

FREQUENCY DOMAIN ESTIMATION USING ORTHONORMAL BASES

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Abstract. This paper examines the use of general orthonormal bases for system identification from frequency domain data. This idea has been studied in great depth for the particular case of the orthonormal trigonometric basis. Here we show that the accuracy of the estimate can be significantly improved by rejecting the trigonometric basis in favour of a more general orthogonal basis that is able to be adapted according to prior information that is available about the system being identified. The usual trigonometric basis emerges as a special case of the general bases employed here.

Keywords. Frequency Response Estimation, System Identification, Parameter Estimation

1. INTRODUCTION

The bulk of system identification theory addresses the problem of estimating system models on the basis of observed time domain data (Ljung, 1987; T.Söderström and P.Stoica, 1989). However, in many cases the available data involves measurements of the systems frequency response (Pintelon *et al.*, 1994; Ljung, 1993). Indeed, using such measurement data with filters to remove harmonics is an effective way to deal with measurement or actuator non-linearities which would otherwise obscure the estimation of an underlying linear system. Estimation from frequency domain data has also been intensively studied as a means for providing estimated models together with error bounds which are suitable for subsequent robust control design (Parker and Bitmead, 1987; Gu and Khargonekar, 1992; Partington, 1991*b*; Partington, 1991*a*).

In this latter work the idea is (essentially) to estimate the systems impulse response by taking the inverse Discrete Fourier Transform of the measured frequency response. This can be interpreted as the process of fitting a linear combination of the trigonometric basis functions $\{e^{j\omega n}\}$ to the available frequency domain data. The orthonormality of the basis can be exploited to simplify

the calculations required and also to make tractable the theoretical analysis of the performance of the scheme. However, use of this basis exploits no a-priori knowledge of the nature of the system being identified (apart from causality).

The idea in this paper is to show how the trigonometric basis can be replaced with a general orthonormal set of basis functions $\{\mathcal{B}_n(e^{j\omega})\}$, of which the trigonometric set $\{e^{j\omega n}\}$ is a special case. The utility of this substitution is that we show how the bases $\{\mathcal{B}_n(e^{j\omega})\}$ can be constructed so as to incorporate prior knowledge of the system being identified whilst still preserving orthonormality. This leads to more accurate estimation.

This idea of using orthonormal bases for system identification has been pursued vigorously in the case of estimation from time domain data. In (Wahlberg, 1991) the so-called Laguerre basis which incorporated prior knowledge of one real pole has been analyzed. In (Wahlberg, 1994) this analysis was extended to the knowledge of one resonant mode via the use of the ‘Kautz’ basis. Recently Heuberger, Van den Hof and co-workers (Heuberger *et al.*, 1995) have extended this work by using constructions that exploit the properties of balanced realization of all pass systems. The general orthonormal bases they develop can incorporate prior knowledge of any fixed set of poles which are repeated. Use of these bases for identification from FFT transformed time domain data is considered in (de Vries, 1994). This paper considers a different scenario where the measurement technique

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involves ‘sine sweep’ experiments providing frequency response measurements as the available data set.

The bases $\{\mathcal{B}_n\}$ employed in this paper incorporate the Laguerre and Kautz bases as special cases since, and in common with the balanced realization construction, they allow the incorporation of prior knowledge of any number and type of poles in the system being identified. However, in contrast with the balanced realization construction of (Heuberger *et al.*, 1995) the basis functions used in this paper do not require these poles to be repeated as the order of the identified model grows. The cost for this is that the theoretical analysis is made much more difficult, essentially because the bases we employ do not form a group under pointwise multiplication. Balanced against this latter difficulty are the advantages of a high degree of generality and a very simply motivated and understood construction for the bases we study.

It is important to point out that the basis construction used here has been known for many years, but only in contexts different to the system identification one of this paper. To the author’s knowledge it dates back to Malmquist (Malmquist, 1925) and was taken up by Walsh (Walsh, 1935) in the context of complex rational approximation theory. Later Wiener (Wiener, 1949) used the same idea to explain Laguerre functions for the purposes of network synthesis. Kautz (Kautz, 1952) also used these methods in the context of continuous time network synthesis, and Broome (Broome, 1965) has used the ideas for discrete time network synthesis. In spite of this, although it has been mentioned in (Wahlberg, 1994), the general basis construction method investigated here does not seem to have been explored elsewhere in the context of system identification.

