Integral Constraints on the Accuracy of Least Squares Estimation

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ABSTRACT

It is common to need to estimate the frequency response of a system from observed input-output data. In this paper we characterise, via integral constraints, the undermodelling induced errors involved in solving this problem via parametric least squares methods. Our approach is to exploit the Hilbert Space structure inherent in the least squares solution in order to provide a geometric interpretation of the nature of frequency domain errors. This allows an intuitive process to be applied in which for a given data collection method and model structure, one identifies the sides of a right triangle, and then by noting the hypotenuse to be the longest side, integral constraints on magnitude estimation error are obtained. By also noting that the triangle sides both lie in a particular plane, integral constraints on phase estimation error are derived. This geometric approach is in contrast to earlier work in this area which has relied on algebraic manipulation.

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1 Introduction

A topic of recent interest in the system identification community has been that of examining the performance of estimation algorithms when the underlying true system cannot be exactly captured by the chosen model structure. This situation has been colloquially termed 'undermodelling', and when it exists it introduces a source of estimation error (called 'bias error'), the nature of which is not nearly so well understood as the so called 'variance error' that arises from the corrupting influence of measurement noise. Nevertheless, this source of error is of considerable importance, particularly in the context of frequency response estimation for subsequent use in robust control system design. The most widespread approach to this problem has been to tailor the design of new estimation schemes towards the goal of making both the bias and variance error quantifiable; see [21, 1, 16] for an overview of these efforts. This paper takes a different approach by trying to gain some insight, via the development of integral constraints, into the nature of undermodelling induced bias error in existing classical least squares estimation schemes.

This problem has been tackled by other authors [19, 5, 6, 3, 4] for the particular case of using ARX type model structures in combination with non-noisy measurements collected in open loop. The methods used by those authors are algebraic. In [3, 4] Lagrange multiplier techniques are used, whereas in [19, 5, 6] the authors complete squares with a term which is zero since it is the gradient of the least squares cost function evaluated at the estimate.

In contrast, this paper shows how a geometric approach may be taken. To be more specific, we show that a general principle is that the estimation error is orthogonal to a particular subspace or linear variety living in a frequency domain space. This provides two sides of a right triangle, and by noting that the longest side of such a triangle is the hypotenuse, the integral constraints on magnitude estimation error are obtained. Although this geometric principle has been heuristically stated, as we shall show it appears in more rigorous but equally simple form as the Cauchy-Schwarz inequality.

The benefit of recognising the less formal interpretation is that it gives an intuitive principle that can be easily transferred across different model structures and data collection setups. All that needs to be done is to recognise what terms make up the sides of the triangle, no algebraic calculation is required. In this paper we take advantage of this by extending the results of [19, 5, 6, 3, 4] to closed loop data collection settings and to 'basis function' type (Laguerre, Kautz etc.) model structures.

For the particular case of ARX type model structures, in [19, 5] the case of terms in numerator or denominator being fixed was considered, and in [3, 4] the integral constraint analysis was extended to the case of quadratic constraints on the numerator or denominator terms. Another insight arising from the geometric viewpoint is that it immediately becomes obvious that the integral constraints hold regardless of the nature of constraints on the numerator or denominator terms. Whether they are quadratic or something much more complicated is irrelevant, all that matters is that if the denominator terms are constrained, then all the numerator terms need to be free so as to form a subspace (and vice-versa if numerator terms are constrained).

It is also possible to develop integral constraints on phase estimation error from a geometric perspective. The same orthogonality condition that leads to recognising the sides of a right triangle also provides the angle between two other sides of the triangle and a characterisation of what plane both these sides live in. For the case of ARX type models this provides a very direct way of providing the phase error constraint derived algebraically in [19]. However, because the geometric method is so intuitive, one immediately sees another phase error characterisation that was not presented in [19], and it is also trivial to extend these phase error characterisations to other situations such as closed loop estimation, and estimation with other model structures.

We will discuss all these issues in the following format. Firstly we will define the problem assumptions, the notation, and the model structures we will be considering. This will occupy section 2. In section 3 we will begin our discussion of how a geometric interpretation may be arrived at by describing how the estimated frequeny response may be considered as the solution to a norm minimisation problem in a certain Hilbert Space, for which a an orthogonality condition applies to characterise the solution. The nature of this solution depends on the model structure and the data collection mechanism. We treat the cases of open and closed loop data collection, and 'basis function' and ARX type model structures in turn. In each case we characterise both the gain and phase estimation error, and emphasise how the results may be intuitively derived via a geometric picture; where appropriate we document where the results have been previously derived by other authors using different methods. The paper closes with a discussion of the shortcomings of the integral constraints we derive here.

2 Problem Setting

In order to discuss the ideas mentioned in the introduction let us first fix the problem and the notation. Suppose that frequency response estimates of a system are to be derived on the basis of observing an N point input-output data set $Z_N = [\{u_k\}, \{y_k\}]$. Suppose further that the measurement conditions are such that this observed data set is generated as follows:

$$y_k = G_T(q)u_k + H(q)\nu_k. \tag{1}$$

Here $G_T(q)$ and $H(q)$ are strictly stable SISO transfer functions in the forward shift operator q, $\{u_k\}$ is a known input sequence and $\{\nu_k\}$ is a disturbance sequence which we elect to describe as a zero mean i.i.d. stochastic process with $\mathsf{E}\{\nu_k^2\} = \sigma_\nu^2 < \infty$. The frequency response of the true system $G_T(e^{j\omega})$ is the quantity to be estimated from Z_N.

There are non-parametric methods available for estimating $G_T(e^{j\omega})$ [9, 10], but in this paper we only consider parametric methods. These involve fitting a model $G(q,\theta)$, parameterised by a vector $\theta \in \mathbf{R}^p$, to the available data Z_N and then taking $G(e^{j\omega},\widehat{\theta}_N)$ as the estimate of $G_T(e^{j\omega})$. The parameter vector estimate $\widehat{\theta}_N$ could be obtained in a variety of ways [10, 22], but again we focus on a specific case by looking at 'Least Squares' estimation:

$$\widehat{\theta}_N \triangleq \arg\min_{\theta\in\mathbf{R}^p} \left\{ \frac{1}{N}\sum_{k=0}^{N-1} \varepsilon_k^2(\theta) \right\}, \tag{2}$$

$$\varepsilon_k(\theta) \triangleq y_k^f - \widehat{y}_{k|k-1}^f(\theta), \tag{3}$$

$$y_k^f \triangleq F(q)y_k, \tag{4}$$

$$\widehat{y}_{k|k-1}^f(\theta) \triangleq F(q)\widehat{y}_{k|k-1}(\theta). \tag{5}$$

Here $F(q)$ is some user chosen strictly stable data pre-filter and $\widehat{y}_{k|k-1}(\theta)$ is some predictor of y_k given the past data Z_{k-1} and a parameter vector θ. The choice of this prefilter has been the subject of a large literature, we refer the reader to [10, 22] for excellent overviews. If F and H are both stably invertible, then from (1),(4) we have

$$y_k^f = \left(F - H^{-1}\right) y_k + H^{-1}G_T u_k + \nu_k.$$

Provided the noise variance σ_ν^2 has been scaled so that $H(q)F(q)$ has a monic impulse response this expresses y_k^f in terms of past data and a white noise innovations so that the mean square optimal predictor $\widehat{y}_{k|k-1}^f$ is

$$\widehat{y}_{k|k-1}^f = \left(F - H^{-1}\right) y_k + H^{-1}G_T u_k.$$

It therefore makes sense to define the predictor in (5) as

$$\widehat{y}_{k|k-1}^f(\theta) = \left(F - H^{-1}\right) y_k + H^{-1}G(q,\theta)u_k \tag{6}$$

so that the form of the predictor depends on the model structure $G(q,\theta)$ and the filter $F(q)$ that is chosen. In this paper, two model structures are of interest to us, ARX models and fixed denominator models, in which cases the forms of the predictors are as follows.

2.1 ARX Modelling

This is a very common model structure [10, 22] described as

$$G(q,\theta) = \frac{B(q,\theta)}{A(q,\theta)} \tag{7}$$

where

$$\begin{aligned} A(q,\theta) &= a_n q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + 1, \\ B(q,\theta) &= b_m q^m + a_{m-1}q^{m-1} + \cdots + b_1 q + b_0, \\ \theta^T &= \left[a_n, a_{n-1}, \cdots, a_1, b_m, b_{m-1}, \cdots, b_1, b_0\right]. \end{aligned} \tag{8}$$

Now, because of the rational structure of $G(q,\theta)$ in (7), calculation of $\widehat{\theta}_N$ given by (2)-(5) and using the predictor (6) is a non-linear, non-convex optimization problem requiring some sort of numerical search technique which may arrive at local minima rather than the global solution.

