On the role of sampling zeros in robust sampled-data control design

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Abstract

In this paper, we investigate the implications for robust sampled-data feedback design of minimum phase sampling zeros appearing in the transfer function of discrete-time plants. Such zeros may be obtained by zero-order hold (ZOH) sampling of continuous-time models having relative degree two or greater. In particular, we address the robustness of sampled-data control systems to multiplicative uncertainty in the model of the continuous-time plant. We argue that lightly damped controller poles, which may arise from attempting to cancel, or almost cancel, sampling zeros of the discretized plant are likely to introduce peaks into the fundamental complementary sensitivity function near the Nyquist frequency. This in turn makes the satisfaction of necessary conditions for robust stability difficult for all but the most modest amounts of modeling uncertainty in the continuous-time plant. Some H_2- and H_∞-optimal discrete-time and sampled data designs may lead to (near-) cancellation, and we therefore argue that their suitability is restricted.

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1 Introduction

For single-input, single-output (SISO) systems of relative degree p, the corresponding discrete-time transfer function arising from zero-order hold (ZOH) sampling has unity relative degree for all but a finite set of sampling periods. The additional $p-1$ discrete-time zeros are called the sampling zeros [1], [2], and the cancellation of minimum phase sampling zeros by lightly damped controller poles has for many years been closely linked to problems with undesirable intersample ripple of either the regulated output or the control signal—for a partial list, see [3, p. 116, pp. 226–227, pp. 232–234], [4, pp. 169–170], [5, pp. 156–161], [6, p. 648], [7], [8], [9, pp. 972–977], [10]. In recent years, however, renewed interest in analysis and synthesis methods which directly take into account intersample behaviour has lead to direct sampled-data control synthesis techniques in which notions of pole-zero cancellations and sampling zeros play no role [11], [12], [13], [5], [14].

In this paper, we focus on the role of sampling zeros and their effect on the robust stability of sampled-data control systems, in which continuous-time plants are controlled by digital compensators in conjunction with appropriate sample and hold devices. While modern sampled-data control synthesis techniques typically avoid consideration of sampling zeros, it is argued in this paper that these zeros can have a substantial effect on the robust stability of sampled-data feedback systems, whether or not they arise explicitly during the synthesis procedure.

The key tools used in this paper are the fundamental sensitivity and complementary sensitivity functions (denoted $S_{\text{fun}}(s)$ and $T_{\text{fun}}(s)$) discussed in Freudenberg et al. [15] in the study of fundamental design limitations for sampled-data feedback control systems; see also [16], [17]. While these functions are not transfer functions in the usual sense, they do play a key role in governing the tracking and disturbance rejection response of sampled-data systems, and are more readily calculated than the complete sampled-data frequency response [18]. Most importantly for this paper, however, is that a necessary condition for stability in the presence of multiplicative uncertainty in the continuous-time plant can be stated in terms of the fundamental complementary sensitivity function [15, Theorem 1].

In this paper, we show that digital controllers which rely on cancellation of minimum phase sampling zeros by lightly damped controller poles have poor robustness to unmodeled high-frequency plant dynamics. This has direct implications for those formulations of discrete-time H_2- and H_∞-optimal control synthesis problems which lead to cancellation of all minimum-phase plant zeros—including those arising through sampling—unless otherwise constrained [19]. Furthermore, the frequency-domain formulation removes the need for dealing explicitly with pole-zero cancellations, so that the robustness of modern sampled-data H_2- and H_∞-optimal controllers can also be addressed. For controllers designed by these direct methods which lead to near (as opposed to exact) cancellations between minimum phase plant zeros and controller poles, this suggests further research is required to clarify the robustness margins of direct sampled-data controllers to unstructured uncertainty in the continuous-time plant model. The results of this paper might therefore be used in conjunction with known necessary and sufficient conditions for robust stability in the presence of linear, time-invariant perturbations, which require the solution of infinite dimensional structured singular value problems [20], [21].