2. PROBLEM SETTING

In this paper it will be assumed that information in the form of n frequency response measurements $\{f_m\}$ of a linear time invariant system are available. Furthermore, it will be assumed that the measurements are obtained via sampling of continuous time signals and are equally spaced in the interval from d.c. to the folding frequency. This regular spacing requirement is imposed to make the analysis of the estimation methods we propose tractable, however it is not a necessary condition for the employment of the algorithms we propose.

Since discrete time measurements are assumed, the true system is described via a \mathcal{Z} transform model $G_T(z)$. The frequency domain estimation idea pursued in this paper is to parameterise the model structure $G(z, \hat{\theta})$ that is approximating $G_T(z)$ as a linear combination of user chosen basis functions $\{\mathcal{B}_k(z)\}$

$$G(z, \theta) = \sum_{k=0}^{p-1} \theta_k \mathcal{B}_k(z) \quad (1)$$

with $\theta^T \triangleq [\theta_0, \theta_1, \dots, \theta_{p-1}]$ being the vector of parameters that needs to be estimated from observed frequency domain data. The most obvious way of performing this approximation is to try to choose θ such that the estimated model’s frequency response matches that of the observations

$$\sum_{k=0}^{p-1} \theta_k \mathcal{B}_k(e^{jm\omega_s}) = f_m; \quad \omega_s \triangleq \frac{2\pi}{n}, m \in [0, n]. \quad (2)$$

This system of n equations in p unknowns can be more succinctly represented in matrix vector form as

$$\Omega_n \theta = f \quad (3)$$

where

$$[\Omega_n]_{m,\ell} = \mathcal{B}_\ell(e^{jm\omega_s})$$

$$f^T \triangleq (f_0, f_1, \dots, f_{n-1}).$$

If $n > p$ then this is an overdetermined set of linear equations. A natural estimate $\hat{\theta}$ then is the least squares one that minimises the quadratic error in achieving (3) by choosing $\hat{\theta}$ as

$$\hat{\theta} = (\Omega_n^* \Omega_n)^{-1} \Omega_n^* f \quad (4)$$

provided that the indicated inverse exists. Notice that up to this point there is no need for the frequency response measurements to be regularly spaced in the normalised frequency range $[0, 2\pi)$ since the estimate $\hat{\theta}$ is still perfectly well defined otherwise. However, there are two important advantages accrued if it can be so arranged that the measurement be linearly spaced:

- (1) As will be shortly demonstrated, for linearly spaced measurements a feature of the use of orthonormal basis functions is that $\Omega_n^* \Omega_n \approx nI$ (for the trigonometric FIR basis this is exact) so that no matrix inversion is required and $\hat{\theta}$ can be taken as $\frac{1}{n} \Omega_n^* f$ which can be thought of as the ‘generalised’ inverse DFT of the measurements.
- (2) With $\hat{\theta}$ as defined in (4) there is no guarantee of an estimate with real valued impulse response. However, if the measurements can be arranged to be linearly spaced, then only $n/2$ need be obtained in the normalised frequency range $[0, \pi)$, and another $n/2$ are then formed in $[\pi, 2\pi)$ by complex conjugation of these original ones. This will then lead to a real valued impulse response estimate.

The remainder of this paper will examine the properties of the estimated frequency response

$$G(e^{j\omega}, \hat{\theta}) = \sum_{k=0}^{p-1} \hat{\theta}_k \mathcal{B}_k(e^{j\omega})$$

when the measurement sequence $\{f_k\}$ is noise corrupted. Obviously it is desirable to choose the $\{\mathcal{B}_k\}$ basis functions such that the expansion $G_T = \sum \theta_k \mathcal{B}_k$ converges as quickly as possible so that the truncated expansion (1) is as accurate as possible. Intuitively, this can be achieved by choosing (via prior knowledge) the poles of the $\{\mathcal{B}_k\}$ basis functions to be as close as possible to the poles of G_T . How this may be achieved, while retaining orthonormality of the basis (so that numerical properties are enhanced and theoretical analysis is simplified) is documented in the next section.