We can obviate this by choosing $F(q)$ in (6) as

$$F(q) = \frac{E(q)}{H(q)A(q,\theta)} \tag{9}$$

where $E(q)$ is Schur and is of the form

$$E(q) = e_r q^r + e_{r-1}q^{r-1} + \cdots + e_1 q + 1.$$

This so called 'observer polynomial' is often set[2] to $E(q) = 1$. For a discussion of more thoughtful choices, see [13]. In any event (6) becomes

$$\begin{aligned} \widehat{y}^f_{k|k-1}(\theta) &= \left(\frac{E(q) - A(q,\theta)}{E(q)}\right) y^f_k + \frac{B(q,\theta)}{E(q)} u^f_k & (10) \\ &= \phi_k^T \theta & (11) \end{aligned}$$

where

$$\phi_k^T = \left[\frac{y^f_{k-r}}{E(q)}, \ldots, \frac{y^f_{k-1}}{E(q)}, \frac{u^f_{k-m}}{E(q)}, \ldots, \frac{u^f_k}{E(q)}\right]$$

and θ is re-defined to be

$$\theta^T = [e_r, \ldots, e_{n+1}, e_n - a_n, \ldots, e_1 - a_1, b_m, \ldots, b_0]$$

so that $\widehat{\theta}_N$ satisfying (2) can be written in closed form

$$\hat{\theta}_N = \left(\sum_{k=1}^{N} \phi_k \phi_k^T\right)^{-1} \sum_{k=1}^{N} \phi_k y^f_k \tag{12}$$

provided sufficient excitation exists for the indicated inverse to also exist. Now, of course, we cannot choose $F(q)$ as given in (9) since $A(q,\theta)$ is unknown. Nevertheless, it motivates the linear regressor form of the predictor (11) and since this leads to the simple expression (12) for $\widehat{\theta}_N$,the formulation (10)-(12) is very common in estimation of ARX type model structures [13, 10]; it is commonly known as an 'equation error' type model structure. In section 3.2 we will restrict ourselves, as other authors on the subject have done [19, 5, 6, 3, 4], to examining the effect of the choice of (10) as the predictor for ARX model estimation.

[2]Note that the normalised term in $E(q)$ is set conformably to the one in $A(q)$ as $e_0 = a_0 = 1$. Any term (or set of terms) in $A(q)$ can be normalised, but the conformal nature of the normalisation with $E(q)$ should be maintained in order to keep the size of the parameter vector θ as small as possible.

2.2 Fixed Denominator Modelling

In this structure we have

$$G(q,\theta) = \frac{B(q,\theta)}{A(q)} \tag{13}$$

where $A(q)$ is fixed according to some a-priori knowledge of the dynamics of $G_T(q)$. The advantage of this model structure is that finite data statistical properties of $\widehat{\theta}_N$ can be specified since the predictor $\widehat{y}^f_{k|k-1}(\theta)$ can be cast in the linear form (11) with the regression vector ϕ_k containing no stochastic components. This linear form is achieved by motivating the predictor $\widehat{y}^f_{k|k-1}(\theta)$ from (6) with the filter

$$F(q) = H(q)^{-1} \tag{14}$$

to give

$$\widehat{y}^f_{k|k-1}(\theta) = \frac{B(q,\theta)}{A(q)} u^f_k = \phi_k^T \theta \tag{15}$$

where

$$\begin{aligned} \phi_k^T &= \left[\frac{1}{A(q)}u_k, \frac{q}{A(q)}u_k, \cdots, \frac{q^p}{A(q)}u_k\right], \\ \theta^T &= [b_0, \cdots, b_p], \qquad p \le m. \end{aligned} \tag{16}$$

Now, because of the linearity of this model structure, we will not change the frequency response estimate by linearly re-parameterising as

$$y = \psi_k^T \beta, \qquad \beta = M\theta,$$

$$\psi_k^T = \phi_k^T M^{-1} = [\mathcal{B}_0(q)u_k, \mathcal{B}_1(q)u_k, \cdots, \mathcal{B}_p(q)u_k]$$

where the transfer functions $\{\mathcal{B}_0(q), \cdots, \mathcal{B}_p(q)\}$ are given by

$$[\mathcal{B}_0(q), \mathcal{B}_1(q), \cdots, \mathcal{B}_p(q)] = \left[\frac{1}{A(q)}, \frac{q}{A(q)}, \cdots, \frac{q^p}{A(q)}\right] M^{-1}$$

and M is any non-singular matrix. The filters $\{\mathcal{B}_k(q)\}$ can then be thought of as user chosen 'basis functions' for the expansion of $G_T(q)$. In the sequel, when talking of fixed denominator models we will use this $\{\mathcal{B}_0(q), \cdots, \mathcal{B}_p(q)\}$ basis function notation since it is the most general.

Fixed denominator models thought of in these terms have been studied for some time now under the names of Laguerre models [25] and Kautz models [26]. More recently Heuberger, Van den Hof and co-workers [23, 24] have considered general orthonormal constructions of these basis functions from balanced realisations of user chosen dynamics. Finally, in [15, 17] a unifying construction encompassing all these cases has been shown to be

$$\mathcal{B}_n(z) = \left(\frac{\sqrt{1-|\xi_n|^2}}{z-\xi_n}\right) \prod_{k=0}^{n-1} \left(\frac{1-\overline{\xi_k}z}{z-\xi_k}\right)$$

where the set of user chosen poles $\{\xi_0, \xi_1, \cdots, \xi_p\}$ is selected according to prior information about the dynamics of the system to be identified. The point is that these fixed denominator models form a rich structure that has attracted enough interest that properties of the estimation error in using them is pertinent.

3 Geometric Characterisations

Our method of examining the relationship between $G_T(e^{j\omega})$ and $G(e^{j\omega}, \widehat{\theta}_N)$ via integral constraints is motivated by the work in [19, 5, 6, 3, 4]. The difference to that work is that we develop our results using geometric principles rather than by the clever algebraic constructions of the previous workers. We believe this adds extra insight and also allows easy extension of the results to settings (such as closed loop and fixed denominator model structures) that are not considered in [19, 5, 6, 3, 4].

In order to develop our geometric interpretation we embed the estimation problem in a Hilbert space. Now this is a very old and by now well known idea when the embedding space is a time domain signal space ℓ_2 and the inner product is defined via expected values of these signals - see, for example [28, 22, 2]. In this case, one is led to conclude that the least-squares solution involves an innovations sequence that is orthogonal to a subspace of ℓ_2 that is spanned by the observed data.

The key idea in this paper is not to embed in ℓ_2 but rather to embed in the Hardy space H_2 that various frequency responses live in. The consequence of this choice is that we can then directly make statements about the nature of frequency response estimates instead of indirectly inferring them from properties of the parameter estimates, as other authors have done [19, 5, 6, 3, 4]. In essence, one simply notes that the frequency response estimation error is orthogonal to the estimate itself, this gives two sides of a right triangle, and identifying the third side (which is model structure and measurement set-up dependent) gives the integral constraints.

To see how this embedding in H_2 is set up, we first assume that the excitation signal $\{u_k\}$ is stationary in the sense introduced by Wiener [28] of possessing a limiting sample autocorrelation function $R_u(\tau)$

$$R_u(\tau) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} \mathbf{E}\{u_k u_{k+\tau}\} \tag{17}$$

which decays sufficiently quickly with increasing τ that a spectral density $\Phi_u(\omega)$ given by

$$\Phi_u(\omega) = \sum_{\tau=-\infty}^{\infty} R_u(\tau) e^{-j\omega\tau}, \qquad R_u(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi_u(\omega) e^{j\omega\tau}\, d\omega \tag{18}$$

also exists. In this case, under some further mild regularity conditions on $\{u_k\}$ and $\{\nu_k\}$, the least squares estimate $\hat{\theta}_N$ given by (2) converges as follows [11, 2]

$$\hat{\theta}_N \xrightarrow{a.s.} \theta_\star \quad \text{as } N \to \infty$$

where

$$\theta_\star = \arg\min_{\theta\in\mathbf{R}^p}\left\{\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\mathsf{E}\left\{\varepsilon_k^2(\theta)\right\}\right\} = \arg\min_{\theta\in\mathbf{R}^p}\left\{R_\varepsilon(0,\theta)\right\}. \tag{19}$$

Now, the right hand side of this expression is the autocorrelation function for $\varepsilon_k(\theta)$ evaluated at lag $\tau = 0$. Since by (18) this is also the zeroth coefficient of the Fourier series representation of the spectral density $\Phi_\varepsilon(\omega,\theta)$ we can use the idea in [27] and rewrite the criterion (19) as

$$\theta_\star = \arg\min_{\theta\in\mathbf{R}^p}\left\{\int_{-\pi}^{\pi}\Phi_\varepsilon(\omega,\theta)\,\mathrm{d}\omega\right\}. \tag{20}$$

In turn, the spectral density $\Phi_\varepsilon(\omega,\theta)$ depends on the model frequency response $G(e^{j\omega},\theta)$, the true frequency response $G_T(e^{j\omega})$, the input spectral density $\Phi_u(\omega)$ and the noise spectral density $|H(e^{j\omega})|^2\sigma_\nu^2$. Finally, this dependence is a function of the model structure.