The paper is organized as follows. In § 2, we review the notion of the frequency responmse of sampled-data systems as presented in the work of Freudenberg, Middleton

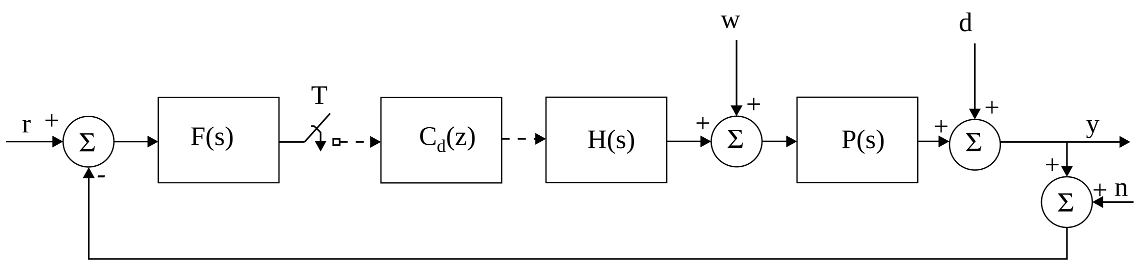

Figure 1: Unity feedback sampled-data control system

and Braslavsky [15], [16], [22]. In § 3 we use this frequency-domain based framework to investigate the implications of cancellation (or near-cancellation) of sampling zeros on the fundamental complementary sensitivity operator $T_{\text{fun}}(s)$. In § 4, we apply the results of § 3 to a recent example of a direct sampled-data H_2-optimal controller [23]. While this example is striking in its demonstration of how sampled-data design can substantially improve intersample ripple in comparison with the associated classical controller, the corresponding fundamental complementary sensitivity function indicates very poor stability robustness to multiplicative plant uncertainty, a fact borne out by simulation experiments.

2 Frequency response of sampled-data systems

The steady-state response of a stable sampled-data feedback system to a sinusoidal input consists of a fundamental component at the frequency of the input, together with all of its aliases, i.e. harmonics located at integer multiples of the sampling frequency [3], [24]. In this section, we introduce the assumptions, notation and definitions needed in the sequel for dealing with the fundamental components of the response of $y(t)$ in Figure 1 to output disturbances $d(t)$, measurement noise $n(t)$ and commands $r(t)$. The reader is referred to [15] for a more complete exposition.

Consider the single-input, single-output sampled-data feedback system in Figure 1, where $P(s)$ and $F(s)$ are the transfer functions of the continuous-time plant and anti-aliasing filter, $C_{\text{d}}(z)$ is the transfer function of the digital controller, $r(t)$, $d(t)$ and $n(t)$ are the command, output disturbance and noise signals, $u(t)$ is the control input, and $y(t)$ is the system output. The sampling period is denoted by T, the sampling frequency by $\omega_{\text{s}} = 2\pi/T$, and the Nyquist frequency by $\omega_{\text{N}} = \pi/T$. The frequency range $\Omega_{\text{N}} \triangleq (-\omega_{\text{N}}, \omega_{\text{N}}]$ is termed the baseband.

A rational function of s (respectively, z) is *minimum phase* if it has no zeros in the open right half-plane (respectively, in the complement of the closed unit disk $\overline{D} \triangleq \{z : |z| \leq 1\}$). Likewise, a rational function of s (respectively, z) is *stable* if it has no poles in the closed right half-plane (respectively, in the complement of the open unit disk $D \triangleq \{z : |z| < 1\}$.

We shall assume that the plant, prefilter and controller are each free of unstable hidden modes, that $P(s)$ is rational and proper, $F(s)$ is rational, strictly proper, and has no closed right half-plane poles or zeros, and that $C_{\text{d}}(z)$ is rational and proper. We restrict attention to a zero-order hold (ZOH) defined by

$$u(t) = u_k, \qquad \text{for } kT \leq t < (k+1)T,$$

for a discrete input sequence $\{u_k\}_{k=0}^{\infty}$. The associated frequency response function of the ZOH is

$$H(s) = \frac{1 - e^{-sT}}{s}. \tag{1}$$

The discrete transfer function of the series connection of hold, plant, prefilter and sampler is given by [25]