3. ORTHONORMAL BASIS CONSTRUCTION

The orthonormal bases examined in this paper are constructed to be orthonormal with respect to the usual inner product that is useful in discrete time system analysis and which is defined by

$$\langle \mathcal{B}_n, \mathcal{B}_m \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathcal{B}_n(e^{j\omega}) \overline{\mathcal{B}_m(e^{j\omega})} d\omega. \quad (5)$$

The idea is that this orthonormality is to be preserved while at the same time incorporating prior knowledge in the estimation problem by choosing the poles $\{\xi_0, \dots, \xi_{p-1}\}$ of the $\{\mathcal{B}_n\}$ basis function to be near where the poles of the identified system G_T are believed to lie.

The construction of an orthonormal basis that satisfies the above requirements has been developed elsewhere (Ninness, 1994; Ninness and Gustafsson, 1994; Ninness and Gustafsson, 1996) where the following basis functions are presented

$$\mathcal{B}_n(z) = z^d \left(\frac{\sqrt{1 - |\xi_n|^2}}{z - \xi_n} \right) \prod_{k=0}^{n-1} \left(\frac{1 - \bar{\xi}_k z}{z - \xi_k} \right); \quad d = 0, 1. \quad (6)$$

The choice of d corresponds to a simple time shift on the impulse response of \mathcal{B}_0 and depends on whether the user feels that a causal or strictly causal model is most appropriate. In the remainder of this paper, these bases with $d = 0$ will be used.

Notice that for the choice $\xi_k = 0$ for all k this basis degenerates to that corresponding to an FIR model structure. For the choice of $\xi_k = \xi \in \mathbf{R}$ of all poles being the same and real, the Laguerre basis (Wahlberg, 1991) is obtained as a special case. However, it would seem more

appropriate to not use this basis in such a restricted setting, but rather choose the poles $\{\xi_k\}$ in a distributed fashion that most accurately reflects prior knowledge of $G_T(z)$.

Notice that if any of the poles $\{\xi_k\}$ are chosen as complex, then the formulation (6) has a complex valued impulse response which is inappropriate. In (Ninness and Gustafsson, 1994; Ninness and Gustafsson, 1996) it is shown how this may easily be circumvented by still using the construction (6), but if ξ_n is chosen as complex, then another pole ξ_{n+1} must also be chosen as the complex conjugate $\xi_{n+1} = \bar{\xi}_n$. This leads now to two basis function \mathcal{B}_n and \mathcal{B}_{n+1} with complex valued impulse responses. The idea now is that these may be linearly combined in an infinite variety of ways to yield two new basis function $\mathcal{B}'_n, \mathcal{B}''_n$ which have the same complex valued poles, are orthonormal to one another, and have **real** valued impulse responses. These latter two basis function are the ones used in the identification procedure. The details of this linear combination are unimportant here, but lead to \mathcal{B}'_n having the form

$$\mathcal{B}'_n(z) = \frac{\sqrt{1 - |\xi_n|^2}(\beta z + \mu)}{z^2 + (\xi_n + \bar{\xi}_n)z + |\xi_n|^2} \prod_{k=0}^{n-1} \left(\frac{1 - \bar{\xi}_k z}{z - \xi_k} \right)$$

where $x^T = (\beta, \mu)$ is any choice lying on the ellipse

$$x^T M x = |1 - \xi_n^2|^2 \quad (7)$$

with

$$M \triangleq \begin{pmatrix} 1 + |\xi_n|^2 & 2\text{Re}\{\xi_n\} \\ 2\text{Re}\{\xi_n\} & 1 + |\xi_n|^2 \end{pmatrix}.$$