However, regardless of the model structure, $\Phi_\varepsilon(\omega,\theta)$ can be expressed in terms of the frequency responses of linear discrete time systems, and so we can embed the expression (20) in a particular Hilbert Space $(H_2(\mathbf{T}),\|\cdot\|_\chi)$ of complex valued functions on the unit circle $\mathbf{T}$. This space is defined such that a function f is in $(H_2(\mathbf{T}),\|\cdot\|_\chi)$ if the Fourier coefficients $\{c_k\}$ of f are zero for $k < 0$ (this corresponds to only considering causal systems) and if $\|f\|_\chi < \infty$ where the seminorm $\|\cdot\|_\chi$ is given by

$$\|f\|_\chi^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|f(e^{j\omega})\right|^2\chi(\omega)\,\mathrm{d}\omega \tag{21}$$

with $\chi(\omega)$ a non-negative definite function [3]. The geometric structure of $(H_2(\mathbf{T}),\|\cdot\|_\chi)$ then comes from the inner product that can be defined on the space:

$$\langle f,g\rangle_\chi = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{j\omega})\overline{g(e^{j\omega})}\chi(\omega)\,\mathrm{d}\omega. \tag{22}$$

Now this is all very abstract, but the point is that by recognising this embedding, (20) becomes an expression for $\theta_\star$ as a norm minimisation problem in H_2, the solution of which is characterised by an orthogonality condition in terms of the above defined inner product; the constraints on the estimation error then follow immediately. Just what $\chi(\omega)$ is and what is being minimised depends on the model structure, so to explain the geometric interpretation further we have to be more specific about this structure.

3.1 Fixed Denominator Modelling

Suppose to begin with that we elect to use the fixed denominator model structure that was detailed in section 3.1. Then assuming $\{u_k\}$ is uncorrelated with $\{\nu_k\}$ (so that closed

[3] If $\chi(\omega)$ were positive definite, then $\|\cdot\|_\chi$ would be a true norm, but in the sequel $\chi(\omega)$ will come from a spectral density and since we want to allow for spectral densities that may be zero at some points, we technically only have a seminorm. This pedantry is irrelevant anyway since the projection theorem, which will be our major tool, holds in either case.

loop collection of data is ruled out for the moment) we can use Lemma B.1 to calculate $\Phi_\varepsilon(\omega,\theta)$ as

$$\Phi_\varepsilon(\omega,\theta) = \left|G_T(e^{j\omega}) - G(e^{j\omega},\theta)\right|^2 \left|F(e^{j\omega})\right|^2 \Phi_u(\omega) + \left|F(e^{j\omega})H(e^{j\omega})\right|^2 \sigma_\nu^2$$

and hence rewrite the minimisation criterion (20) as

$$\begin{aligned}\theta_\star &= \arg\min_{\theta\in\mathbf{R}^p} \left\{\int_{-\pi}^{\pi} \left|G_T(e^{j\omega}) - G(e^{j\omega},\theta)\right|^2 \left|F(e^{j\omega})\right|^2 \Phi_u(\omega) + \left|F(e^{j\omega})H(e^{j\omega})\right|^2 \sigma_\nu^2 \, \mathrm{d}\omega\right\} \\ &= \arg\min_{\theta\in\mathbf{R}^p} \left\{\int_{-\pi}^{\pi} \left|G_T(e^{j\omega}) - G(e^{j\omega},\theta)\right|^2 \left|F(e^{j\omega})\right|^2 \Phi_u(\omega)\, \mathrm{d}\omega\right\}. \qquad (23)\end{aligned}$$

However, if we use the embedding (21),(22) then we can also consider (23) as the norm minimisation problem

$$\theta_\star = \arg\min_{\theta\in\mathbf{R}^p} \left\{\|G_T - G(\theta)\|_\chi\right\} \qquad (24)$$

where $\chi(\omega) = |F(e^{j\omega})|^2\Phi_u(\omega)$. Furthermore, in the case of fixed denominator modelling, this minimisation is over the subspace M of $(H_2(\mathbf{T}), \|\cdot\|_\chi)$ spanned by the basis functions $\{\mathcal{B}_0(e^{j\omega}), \cdots, \mathcal{B}_p(e^{j\omega})\}$, and so the classical Projection Theorem can be applied to characterise the asymptotic estimate $G(e^{j\omega},\theta_\star)$ as the unique element of M satisfying the orthogonality condition[4]

$$\langle G_T - G(\theta_\star), G(\theta_\star)\rangle_u = 0. \qquad (25)$$

This immediately gives the following result characterising the fact that undermodelling induced bias error is such that the gain of the system averaged over all frequencies will be underestimated.

Theorem 1.

$$\int_{-\pi}^{\pi} \left|G(e^{j\omega},\theta_\star)\right|^2 \left|F(e^{j\omega})\right|^2 \Phi_u(\omega)\,\mathrm{d}\omega \le \int_{-\pi}^{\pi} \left|G_T(e^{j\omega})\right|^2 \left|F(e^{j\omega})\right|^2 \Phi_u(\omega)\,\mathrm{d}\omega$$

with equality if and only if $G_T(e^{j\omega}) = G(e^{j\omega},\theta_\star)$ *for almost all* ω.

Proof. The proof is a trivial consequence of the Cauchy-Schwarz inequality. From (25)

$$\langle G_T, G(\theta_\star)\rangle_u = \|G(\theta_\star)\|_u^2 \qquad (26)$$

But by the Cauchy-Schwarz inequality $\langle G_T, G(\theta_\star)\rangle_u \le \|G_T\|_u \|G(\theta_\star)\|_u$ with equality if and only if $G_T = G(\theta_\star)$. ◻◻◻

The geometric interpretation is that (25) tells us that $G_T - G(\theta_\star)$ and $G(\theta_\star)$ are adjacent orthogonal sides of a right triangle with hypotenuse G_T. The fact that the hypotenuse is the longest side in a right triangle then gives $\|G_T\|_u \ge \|G(\theta_\star)\|_u$ which is Theorem 1.

We can extract more information from the orthogonality condition (25). So far we have only used it as far as it characterises the length of the sides of a triangle, but we note from (26) that it also implies that G_T and $G(\theta_\star)$ lie in the same plane since even though both these elements are complex valued quantities, the angle between them is purely real. This leads to integral characterisations of the phase response estimation error.

[4]Here and in the sequel the subscript u is used as shorthand notation for $\chi(\omega) = |F(e^{j\omega})|^2\Phi_u(\omega)$

Corollary 1.

$$\int_{-\pi}^{\pi} |G(\theta_\star)G_T|\,|F|^2\Phi_u \cos(\measuredangle G(\theta_\star) - \measuredangle G_T)\,\mathrm{d}\omega = \int_{-\pi}^{\pi} |G(\theta_\star)|^2\,|F|^2\,\Phi_u\,\mathrm{d}\omega, \tag{27}$$

$$\int_{-\pi}^{\pi} |F|^2\,|G(\theta_\star)G_T|\,\Phi_u \sin(\measuredangle G(\theta_\star) - \measuredangle G_T)\,\mathrm{d}\omega = 0. \tag{28}$$

Proof. Take the real and imaginary parts of both sides of (26). □□□

These two expressions tell us two things about phase errors. Firstly, (27) tells us that gain errors and phase errors must go hand in hand. That is, if there is a magnitude estimation error such that $|G_T|$ exceeds $|G(\theta_\star)|$ in some frequency region (from Theorem 1 we know this must happen unless G_T happens to be in the model set), then this forces a compensatory non-zero phase error in this or some other region. Vice-versa, if there is a non-zero phase error in some region, then the product $|G(\theta_\star)||G_T|$ must exceed $|G(\theta_\star)|^2$ in some region, and hence the system gain will be underestimated there.

Secondly, (28) tells us that over-estimations of phase in one region, must be balanced by underestimations of phase in another, and that this phenomenon is proportional to the gains and spectral densities of the quantities involved; a phase error in a region where $|F|^2|G(\theta_\star)G_T|\Phi_u$ is large requires more compensation by further phase errors than does an error where $|F|^2|G(\theta_\star)G_T|\Phi_u$ is small.

3.2 ARX Modelling

For the case of ARX modelling the story is similar to the foregoing. In this section we re-derive most of the results in [19, 5, 3, 4]. We believe the re-derivation is of interest because it shows an alternative and perhaps simpler method of obtaining the results, while at the same time introducing an intuitive geometrical interpretation of the problem.