$$(FPH)_{\mathrm{d}}(z) \triangleq \mathcal{Z}\{\mathcal{S}_{\mathrm{T}}\{\mathcal{L}^{-1}\{F(s)P(s)H(s)\}\}\}, \tag{2}$$

and is referred to as the *discretized plant.* Opting for this somewhat unconventional notation has the distinct advantage of allowing the role of the anti-aliasing filter and the frequency response of the hold function to remain completely clear at all times. Define the *discrete sensitivity* and *complementary sensitivity functions*

$$S_{\mathrm{d}}(z) \triangleq \frac{1}{1 + (FPH)_{\mathrm{d}}(z)C_{\mathrm{d}}(z)} \tag{3}$$

and

$$T_{\mathrm{d}}(z) \triangleq (FPH)_{\mathrm{d}}(z)C_{\mathrm{d}}(z)S_{\mathrm{d}}(z). \tag{4}$$

Definition 2.1 (Hybrid sensitivity and complementary functions) *Define the* fundamental sensitivity *and* complementary sensitivity functions *by*

$$S_{\mathrm{fun}}(s) \triangleq 1 - \frac{1}{T}P(s)H(s)C_{\mathrm{d}}(e^{sT})S_{\mathrm{d}}(e^{sT})F(s) \tag{5}$$

and

$$T_{\mathrm{fun}}(s) \triangleq \frac{1}{T}P(s)H(s)C_{\mathrm{d}}(e^{sT})S_{\mathrm{d}}(e^{sT})F(s) \tag{6}$$

respectively.

The functions $S_{\mathrm{fun}}(s)$ and $T_{\mathrm{fun}}(s)$ are not transfer functions in the conventional sense, since they are not equal to the ratio of transformed input and outputs signals. Nevertheless, they do govern the baseband component of the steady-state response to sinusoidal inputs, and therefore play a key role in the overall response. To discuss steady-state behaviour, we assume the absence of unstable pole-zero cancellations in the product $(FPH)_{\mathrm{d}}(z)C_{\mathrm{d}}(z)$, that all poles of $S_{\mathrm{d}}(z)$ lie within D, and that the standard non-pathological sampling conditions are satisfied [26], from which exponential and $\mathcal{L}_2$ input–output stability follow [27], [28].

Denote the responses of $y(t)$ to each of $d(t), n(t)$ and $r(t)$ by $y_{\mathrm{d}}(t), y_{\mathrm{n}}(t)$ and $y_{\mathrm{r}}(t)$, respectively, where $d(t) = e^{j\omega t}, t \geq 0, n(t) = e^{j\omega t}, t \geq 0$ and $r(t) = e^{j\omega t}, t \geq 0$. Then, as $t \to \infty$ [15]:

$$y_{\mathrm{d}}(t) \to y_{\mathrm{dss}}(t), y_{\mathrm{n}}(t) \to y_{\mathrm{nss}}(t), \text{ and } y_{\mathrm{r}}(t) \to y_{\mathrm{rss}}(t),$$

where

$$y_{\mathrm{dss}}(t) = S_{\mathrm{fun}}(j\omega)e^{j\omega t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T_k(j\omega)e^{j(\omega + k\omega_{\mathrm{s}})t}, \tag{7}$$

$$y_{\mathrm{nss}}(t) = -T_{\mathrm{fun}}(j\omega)e^{j\omega t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T_k(j\omega)e^{j(\omega + k\omega_{\mathrm{s}})t}, \tag{8}$$

and

$$y_{\rm rss}(t) = T_{\rm fun}(j\omega)e^{j\omega t} - \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} T_k(j\omega)e^{j(\omega+k\omega_{\rm s})t}, \tag{9}$$

where

$$T_k(s) \triangleq \frac{1}{T}P(s+jk\omega_{\rm s})H(s+jk\omega_{\rm s})C_{\rm d}(e^{sT})S_{\rm d}(e^{sT})F(s), \qquad k \neq 0 \tag{10}$$

is the kth harmonic response function.