The next basis function \mathcal{B}''_n is then found as

$$\mathcal{B}''_n(z) = \frac{\sqrt{1 - |\xi_n|^2}(\beta' z + \mu')}{z^2 + (\xi_n + \bar{\xi}_n)z + |\xi_n|^2} \prod_{k=0}^{n-1} \left(\frac{1 - \bar{\xi}_k z}{z - \xi_k} \right)$$

with (β', μ') related to the initial choice of (β, μ)

$$\begin{pmatrix} \beta' \\ \mu' \end{pmatrix} = \frac{1}{\sqrt{1 - \alpha^2}} \begin{pmatrix} \alpha & 1 \\ -1 & -\alpha \end{pmatrix} \begin{pmatrix} \beta \\ \mu \end{pmatrix}; \quad \alpha \triangleq \frac{\xi_n + \bar{\xi}_n}{1 + |\xi_n|^2}. \quad (8)$$

A special case of this construction is when only one fixed complex mode $\xi_k = \xi$ is considered and where the following special choice satisfying (7) is made

$$(\beta, \mu) = \left(0, \sqrt{(1 - \alpha^2)(1 + |\xi_n|^2)} \right)$$

in which case (8) gives

$$(\beta', \mu') = \sqrt{(1 + |\xi_n|^2)}(1, -\alpha).$$

to the Kautz basis (Wahlberg, 1994). Different initial choices for (β, μ) satisfying (7) give an infinite number of second order bases other than the Kautz one.

4. PROPERTIES OF THE ESTIMATE

Having defined the form of the estimated frequency response via the model structure (1) and the general orthonormal basis construction (6) this section examines the qualities of the least squares estimate (4). In doing so, the assumption that the frequency response measurements $\{f_k\}$ are corrupted is introduced:

$$f_k = G_T(e^{jk\omega_s}) + \nu_k.$$

The corruption $\{\nu_k\}$ can be used to model a number of different error sources. For example, the frequency response estimates may have been collected without sufficient time being allowed for any initial condition transients to die out. In this case, a deterministic bounded magnitude model for the error sequence $|\nu_k|$ would be appropriate (see (Helmicki *et al.*, October 1991) for a discussion on how to quantify such a bound). Alternatively, if the errors in the measurements are assumed to arise from random effects then a stochastic model for $\{\nu_k\}$ may be a better choice. In the sequel both choices will be examined.

A key feature of the least squares estimation scheme being examined is that whatever the model for the disturbance sequence $\{\nu_k\}$, it affects the resultant frequency response estimate in a linear fashion, since according to (4)

$$G(e^{j\omega}, \hat{\theta}_n) = T_n(\omega)g + T_n(\omega)\nu$$

where $T_n(\omega)$ is a linear functional on \mathbf{C}^n defined by

$$T_n(\omega) = [\mathcal{B}_0(e^{j\omega}), \dots, \mathcal{B}_{p-1}(e^{j\omega})] (\Omega_n^* \Omega_n)^{-1} \Omega_n^*$$

and g and ν are vectors

$$g^T = [G_T(1), G_T(e^{j\omega_s}), G_T(e^{j2\omega_s}), \dots, G_T(e^{j(n-1)\omega_s})]$$

$$\nu^T = [\nu_0, \nu_1, \nu_2, \dots, \nu_{n-1}]$$

and the vector of observed noise corrupted measurements f is given by $f = g + \nu$.

Because of this linearity, the frequency response estimation error $G_T(e^{j\omega}) - T_n(\omega)f$ can be decomposed into two components which can be treated separately. One error component $G_T(e^{j\omega}) - T_n(\omega)g$ is due to the finite order p of the model structure (1). This will be termed the ‘undermodelling error’. The other error component $T_n(\omega)\nu$ is due to errors in the measurements.

In order to quantify this latter error component it is convenient to perform a simplification of the estimation operator $T_n(\omega)$ by noticing that the (m, ℓ) th element of $\Omega_n^* \Omega_n$ is given by

$$\frac{1}{n} [\Omega_n^* \Omega_n]_{m,\ell} = \delta(\ell - m) + o(\eta^n)$$

where $\eta = \max_{0 \leq k < p} |\xi_k|$.

Therefore, since η^n decays so quickly with n a very good approximation to the $(\Omega_n^* \Omega_n)^{-1}$ in $T_n(\omega)$ is $n^{-1}I$.

Using this approximation, use a bounded magnitude disturbance model for the corrupting sequence $\{\nu_k\}$, the error in estimation $T_n(\omega)\nu$ due to this corruption can be bounded as follows.