To begin, we note that for equation error type ARX model structures the asymptotic estimate still satisfies (20) but since the predictor is now of the form given in (10) we have

$$\begin{aligned}\varepsilon_k(\theta) &= y_k^f - \widehat{y}_{k|k-1}^f(\theta)\\ &= y_k^f - \left(\frac{E - A(\theta)}{E}\right) y_k^f - \frac{B(\theta)}{E}u_k^f\\ &= \frac{A(\theta)}{E}y_k^f - \frac{B(\theta)}{E}u_k^f.\end{aligned}$$

Furthermore, $\{y_k\}$ is generated according to (1) as

$$\varepsilon_k(\theta) = \frac{A(\theta)}{E}\left[G_T - G(\theta)\right]u_k^f + \frac{FHA(\theta)}{E}\nu_k.$$

So again, assuming for the moment that the data is collected in open loop so that $\{u_k^f\}$ and $\{\nu_k^f\}$ are uncorrelated, Lemma B.1 and (20) lead us to

$$\theta_\star = \arg\min_{\theta\in\mathbf{R}^p}\left\{\int_{-\pi}^{\pi} |G_T - G(\theta)|^2 \left|\frac{A(\theta)F}{E}\right|^2 \Phi_u + \left|\frac{FHA(\theta)}{E}\right|^2 \sigma_\nu^2\,\mathrm{d}\omega\right\}. \tag{29}$$

In the noise free case this becomes

$$\theta_\star = \arg\min_{\theta\in\mathbf{R}^p}\left\{\int_{-\pi}^{\pi}|G_T - G(\theta)|^2\left|\frac{A(\theta)F}{E}\right|^2\Phi_u\,\mathrm{d}\omega\right\},$$

$$= \arg\min_{\theta\in\mathbf{R}^p}\left\{\int_{-\pi}^{\pi}\left|\frac{G_TA(\theta)}{E} - \frac{B(\theta)}{E}\right|^2|F|^2\Phi_u\,\mathrm{d}\omega\right\}, \tag{30}$$

$$= \arg\min_{\theta\in\mathbf{R}^p}\left\{\left\|\frac{G_TA(\theta)}{E} - \frac{B(\theta)}{E}\right\|_\chi^2\right\}, \quad \chi = |F|^2\Phi_u. \tag{31}$$

As Mullis and Roberts point out [14], in the context of system realisation theory, (30) with $|F|^2\Phi_u = 1$ has been suggested since the early work of Kalman as a means of tractably approximating the solution to the non-convex problem

$$\arg\min_{\theta\in\mathbf{R}^p}\left\{\int_{-\pi}^{\pi}|G_T(e^{j\omega}) - G(e^{j\omega},\theta)|^2\,\mathrm{d}\omega\right\}.$$

The following analysis of the nature of solutions to (30) is therefore also of relevance in this realisation theory context in addition to our motivating system identification one.

Now, at first glance (31),(30) would seem to have the trivial solution $A(\theta) = B(\theta) = 0$. However, this is not allowed due to the normalisation specified in (8) that $a_0 = 1$. In this case, $G_TA(\theta)/E$ is a linear variety in $H_2(\mathbf{T})$ and $B(\theta)/E$ is a closed subspace, with $\theta_\star$ chosen to minimise the distance between them.

A projection theorem (Lemma A.1 which is a slight generalisation from the classical case of points and subspaces to linear varieties and subspaces) again holds stating that the error must be orthogonal to both the variety and the subspace. This leads again to a geometrical interpretation shown in figure 1 and explained in the proof of the next Theorem[5].

What is of interest, and was first proved in [5] using a clever algebraic argument invented by Salgado [19] and then extended in [3, 4] using Lagrange multiplier techniques is that whether this normalisation occurs in the numerator or denominator affects whether the average (over frequency) gain is over or under-estimated.

Previously, this result may have seemed surprising, but with the geometric approach presented here the result becomes intuitively obvious since the choice of normalisation affects whether $A(\theta)/E$ (resp. $B(\theta)/E$) parameterises a linear variety or a subspace, which in turn affects whether $\|G_T\|_u$ (resp. $\|G(\theta_\star)\|_u$) represents the length of the hypotenuse or the length of one of the orthogonal sides in a right triangle.

Theorem 2. *If there are terms in $A(\theta)$ fixed, but not in $B(\theta)$ then*

$$\int_{-\pi}^{\pi}\left|G(e^{j\omega},\theta_\star)\right|^2\left|\frac{A(e^{j\omega},\theta_\star)}{E(e^{j\omega})}\right|^2|F|^2\Phi_u(\omega)\,\mathrm{d}\omega \le \int_{-\pi}^{\pi}\left|G_T(e^{j\omega})\right|^2\left|\frac{A(e^{j\omega},\theta_\star)}{E(e^{j\omega})}\right|^2|F|^2\Phi_u(\omega)\,\mathrm{d}\omega. \tag{32}$$

[5]In figure 1 and in Theorem 2 we have assumed the normalisation to be in the a_0 term of $A(q)$. That is, $a_0 = 1$. Of course, normalisation in any term, or set of terms is possible and will lead to the same conclusion in Theorem 2 since all that matters is orthogonality between a subspace and **any** line.

If terms in $B(\theta)$ are fixed, but not in $A(\theta)$ then the inequality goes the other way. Equality occurs if and only if $G_T(e^{j\omega}) = G(e^{j\omega}, \theta_\star)$ for almost all ω.

Proof. The proof is an elementary application of an extended version of the Projection Theorem (Lemma A.1) and the Cauchy-Schwarz inequality. Specifically, if we fix terms in $A(\theta)$ then $A(\theta)G_T/E$ is a linear variety in $H_2(\mathbf{T})$ and $B(\theta)/E$ is a closed subspace in $H_2(\mathbf{T})$. A trivial extension of the classical Projection Theorem (see Lemma A.1) gives that the line of minimum length connecting a line and a plane is perpendicular to both, and in particular, perpendicular to the plane:

$$\langle A(\theta_\star)G(\theta_\star), A(\theta_\star)G(\theta_\star) - A(\theta_\star)G_T\rangle_u = 0. \tag{33}$$

So geometrically[6] $A(\theta_\star)G(\theta_\star)$ and $A(\theta_\star)G(\theta_\star) - A(\theta_\star)G_T$ form orthogonal sides of a right triangle with $A(\theta_\star)G_T$ as the hypotenuse - see figure 1. Consequently we must have $\|A(\theta_\star)G(\theta_\star)\|_u \leq \|A(\theta_\star)G_T\|_u$. More formally, (33) and the Cauchy-Schwarz inequality give:

$$\|A(\theta_\star)G(\theta_\star)\|_u^2 = \langle A(\theta_\star)G(\theta_\star), A(\theta_\star)G_T\rangle_u \leq \|A(\theta_\star)G(\theta_\star)\|_u \|A(\theta_\star)G_T\|_u \tag{34}$$

with equality on the right if and only if $G_T(e^{j\omega}) = G(e^{j\omega}, \theta_\star)$ for almost all ω. If terms in $B(\theta)$ are fixed, then $A(\theta)G_T$ becomes a subspace and $B(\theta)$ a variety so that instead of (33) we get

$$\langle A(\theta_\star)G_T, A(\theta_\star)G(\theta_\star) - A(\theta_\star)G_T\rangle_u = 0 \tag{35}$$

to give analogously to (34)

$$\|A(\theta_\star)G_T\|_u^2 = \langle A(\theta_\star)G(\theta_\star), A(\theta_\star)G_T\rangle_u \leq \|A(\theta_\star)G(\theta_\star)\|_u \|A(\theta_\star)G_T\|_u.$$

□□□

Notice that in Theorem 2, we've only used the orthogonal to subspace property, but as mentioned in the proof of Theorem 2 and as shown in figure 1 there is also an orthogonal to linear variety property we haven't used. Unfortunately, it does not seem to be useful. For example, in the case of normalisation in the denominator of the form $a_0 = 1$, recognising the hypotenuse to be the longest side of the triangle in figure 1 gives the integral constraints

$$\int_{-\pi}^{\pi} |G_T - G(\theta_\star)|^2 \left|\frac{A(\theta_\star)}{E}\right|^2 |F|^2 \Phi_u \, d\omega \leq \int_{-\pi}^{\pi} |G_T - A(\theta_\star)G(\theta_\star)|^2 \left|\frac{F}{E}\right|^2 |F|^2 \Phi_u \, d\omega$$

and

$$\int_{-\pi}^{\pi} |A(\theta_\star) - 1|^2 |G_T|^2 \left|\frac{A(\theta_\star)}{E}\right|^2 |F|^2 \Phi_u \, d\omega \leq \int_{-\pi}^{\pi} |G_T - A(\theta_\star)G(\theta_\star)|^2 \left|\frac{F}{E}\right|^2 |F|^2 \Phi_u \, d\omega.$$