3 Implications of sampling zero cancellation

We now consider the consequences for robust stability of cancellations between minimum phase sampling zeros appearing in the discretized plant, and poles of the digital controller $C_{\rm d}$. While state-space frameworks sometimes obscure the role of pole-zero cancellations, it is nevertheless well known that cancellations of the form just described are a feature in several different classes of discrete-time H_2- and H_∞-optimal control problems [29], [30], [19]. Moreover, for direct sampled-data control synthesis in which no exact cancellations occur, the frequency-domain approach of this paper nonetheless provides quantitative and qualitative information about the likely implications for robustness of near pole-zero cancellations.

We assume that a controller $C_{\rm d}$ has been designed to ensure the nominal stability of the feedback system in Figure 1, and consider the effect of multiplicative uncertainty of the form

$$P'(s) = P(s)(1 + W(s)\Delta(s)), \tag{11}$$

where $\Delta(s)$ is proper and stable, and $W(s)$ is a stable weighting function used to represent the frequency dependence of the modeling error. It was shown in [15] that a *necessary* condition for the closed loop system to remain stable for all $\Delta(s)$ satisfying

$$|\Delta(j\omega)| < 1, \qquad \forall\, \omega \in \mathbf{R}$$

is that

$$|W(j\omega)T_{\rm fun}(j\omega)| \leq 1, \qquad \forall\, \omega \in \mathbf{R}. \tag{12}$$

Since $T_{\rm fun}(j\omega)$ is readily calculated from (6), and the weighting function $W(s)$ reflects the designers uncertainty in the continuous-time plant model, condition (12) is very useful for assessing the robust stability properties of a given controller to unstructured plant uncertainty of various kinds. For instance, by representing a single high-frequency pole as multiplicative uncertainty, it is possible to estimate the maximum time constant of a neglected lag for which a given controller maintains stability. If the true plant has transfer function

$$P'(s) = P(s)\frac{1}{\tau_p s + 1}, \qquad 0 \leq \tau_p \leq \tau_{\max},$$

it follows that an appropriate choice of weight is [31, p. 267]:

$$W(s) = \frac{\tau_{\max} s}{\tau_{\max} s + 1}. \tag{13}$$

Tuning $\tau_{\max}$ until condition (12) is just barely satisfied then gives a useful indication of the likely robustness (or otherwise) of a given control design.

The simplicity of condition (12) is a direct consequence of ignoring the effect of aliases on the closed-loop response. By taking these aliases into account, it is possible to use the results of [20] to state a stronger necessary condition for closed-loop stability [17] whose testing requires only a little more effort than (12):

$$|W(j\omega)T_{\text{fun}}(j\omega)| + \underline{w}\,|S_{\text{fid}}(j\omega)| \leq 1, \qquad \omega \in \Omega_{\text{N}}, \tag{14}$$

where

$$\underline{w} \triangleq \inf_{\omega \notin \Omega_{\text{N}}} |W(j\omega)|, \tag{15}$$

and $S_{\text{fid}}(s)$ is the *fidelity function*

$$\begin{aligned} S_{\text{fid}}(s) &\triangleq S_{\text{fun}}(s) - S_{\text{d}}(e^{sT}) \\ &= -T_{\text{fun}}(s) + T_{\text{d}}(e^{sT}). \end{aligned} \tag{16}$$

Conditions (12) and (14) both indicate that controllers leading to large peaks in $|T_{\text{fun}}(j\omega)|$ are likely to have their nominal stability destroyed by only modest high-frequency deviations of the continuous-time plant from the nominal model $P(s)$. In the remainder of this section, we show how controllers which cancel, or almost cancel, minimum phase sampling zeros of the discretized plant near -1 necessarily lead to these undesirable peaks in $|T_{\text{fun}}(j\omega)|$ near the Nyquist frequency.