Lemma 1. Suppose that the disturbance sequence $\{\nu_k\}$ is bounded in magnitude as $|\nu_k| \leq \varepsilon$. Then under the approximation $\Omega_n^* \Omega_n \approx nI$

$$\|T_n(\omega)\nu\|_\infty \leq \varepsilon \sqrt{\sum_{k=0}^{p-1} |\mathcal{B}_k(e^{j\omega})|^2}$$

On the other hand, if a uncorrelated stochastic model for the corruption $\{\nu_k\}$ is deemed more appropriate, then the error component due to this noise can be quantified as

Lemma 2. Suppose that $\{\nu_k\}$ is zero mean and uncorrelated with $\mathbf{E}\{|\nu_k|^2\} \leq \sigma_\nu^2 < \infty$. Then under the approximation $\Omega_n^* \Omega_n \approx nI$

$$\mathbf{P}\{|T_n(\omega)\nu| \geq \kappa\} \leq \frac{\sigma_\nu^2}{\kappa^2} \frac{1}{n} \sum_{k=0}^{p-1} |\mathcal{B}_k(e^{j\omega})|^2.$$

Therefore, regardless of whether we employ a deterministic or stochastic model for disturbances, the error induced by them is proportional to the function

$$\sum_{k=0}^{p-1} |\mathcal{B}_k(e^{j\omega})|^2. \quad (9)$$

Recognising this allows the poles in the basis functions to be made so as to minimise this function in regions where low sensitivity to noise is required.

5. SIMULATION EXAMPLE

In this section we present a simulation study that examines the utility of frequency domain system identification using the general orthonormal basis presented

in section 3. Suppose the system to be identified is the following continuous time one

$$G_T(s) = \frac{e^{-2s}}{(s+1)(10s+1)}$$

where the observations are collected by sampling every 1 second. In this case the zero order hold equivalent discrete time model for the system then is

$$G_T(q^{-1}) = \frac{q^{-2}(0.0355q + 0.0247)}{(q - 0.9048)(q - 0.3679)}$$

Suppose that we experiment on the plant $G_T(s)$ by inputting a sine wave of unit amplitude and frequency some integer multiple of $\omega_s = 2\pi/n$ for 500 samples. Suppose our measurements of the plant response $\{y_k\}$ are corrupted by measurement noise bounded in magnitude by 0.85. Finally, suppose we accept the suggestion of Helmicki, Jacobson and Nett in (Helmicki *et al.*, October 1991) and estimate the system frequency response by taking the DFT of the observed output signal $\{y_k\}$. According to Theorem 3.3 of (Helmicki *et al.*, October 1991) this leads to a bound $\varepsilon = 0.144$ for $|\nu_k|$.

Now suppose that we repeat this experiment $n = 20$ times, and use the conjugated data to form another 20 data points so as to ensure a real valued impulse response measurement. Finally, suppose that we form the estimate as in (4). Then the results for various choices of basis function are shown in figures 1 and 2. In these figures we show the true frequency response as a solid line, and ten examples of the estimated frequency response as dash-dot lines - the ten examples correspond to 10 different realisations of the measurement noise and so the results give some idea of the sensitivity of the estimate to this random effect.

On the left of figure 1 we employ the suggestion of (Parker and Bitmead, 1987; Gu and Khargonekar, 1992; Partington, 1991*b*; Partington, 1991*a*) and employ a trigonometric basis (FIR model structure) corresponding to the choice $\xi_k = 0$ for every basis element \mathcal{B}_k . In this example we were forced to choose $p = 30$ in order for the error component due to undermodelling to be acceptably small. On the right of figure 1 we incorporate prior knowledge in the estimation problem by employing a Laguerre basis which corresponds to the choice $\xi_k = 0.7$ for every basis function element. As can be seen, the sensitivity to noise is substantially reduced, especially at high frequencies where plant knowledge is particularly important if subsequent high performance control design is to be performed.