[6]Note that, strictly speaking, in the proof of Theorem 2 and in the sequel when we consider ARX models in closed loop settings, $A(\theta_\star)$ should be replaced by $A(\theta_\star)/E$, but we have not done so since if we did, it would compromise readability

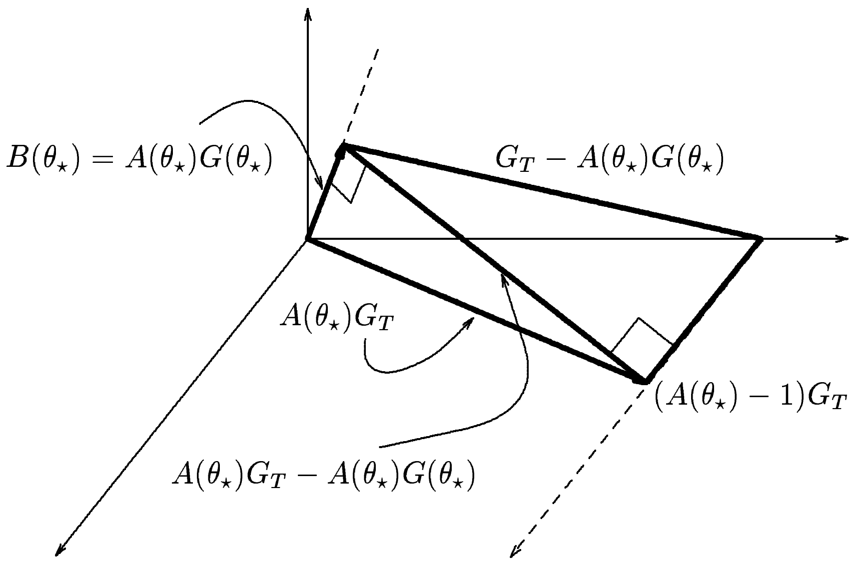

Figure 1: *Geometric Interpretation for ARX Estimation: The simple case of the restriction on the denominator terms being a normalisation of one (or more) of the denominator coefficients ($a_0 = 1$ is shown).*

It seems difficult to draw any useful conclusions from these constraints on how the frequency response of the estimated model will relate to the frequency response of the true system. The only point we can really make is that the expressions extend Theorem 2 slightly by showing that not only does it matter whether we normalise in numerator or denominator, it also matters which particular term (or group of terms) in the numerator or denominator we choose to normalise, although the nature of this latter dependence is unclear.

In [3, 4] the authors extend these integral constraint type error formulae for ARX model structures to cases with various non-linear constraints on the normalisation of the numerator and denominator terms. They use Lagrange multiplier techniques, but a generalisation of the simplest non-linear constraint case they consider can also be obtained using our geometric principles.

Corollary 2. *Theorem 2 holds under any constraints on the numerator or denominator. Specifically, if all the terms are free in $B(\theta)$ and there is any constraint on those in $A(\theta)$ then (34) applies. If all the terms in $A(\theta)$ are free and there is any constraint on those in $B(\theta)$ then the inequality in (34) flips. Equality in (34) occurs if and only if $G_T(e^{j\omega}) = G(e^{j\omega}, \theta_\star)$ for almost all ω.*

Proof. This is not so much a corollary to Theorem 2, as a result with identical proof. The reader will notice that in the proof of Theorem 2, all that mattered was that given **any** point, no matter what manifold it might be constrained to lie on, the minimum distance line from this point to a subspace is characterised by it's orthogonality to the subspace. Once this is recognised, the proof is immediate. See figure 2 for a visual proof. ◻◻◻

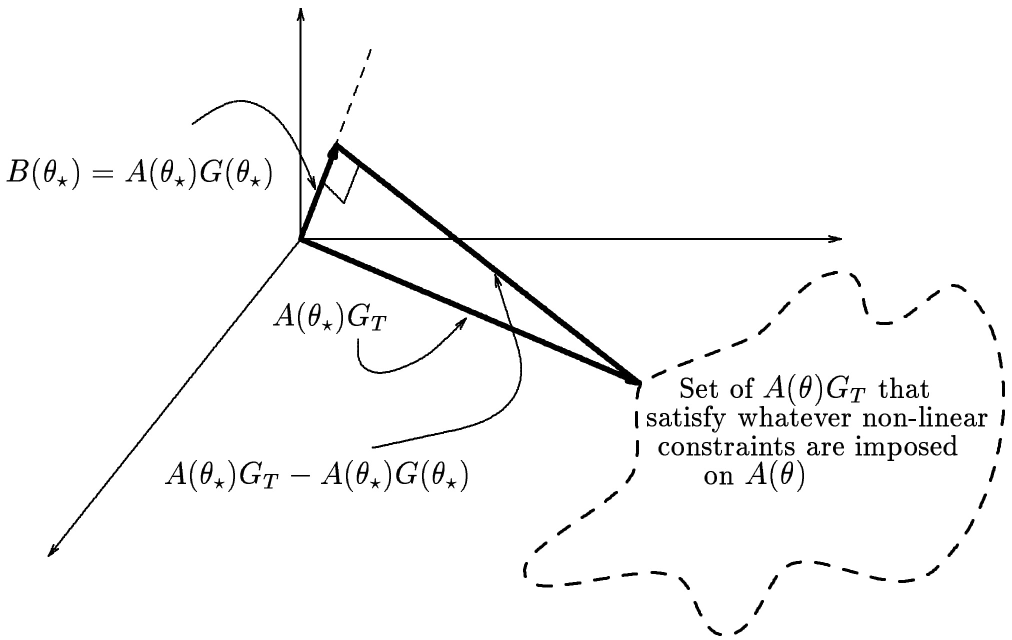

Figure 2: *Geometric Interpretation for ARX Estimation: The case of arbitrary non-linear constraints on the denominator terms.*

Although this corollary holds for any arbitrary non-linear constraint, it may be very difficult to solve (2) and hence find the estimate $\widehat{\theta}_N$ for such a constraint. In [3, 4] the solution to (2) under convex quadratic constraints is derived, and for this special case the preceding corollary is derived algebraically. Also in [3, 4] constraints that are coupled between numerator and denominator terms are considered, and it is shown that in some of these cases equality in (32) eventuates. It is not apparent how these latter results can be proven using the geometric ideas of this paper, the algebraic tools of [3, 4] seem to be the only appropriate ones.

As per the previous section, up to this point we have only used geometric properties about the length of certain sides in a triangle in order to gain information about errors in estimating the magnitude frequency response. Insight into errors in estimating the phase frequency response can be obtained by noting that the orthogonality condition (33) tells us that the true and estimated response both lie in a certain plane, since the inner product measurement of the angle between them is purely real.

Corollary 3. *If all the terms in $B(\theta)$ are free, and there is any constraint on those in $A(\theta)$ then*

$$\int_{-\pi}^{\pi} \left|\frac{A(\theta_\star)}{E}\right|^2 |F|^2 \Phi_u \left|G(\theta_\star)G_T\right| \cos(\measuredangle G(\theta_\star) - \measuredangle G_T)\, \mathrm{d}\omega = \int_{-\pi}^{\pi} \left|\frac{A(\theta_\star)}{E}\right|^2 |F|^2 \Phi_u \left|G(\theta_\star)\right|^2 \mathrm{d}\omega, \tag{36}$$

$$\int_{-\pi}^{\pi} \left|\frac{A(\theta_\star)}{E}\right|^2 |F|^2 \Phi_u \left|G(\theta_\star)G_T\right| \sin(\measuredangle G(\theta_\star) - \measuredangle G_T)\, \mathrm{d}\omega = 0. \tag{37}$$

If all the terms in $A(\theta)$ are free and there is any constraint on those in $B(\theta)$ then (36) becomes

$$\int_{-\pi}^{\pi}\left|\frac{A(\theta_\star)}{E}\right|^2 |F|^2\Phi_u\,|G(\theta_\star)G_T|\cos(\measuredangle G(\theta_\star)-\measuredangle G_T)\,\mathrm{d}\omega = \int_{-\pi}^{\pi}\left|\frac{A(\theta_\star)}{E}\right|^2 |F|^2\Phi_u\,|G_T|^2\,\mathrm{d}\omega \quad (38)$$

and (37) is unchanged.

Proof. From Theorem 2, if all the terms in $B(\theta)$ are free, then (31) is a norm minimisation problem over the subspace $B(\theta)$ and so the orthogonality condition (33) holds which can be written

$$\langle A(\theta_\star)G(\theta_\star), A(\theta_\star)G_T\rangle_u = \|A(\theta_\star)G(\theta_\star)\|_u^2 .$$

Taking real and imaginary parts of both sides then gives (36) and (37). If terms in $A(\theta)$ are free, then the orthogonality condition (35) applies, and taking real and imaginary parts of both sides of this gives (38) and (37). ◻◻◻

The expression (36) was obtained via algebraic manipulations in [19] where those authors concluded that since the sign of $\cos(\alpha)$ is insensitive to the sign of the angle α then the resultant frequency response error would be 'insensitive to the sign of phase errors'. The new expression (37) shows that phase errors are not quite so benign. Any positive phase error must be balanced by a negative phase error. The other conclusions we can make about phase and magnitude errors not being independent are the same as those made in [19] and in the discussion of Corollary 1.

