag{62}$$

Since $\sup_{|\omega|\geq\omega_m} \overline{\sigma}(\mathcal{M}(j\omega)) \leq \alpha_m$ by design, and since $\sup_{|\omega|<\omega_m} \overline{\sigma}(\mathcal{M}(j\omega)) \leq \|\mathcal{M}(s)\|_\infty$ by definition, it follows that

$$\begin{aligned} &\pi \log \overline{\sigma}(\mathcal{B}_B^{-1}(q)\mathcal{B}_C(q)) \\ &\quad \leq \int_{-\infty}^{-\omega_m} \log(\alpha_m) \frac{x_1}{x_1^2 + (x_2 - \omega)^2} d\omega \\ &\qquad + \int_{\omega_m}^{\infty} \log(\alpha_m) \frac{x_1}{x_1^2 + (x_2 - \omega)^2} d\omega \\ &\qquad + \int_{-\omega_m}^{\omega_m} \log ||\mathcal{M}(s)||_\infty \frac{x_1}{x_1^2 + (x_2 - \omega)^2} d\omega \\ &\quad = \log(\alpha_m)[\pi - \theta(q)] + \log ||\mathcal{M}(s)||_\infty \theta(q). \end{aligned} \tag{63}$$

Exponentiating both sides of (63) gives the first inequality. Inequality (61) follows by the same process, beginning instead with (51). □

Once again, the importance of directionality is apparent from (61). The following Lemma makes this connection more explicit.

Lemma 4. *Consider any bounded error estimator such that $\mathcal{M}(s)$ is proper. Let the assumptions in Lemma 3 hold. Let $q = x_1 + jx_2$ be any element of $\mathcal{Z}_P$ with associated output direction w, and suppose $p_1, \ldots, p_k$ are the ORHP zeros of* $\det B(s)$*, having input directions $\eta_1, \ldots, \eta_k$, respectively. Then*

$$||\mathcal{M}(s)||_\infty \geq \max_{1\leq i\leq k} \sqrt{\cos^2\angle(w,\eta_i) \left| \frac{q + \overline{p}_i}{q - p_i} \right|^2 + \sin^2\angle(w,\eta_i)}. \tag{64}$$

Proof. This result follows from Lemma 3 and the following inequality:

$$\begin{aligned} &\int_{-\infty}^{\infty} \log \overline{\sigma}(\mathcal{M}(j\omega)) \frac{x_1}{x_1^2 + (x_2 - \omega)^2} d\omega \\ &\quad \leq \log ||\mathcal{M}(s)||_\infty \int_{-\infty}^{\infty} \frac{x_1}{x_1^2 + (x_2 - \omega)^2} d\omega \\ &\quad = \pi \log ||\mathcal{M}(s)||_\infty. \end{aligned} \tag{65}$$

□

While Theorem 5.1 has been derived using the simple design specification (58), the method easily extends to more general specifications. In general, the lower bounds in Theorem 5.1 will be different for each element of $\mathcal{Z}_P$, that is, each distinct unstable plant pole. Note that since $\alpha_m < 1, \overline{\sigma}(\mathcal{B}_B^{-1}(q)\mathcal{B}_C(q)) > 1$ and $\theta(q) < \pi$ imply that the peak sensitivity necessarily exceeds one over some frequency range whenever the plant is unstable and there is at least one nonminimum phase zero in the transfer function $T_{yv}(s)$ from process noise to system output.

It should be emphasized that the lower bounds are computed entirely from parameters in the plant model (6)-(7), or as part of the design specification (58). Moreover, the closed-form expression for $\theta(q)$ and the state-space construction of the all-pass factors $\mathcal{B}_B(s)$ and $\mathcal{B}_C(s)$ mean that the lower bounds are easily computed without the need to explicitly compute any MFDs.

6. CONCLUSIONS

In this paper, we have developed integral relations governing the sensitivity of a large class of linear multivariable state estimation procedures to process and measurement disturbances. The results in this paper supplement the multivariable extension of SISO filtering constraints as presented by Braslavsky et al. [10]. The inequalities in this paper may be used to assess the feasibility of various filter design requirements independent of the particular design method used. Moreover, since the sensitivity bounds developed depend only on the plant and the design specification, the bounds can be used to assess the potential benefits of additional (or relocated) sensors independent of any particular choice of linear filter.

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