This decreased sensitivity to noise is due to the fact that with incorporation of prior knowledge by the choice of

pole position at $\xi_k = 0.7$ it is possible to decrease the model order to $p = 8$ with the truncation error (undermodelling error) component still being acceptably small. This decrease in sensitivity to noise is predicted by the noise error bounding term (9) which is shown plotted in the right hand diagram of figure 2. Firstly the bounding term (9) predicts that the noise sensitivity decreases with the number of basis functions employed, which we observe in figure 1. Secondly, the term (9) predicts the noise sensitivity to be proportional to the frequency response of the basis functions. With the FIR choice $\xi_k = 0$ the basis functions are all pass, whereas with the Laguerre choice $\xi_k = 0.7$ the basis functions are low pass (see right hand diagram of figure 2), and so we should expect the high frequency sensitivity to noise to be decreased for the Laguerre choice - we observe this as well in figure 1.

However, the most interesting part of this simulation example arises when we employ the general basis construction of section 3 so as to not be constrained to only incorporate prior knowledge of one pole at $\xi_k = 0.7$. Instead, we use the generalise basis function formulation (6) to allow the incorporation of prior knowledge over a range of frequencies by decreasing p to 5, and linearly spacing the prior knowledge poles $\{\xi_k\}$ from 0.5 to 0.9. In this case the estimation results are shown in the left hand diagram in figure 2. This shows the most accurate estimation of all the results so far, and furthermore, this accuracy can be predicted from the bounding term (9) which is shown in the right hand diagram of figure 2.

6. CONCLUSION

This paper has developed some preliminary results around the idea of using generalised orthonormal bases, which are constructed to allow the incorporation of prior knowledge, for solving the problem of estimation from frequency domain data. It was demonstrated that the accuracy of estimation in the presence of noise can be significantly improved by using such general bases. An extended version of this paper containing more detail and the proofs for the results presented here can be obtained by accessing the authors [www page at file://ee.newcastle.edu.au/brett/brett.html](http://www.ee.newcastle.edu.au/brett/brett.html).

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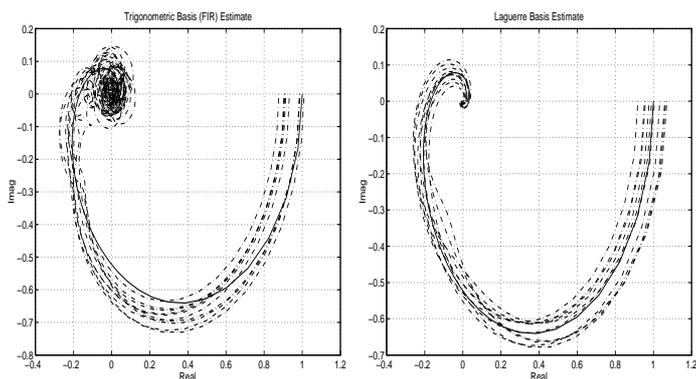


Fig. 1. Monte Carlo estimation results over 10 noise realisations. In all cases the true response is the solid line, and the estimates are dashed. On the left a 30th order FIR model is estimated, on the right an 8th order Laguerre model is estimated.

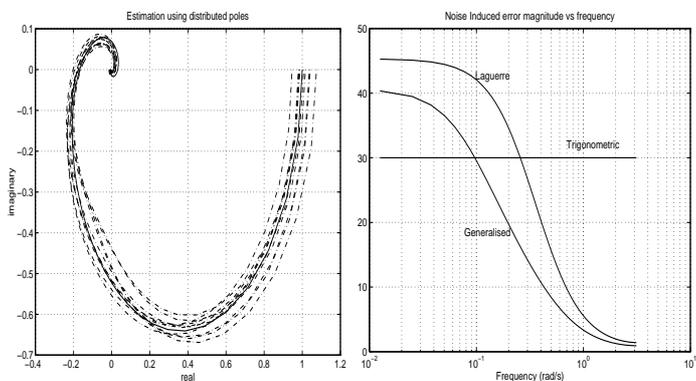


Fig. 2. On the left are the estimation results when using a generalised basis model with five poles at $[0.5, 0.6, 0.7, 0.8, 0.9]$. On the right the bounding term $\sum |\mathcal{B}_k|^2$ is plotted for the Laguerre and Generalised bases. As can be seen, it predicts the more accurate estimation results for the generalised basis.

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