All the analysis in this section has presumed, as previous authors [19, 5, 3, 4] have done, that no measurement noise is present. When measurement noise does exist we have that $\theta_\star$ satisfies

$$\theta_\star = \arg\min_{\theta\in\mathbf{R}^p}\left\{\left\|(G_T - G(\theta))\frac{A(\theta)}{E}\right\|_u^2 + \sigma_\nu^2\left\|\frac{FHA(\theta)}{E}\right\|^2\right\}. \quad (39)$$

The complicated form of (39) makes it difficult for us to say anything about the asymptotic frequency distribution of estimation errors beyond the qualitative analysis made in [27] where (among other things) it was noted that the presence of the measurement noise term in (39) adds a bias to the estimation result. So called 'pseudo-linear' regression methods [7] and AR expansions of $H(q)$ [20] are available to counter this problem, but we will not detail them here.

Regalia [18] has noted that if one knows what $H(q)$ is, then a quadratic constraint can be placed on the terms in $A(q,\theta)$ so that the last term in (39) is independent of θ, hence avoiding bias problems due to noise. For example, with $|HF/E| = 1$ we have

$$\left\|\frac{FHA(\theta)}{E}\right\|^2 = \|A(\theta)\|^2 = \sum_{k=0}^{n} a_k^2$$

so that solution of (29) subject to the constraint $\sum_{k=0}^{n} a_k^2 = 1$ will give a result unbiased by measurement noise. More detailed information about the nature of solutions to (29), such as conditions under which $G(\theta_\star)$ is stable, and Hankle norm characterisations of the estimation error, may be found in [18] where again an algebraic approach to the analysis is undertaken.

3.3 Closed Loop Estimation

The analysis so far has been for the case of data collected in open loop. Let us now apply our geometric methods to the closed loop case. The provision of integral constraint type characterisations of bias error for this case does not seem to have been addressed in the literature so far.

The assumed measurement set up is as shown in figure 3. Here $\{r_k\}$ is a quasi-stationary reference input uncorrelated with the measurement noise $\{\nu_k\}$ and $K(q)$ is a controller. We assume that $\{y_k\}$, the noise corrupted measurements of the true system output $\{z_k\}$ are used both for feedback and identification. We relax the requirement that $G_T(q)$ be stable. Now, defining $S(q)$ and $T(q)$ as the usual sensitivity and complementary sensitivity functions we have

$$\begin{aligned} y_k &= S(q)H(q)\nu_k + T(q)r_k, \\ u_k &= S(q)K(q)(r_k - H(q)\nu_k) \end{aligned}$$

so that for fixed denominator modelling we have a prediction error given by

$$\varepsilon_k(\theta) = (T - G(\theta)SK)Fr_k + (1 + G(\theta)K)SFH\nu_k \tag{40}$$

where $F(q)$ is the usual data prefilter discussed in previous sections. Therefore, since $\{r_k\}$ and $\{\nu_k\}$ are uncorrelated, then provided that the closed loop system is strictly stable we can use Lemma B.1 and (20) to characterise the asymptotic estimate $\theta_\star$ as the one solving the norm minimisation problem[7]

$$\theta_\star = \arg\min_{\theta \in \mathbf{R}^p} \left\{ \|T - G(\theta)SK\|_r^2 + \|(1 + G(\theta)K)S\|_\nu^2 \right\}. \tag{41}$$

Therefore, as is well known (see for example [8, 22]) with no noise $\theta_\star$ is such that it tries to match the estimated closed loop response to the true closed loop response and if there is no reference signal but $\sigma_\nu^2 > 0$ then $\theta_\star$ is such that $G(\theta_\star)$ tries to match $-K^{-1}(e^{j\omega})$. However, using the same geometric argument as in Theorem 1 we can immediately progress from these qualitative statements to the following more quantitative ones

Theorem 3. *For fixed denominator modelling,* $\Phi_r \neq 0$ *and* $\sigma_\nu^2 = 0$

$$\int_{-\pi}^{\pi} |T(e^{j\omega})|^2 \left| \frac{G(e^{j\omega}, \theta_\star)}{G_T(e^{j\omega})} \right|^2 |F(e^{j\omega})|^2 \Phi_r(\omega)\, \mathrm{d}\omega \leq \int_{-\pi}^{\pi} |T(e^{j\omega})|^2 |F(e^{j\omega})|^2 \Phi_r(\omega)\, \mathrm{d}\omega$$

[7] We use the style of notation that we've used before; a subscript r is short for $\chi = |F|^2\Phi_r$ and a subscript ν is short for $\chi = |FH|^2\sigma_\nu^2$.

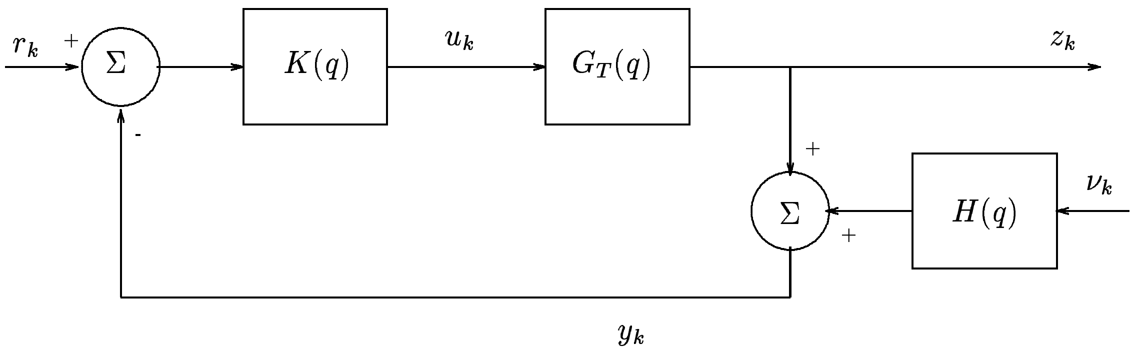

Figure 3: *Assumed Measurement setup for closed loop identification*

with equality occuring if and only if $G_T(e^{j\omega}) = G(e^{j\omega}, \theta_\star)$ for almost all ω. For $\Phi_r = 0$ and $\sigma_\nu^2 > 0$

$$\int_{-\pi}^{\pi} |T(e^{j\omega})|^2 \left|\frac{G(e^{j\omega}, \theta_\star)}{G_T(e^{j\omega})}\right|^2 |F(e^{j\omega})H(e^{j\omega})|^2 \,\mathrm{d}\omega \leq \int_{-\pi}^{\pi} |1 - T(e^{j\omega})|^2 |F(e^{j\omega})H(e^{j\omega})|^2 \,\mathrm{d}\omega$$

Proof. Follows precisely as per Theorem 1 on recognising that when $\sigma_\nu^2 = 0, \Phi_r \neq 0$ the solution of the norm minimisation problem satisfies

$$\langle G(\theta_\star)SK, G(\theta_\star)SK - T\rangle_r = 0 \tag{42}$$

and when $\sigma_\nu^2 > 0, \Phi_r = 0$ the solution of the norm minimisation problem satisfies

$$\langle G(\theta_\star)SK, G(\theta_\star)SK + S\rangle_r = 0. \tag{43}$$

□□□

So, as for the open loop case, with no noise the system gain tends to be underestimated. In the case of no reference signal, but with measurement noise present (41) tells us that we will tend to estimate negative system gains and Theorem 3 goes further by telling us that the magnitude of these gains will be limited very much at low frequencies, but not so much at high frequencies. The implication of this is that we can expect $G(\theta_\star)$ to have a more high pass nature when measurement noise exists than when it doesn't.

Insight into the phase estimation error can also be provided using the same tools we have employed previously.

Corollary 4. *For fixed denominator modelling, $\Phi_r \neq 0$ and $\sigma_\nu^2 = 0$*

$$\int_{-\pi}^{\pi} |T|^2 \left|\frac{G(\theta_\star)}{G_T}\right|^2 |F|^2 \Phi_r(\omega) \cos(\measuredangle G(\theta_\star) - \measuredangle G_T) \,\mathrm{d}\omega = \int_{-\pi}^{\pi} |T|^2 \left|\frac{G(\theta_\star)}{G_T}\right|^2 |F(e^{j\omega})|^2 \Phi_r(\omega) \,\mathrm{d}\omega,$$

$$\int_{-\pi}^{\pi} |T|^2 \left|\frac{G(\theta_\star)}{G_T}\right|^2 |F|^2 \Phi_r(\omega) \sin(\measuredangle G(\theta_\star) - \measuredangle G_T) \,\mathrm{d}\omega = 0$$

and for $\Phi_r = 0$ *and* $\sigma_\nu^2 > 0$

$$\int_{-\pi}^{\pi} |KG(\theta_\star)||SH|^2 \cos(\measuredangle KG(\theta_\star))\,\mathrm{d}\omega = -\int_{-\pi}^{\pi} |KG(\theta_\star)|^2|SH|^2\,\mathrm{d}\omega,$$

$$\int_{-\pi}^{\pi} |KG(\theta_\star)||SH|^2 \sin(\measuredangle KG(\theta_\star))\,\mathrm{d}\omega = 0.$$

Proof. Simply take real and imaginary parts of both sides of (42) and (43). □□□

For the case of no measurement noise, the interpretation of these integral constraints on phase estimation error is much the same as for the open loop case; see the discussion following Corollary 1. However in the opposite situation where there is only measurement noise and no reference excitation, the constraints show that the estimate has essentially nothing to do with the true system response - on average the phase of the estimate is the opposite of that of the controller. The only contribution of the true plant is the effect it has on the sensitivity function S that weights the integral phase constraint. Using this last extreme case, the suggestion is that when we have both excitation and measurement noise, we can expect the plant phase estimate to be biased away from its true value in a manner that is proportional to the phase compensation characteristics of the controller.

For ARX modelling, following the discussion in section 3.2 the predictor is given by

$$\begin{aligned}\varepsilon_k(\theta) &= y_k^f - \widehat{y}_{k|k-1}^f \\ &= \left(G_T\frac{A}{E} - \frac{B}{E}\right) SKFr_k + \left(\frac{A}{E} + K\frac{B}{E}\right) SF\nu_k\end{aligned}$$

so that assuming $\{r_k\}$ and $\{\nu_k\}$ to be uncorrelated allows us to conclude that $\theta_\star$ solves the norm minimisation problem

$$\theta_\star = \arg\min_\theta \left\{ \left\| \left(G_T\frac{A}{E} - \frac{B}{E}\right) SK \right\|_r^2 + \left\| \left(\frac{A}{E} + K\frac{B}{E}\right) S \right\|_\nu^2 \right\}. \tag{44}$$

Following the same geometric ideas as used in the proof of Theorem 2 immediately gives the following quantitative result on the distribution of estimation errors.

Theorem 4. *If there are terms in* $A(\theta)$ *constrained, but not in* $B(\theta)$ *then for* $\Phi_r \neq 0$ *but* $\sigma_\nu^2 = 0$

$$\int_{-\pi}^{\pi} \left|\frac{G(e^{j\omega},\theta_\star)}{G_T(e^{j\omega})}\right|^2 \left|\frac{A(e^{j\omega},\theta_\star)}{E(e^{j\omega})}\right|^2 |T(e^{j\omega})F(e^{j\omega})|^2\Phi_r(\omega)\,\mathrm{d}\omega \le \int_{-\pi}^{\pi} \left|\frac{A(e^{j\omega},\theta_\star)}{E(e^{j\omega})}\right|^2 |T(e^{j\omega})F(e^{j\omega})|^2\Phi_r(\omega)\,\mathrm{d}\omega$$

with equality occuring if and only if $G_T(e^{j\omega}) = G(e^{j\omega},\theta_\star)$ *for almost all* ω. *For* $\Phi_r = 0$, $\sigma_\nu^2 > 0$

$$\int_{-\pi}^{\pi} \left|\frac{G(e^{j\omega},\theta_\star)}{G_T(e^{j\omega})}\right|^2 |T(e^{j\omega})|^2 \left|\frac{A(e^{j\omega},\theta_\star)H(e^{j\omega})}{E(e^{j\omega})}\right|^2 \mathrm{d}\omega \le \int_{-\pi}^{\pi} |1-T(e^{j\omega})|^2 \left|\frac{A(e^{j\omega},\theta_\star)H(e^{j\omega})}{E(e^{j\omega})}\right|^2 \mathrm{d}\omega.$$

If terms in $B(\theta)$ *are constrained, but not in* $A(\theta)$*, then the inequalities go the other way. The result holds regardless of the nature of the constraints imposed on* $A(\theta)$ *or* $B(\theta)$.

Proof. Follows precisely as per Theorem 2 on recognising that when terms in $A(\theta)$ are constrained and $\sigma_\nu^2 = 0, \Phi_r \neq 0$ then by Lemma A.1 with $\chi = |SKF|^2\Phi_r$ the solution of the norm minimisation problem satisfies

$$\langle A(\theta_\star)G(\theta_\star), A(\theta_\star)G(\theta_\star) - A(\theta_\star)G_T\rangle_\chi = 0 \tag{45}$$

so that $\|A(\theta_\star)G(\theta_\star)\|_\chi \leq \|A(\theta_\star)G_T\|_\chi$ with equality if and only if $G_T(e^{j\omega}) = G(e^{j\omega}, \theta_\star)$ for almost all ω. When $\sigma_\nu^2 > 0, \Phi_r = 0$ then with $\chi = |SH|^2\sigma_\nu^2$ the solution of the norm minimisation problem satisfies

$$\langle KB(\theta_\star), KB(\theta_\star) + A(\theta_\star)\rangle_\chi = 0 \tag{46}$$

so that $\|KB(\theta_\star)\|_\chi \leq \|A(\theta_\star)\|_\chi$. If terms in $B(\theta)$ are constrained, but not in $A(\theta)$, then $B(\theta)$ becomes the subspace that the error must be orthogonal to, and so with $\sigma_\nu^2 = 0$, $\Phi_r \neq 0$ and $\chi = |SKF|^2\Phi_r$ we have

$$\langle A(\theta_\star)G_T, A(\theta_\star)G(\theta_\star) - A(\theta_\star)G_T\rangle_\chi = 0 \tag{47}$$

instead of (45), which immediately leads to $\|A(\theta_\star)G(\theta_\star)\|_\chi \geq \|A(\theta_\star)G_T\|_\chi$. With $\sigma_\nu^2 \neq 0$, $\Phi_r = 0$ and $\chi = |SH|^2\sigma_\nu^2$ we have

$$\langle A(\theta_\star), KB(\theta_\star) + A(\theta_\star)\rangle_\chi = 0 \tag{48}$$

instead of (46) which gives $\|KB(\theta_\star)\|_\chi \geq \|A(\theta_\star)\|_\chi$. □□□

Therefore, if $\sigma_\nu^2 = 0$, then as for the open loop case, the system gain is on average under or overestimated according to whether we constrain terms in the denominator or numerator. If $\sigma_\nu^2 \neq 0$, but $\Phi_r = 0$ then as for the fixed denominator case, the closed loop response T generically having a low pass nature will force $G(\theta_\star)$ to have a more high pass nature in the presence of measurement noise.

Finally, as per the previous discussions, we can easily derive a characterisation of the phase error in our estimation procedure.

Corollary 5. *If there are terms in $A(\theta)$ constrained, but not in $B(\theta)$ then for $\Phi_r \neq 0$ but $\sigma_\nu^2 = 0$*

$$\int_{-\pi}^{\pi} \left|\frac{G(\theta_\star)}{G_T}\right| \left|\frac{A(\theta_\star)TF}{E}\right|^2 \Phi_r(\omega)\cos(\measuredangle G(\theta_\star) - \measuredangle G_T)\,\mathrm{d}\omega = \int_{-\pi}^{\pi} \left|\frac{G(\theta_\star)}{G_T}\right|^2 \left|\frac{A(\theta_\star)TF}{E}\right|^2 \Phi_r(\omega)\,\mathrm{d}\omega \tag{49}$$

and

$$\int_{-\pi}^{\pi} \left|\frac{G(\theta_\star)}{G_T}\right| \left|\frac{A(\theta_\star)TF}{E}\right|^2 \Phi_r(\omega)\sin(\measuredangle G(\theta_\star) - \measuredangle G_T)\,\mathrm{d}\omega = 0. \tag{50}$$

For $\Phi_r = 0$, $\sigma_\nu^2 > 0$

$$\int_{-\pi}^{\pi} |KG(\theta_\star)||S|^2 \left|\frac{A(\theta_\star)H}{E}\right|^2 \cos(\measuredangle KG(\theta_\star))\,\mathrm{d}\omega = \int_{-\pi}^{\pi} |KG(\theta_\star)|^2|S|^2 \left|\frac{A(\theta_\star)H}{E}\right|^2 \mathrm{d}\omega. \tag{51}$$

and

$$\int_{-\pi}^{\pi} |KG(\theta_\star)||S|^2 \left|\frac{A(\theta_\star)H}{E}\right|^2 \sin(\measuredangle KG(\theta_\star))\,\mathrm{d}\omega = 0. \qquad (52)$$

If terms in $B(\theta)$ are constrained, but not in $A(\theta)$, then (50) and (52) remain unchanged, but (49) becomes

$$\int_{-\pi}^{\pi} \left|\frac{G(\theta_\star)}{G_T}\right| \left|\frac{A(\theta_\star)TF}{E}\right|^2 \Phi_r(\omega)\cos(\measuredangle G(\theta_\star) - \measuredangle G_T)\,\mathrm{d}\omega = \int_{-\pi}^{\pi} \left|\frac{A(\theta_\star)TF}{E}\right|^2 \Phi_r(\omega)\,\mathrm{d}\omega$$

and (51) becomes

$$\int_{-\pi}^{\pi} |KG(\theta_\star)||S|^2 \left|\frac{A(\theta_\star)H}{E}\right|^2 \cos(\measuredangle KG(\theta_\star))\,\mathrm{d}\omega = \int_{-\pi}^{\pi} |S|^2 \left|\frac{A(\theta_\star)H}{E}\right|^2 \mathrm{d}\omega.$$

These result hold regardless of the nature of the constraints imposed on $A(\theta)$ or $B(\theta)$.

Proof. Take the real and imaginary parts of both sides of (45),(46),(47) and (48). □□□

The interpretation of these results is that same as that for Corollary 4.

4 Conclusion

The aim of this paper was to show that when using least squares methods, an undermodelling induced frequency domain estimation error can be easily and intuitively characterised using geometric principles. This geometric approach is a very old technique when time domain 'signal space' errors are to be characterised. The application of the same idea to a frequency domain setting is therefore a natural extension of existing practice. To the authors knowledge the extension appears to be new, at least in the contexts described in this paper.

Unfortunately, the results arising from this geometric approach, although we believe them to be interesting, are not of a very fine structure. They only quantify average errors in estimation, and as such only act as a general guide to the expected performance of a least squares approach. What is really required for say, robust control system design, is precise quantification of frequency domain errors rather then qualitative indicators of the nature of errors. Many workers are currently addressing this problem, but at the time of writing it still appears to be open.

Appendix A Generalised Projection Theorem

Theorem A.1. *Minimum Distance from Subspace to Linear Variety Let H be a Hilbert Space and M_1,M_2 be closed subspaces of H. Now take $z \in H$ and consider the linear variety $V = z + M_2$. If we seek to minimise $\|x - y\|$ where $x \in M_1$, $y \in V$, then the solutions x_0,y_0 have the following properties:*

$$x_0 \quad \textit{is unique,} \tag{A.53}$$
$$y_0 \quad \textit{is unique,} \tag{A.54}$$
$$x_0 - y_0 \perp M_1, \tag{A.55}$$
$$x_0 - y_0 \perp M_2. \tag{A.56}$$

Proof. Take $y \in V$ fixed. Then by the classical Projection Theorem [12] $\|x - y\|$ has unique minimiser $x_0 \in M_1$ such that $x_0 - y \perp M_1$. This is true $\forall y \in V$ so (A.55) is proved. Now, suppose the contrary to (A.56). That is, assume $\exists v \in V$, $v - z \neq 0$ such that

$$\langle x_0 - y_0, v - z\rangle = \delta\|v - z\|^2 \qquad ; \delta > 0$$

Then

$$\begin{aligned}\|x_0 - y_0 - \delta(v - z)\|^2 &= \|x_0 - y_0\|^2 - \langle x_0 - y_0, \delta(v - z)\rangle - \\ &\quad \langle \delta(v - z), x_0 - y_0\rangle + |\delta|^2\|v - z\|^2 \\ &= \|x_0 - y_0\|^2 - |\delta|^2\|v - z\|^2 < \|x_0 - y_0\|^2\end{aligned}$$

and so (A.56) is proved. Finally, $\forall v \in V$ orthogonality of $x_0 - y_0$ and M_2 gives:

$$\begin{aligned}\|x_0 - v\|^2 &= \|x_0 - y_0 + y_0 - v\|^2 \\ &= \|x_0 - y_0\|^2 + \|y_0 - v\|^2\end{aligned}$$

so $\|x_0 - v\| > \|x_0 - y_0\|^2$ unless $v = y_0$ to give (A.54). □□□

Appendix B Manipulation of Spectral Densities

The condition (17) on $\{u_k\}$ has become colloquially known as quasistationarity [10] and the limiting operation on the right hand side of (17) has been given the symbol $\bar{\mathbf{E}}\{u_k u_{k+\tau}\}$. Using this terminology and notation we can say something about the quasi-stationarity and hence spectral density of signals passing though linear systems.

Lemma B.1. *Let $\{u_k\}$ and $\{y_k\}$ be quasi-stationary signals with spectrums $\Phi_u(\omega)$ and $\Phi_y(\omega)$ respectively, and let $G(q)$ and $H(q)$ be strictly stable transfer functions. Let $\{z_k\}$ be generated according to:*

$$z_k = G(q)y_k + H(q)u_k.$$

Then $\{z_k\}$ is quasi-stationary and

$$\Phi_z(\omega) = \left|G(e^{j\omega})\right|^2 \Phi_y(\omega) + \left|H(e^{j\omega})\right|^2 \Phi_u(\omega) + 2\mathrm{Re}\left\{\overline{H(e^{j\omega})}G(e^{j\omega})\Phi_{yu}(\omega)\right\}$$

Proof. $\{z_k\}$ is quasi-stationary by Theorem 2.2 in [10]. Therefore $\Phi_z(\omega)$ exists and is given by:

$$\Phi_z(\omega) = \sum_{\tau=-\infty}^{\infty} R_z(\tau) e^{-j\omega\tau}$$

but

$$z_k = \sum_{n=0}^{\infty} g_n y_{k-n} + \sum_{m=0}^{\infty} h_m u_{k-m}$$

and

$$R_z(\tau) = \bar{\mathsf{E}} \{z_k z_{k-\tau}\}$$

Therefore

$$\begin{aligned} R_z(\tau) &= \bar{\mathsf{E}} \left\{ \left(\sum_{n=0}^{\infty} g_n y_{k-n} + \sum_{m=0}^{\infty} h_m u_{k-m} \right) \left(\sum_{\ell=0}^{\infty} g_\ell y_{k-\tau-\ell} + \sum_{i=0}^{\infty} h_i u_{k-\tau-i} \right) \right\} \\ &= \sum_n \sum_\ell g_n g_\ell R_y(\tau+\ell-n) + \sum_n \sum_i g_n h_i R_{yu}(\tau+i-n) \\ &\quad + \sum_m \sum_\ell h_m g_\ell R_{yu}(\tau+\ell-m) + \sum_m \sum_i h_m h_i R_u(\tau+i-m) \end{aligned}$$

Now

$$\begin{aligned} \sum_{\tau=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} g_n g_\ell R_y(\tau+\ell-n) e^{-j\omega\tau} &= \sum_{\tau=-\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{\ell=0}^{\infty} g_n e^{-j\omega n} g_\ell e^{j\omega\ell} R_y(\tau+\ell-n) e^{-j\omega(\tau+\ell-n)} \\ &= \sum_{n=0}^{\infty} g_n e^{-j\omega n} \sum_{\ell=0}^{\infty} g_\ell e^{j\omega\ell} \sum_{\tau=-\infty}^{\infty} R_y(\tau+\ell-n) e^{-j\omega(\tau+\ell-n)} \\ &= \left| G(e^{j\omega}) \right|^2 \Phi_y(\omega) \end{aligned}$$

Similarly

$$\sum_{\tau=-\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{i=0}^{\infty} h_m h_i R_u(\tau+i-m) e^{-j\omega\tau} = \left| H(e^{j\omega}) \right|^2 \Phi_u(\omega)$$

Also

$$\begin{aligned} \sum_{\tau=\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} g_n h_i R_{yu}(\tau+i-n) e^{-j\omega\tau} &= \sum_{\tau=\infty}^{\infty} \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} g_n e^{-j\omega n} h_i e^{j\omega i} R_{yu}(\tau+i-n) e^{-j\omega(\tau+i-n)} \\ &= G(e^{j\omega}) \overline{H(e^{j\omega})} \Phi_{yu}(\omega) \\ \sum_{\tau=\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} g_\ell h_m R_{yu}(\tau+\ell-m) e^{-j\omega\tau} &= \sum_{\tau=\infty}^{\infty} \sum_{m=0}^{\infty} \sum_{\ell=0}^{\infty} h_m e^{-j\omega m} g_\ell e^{j\omega\ell} R_{yu}(\tau+\ell-m) e^{-j\omega(\tau+\ell-m)} \\ &= \overline{G(e^{j\omega})} H(e^{j\omega}) \Phi_{yu}(\omega). \end{aligned}$$

□□□

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