We start by writing

$$\begin{aligned} T_{\text{fun}}(s) &= \frac{1}{T}F(s)P(s)H(s)C_{\text{d}}(e^{sT})S_{\text{d}}(e^{sT}) \\ &= \frac{1}{T}\frac{F(s)P(s)H(s)}{(FPH)_{\text{d}}(e^{sT})}(FPH)_{\text{d}}(e^{sT})C_{\text{d}}(e^{sT})S_{\text{d}}(e^{sT}) \\ &= \frac{1}{T}\frac{F(s)P(s)H(s)}{(FPH)_{\text{d}}(z)}T_{\text{d}}(z), \qquad z = e^{sT}. \end{aligned} \tag{17}$$

Suppose that the discretized plant $(FPH)_{\text{d}}(z)$ has a minimum phase zero, z_0 say, near -1, which is exactly cancelled by a pole in the controller C_{d}. Since the zero at z_0 no longer appears in the discrete complementary sensitivity function, $T_{\text{d}}(z_0) \neq 0$, and $T_{\text{fun}}(s)$ consequently has a pole at s_0, where

$$\begin{aligned} s_0 &= \frac{1}{T}\ln|z_0| + j\frac{1}{T}\arg z_0 \\ &= \sigma + j\frac{1}{T}\pi, \qquad \sigma \text{ small and negative,} \\ &= \sigma + j\omega_{\text{N}}. \end{aligned} \tag{18}$$

Thus the pole in $T_{\text{fun}}(s)$ at s_0 ensures $|T_{\text{fun}}(j\omega_{\text{N}})| \gg 1$, with undesirable consequences for robust stability.

Conversely, if the zero of the discretized plant $(FPH)_{\text{d}}(z)$ is *not* cancelled by a controller pole, $T_{\text{d}}(z)$ and $(FPH)_{\text{d}}(z)$ share a common zero at z_0, so no such pole appears in $T_{\text{fun}}(s)$ for $s \approx j\omega_{\text{N}}$. If, however, the controller C_{d} has a pole at z_1, with $z_1 \approx z_0$ (as is the situation with near pole-zero cancellations), $T_{\text{d}}(z_0) = 0$ since the discretized plant zero at z_0 is not cancelled by a controller pole, $T_{\text{d}}(z_1) = 1$ and $|(FPH)_{\text{d}}(z_1)| \approx 0$ since $z_1 \approx z_0$.

Thus depending on the proximity of z_0 and z_1, $T_{\text{fun}}(j\omega)$ might still be unacceptable large over a range of frequencies.

It is also possible to interpret (17) in terms of engineering rules of thumb applied to digital control design. Specifically, since

$$|T_{\text{fun}}(j\omega)| = \frac{1}{T}\left|\frac{F(j\omega)P(j\omega)H(j\omega)}{(FPH)_{\text{d}}(e^{j\omega T})}\right| |T_{\text{d}}(e^{j\omega T})|, \qquad \omega \in \mathbf{R}, \tag{19}$$

large peaks in $T_{\text{fun}}(j\omega)$ are avoided by ensuring that $|T_{\text{d}}(e^{j\omega T})|$ is sufficiently small at frequencies ω where the discrepancy between the frequency responses of the continuous-time and discreted plants, $F(j\omega)P(j\omega)H(j\omega)$ and $(FPH)_{\text{d}}(e^{j\omega T})$ respectively, is large, i.e. near the Nyquist frequency. This is therefore consistent with the guideline of restricting the closed-loop bandwidth to no more than around one-fifth the Nyquist frequency.

4 Example

In this section we show how the fundamental complementary sensitivity function and the robust stability necessary condition (12) can be used to assess the robust stability of both a classical discrete-time H_2-optimal controller and its sampled-data counterpart. The example we consider in this section originally appeared in [23]; see also [32] for details of the same design procedures applied to a different plant.

The plant is open-loop unstable, with transfer function

$$P(s) = \frac{1}{s^2 + 2s - 10}. \tag{20}$$

With a sample period $T = 0.2$ s, zero-order hold, and anti-aliasing filter $F(s) = 1$, the corresponding discretized plant transfer function

$$(FPH)_{\text{d}}(z) = \frac{0.0182(z + 0.8768)}{(z - 0.4218)(z - 1.5893)} \tag{21}$$

clearly exhibits a sampling zero at $z = -0.8768$. The controller

$$C_{\text{dt}}(z) = \frac{282.95(z - 0.3768)}{(z + 2.8880)(z + 0.8768)} \tag{22}$$

minimizes the H_2 norm of the closed-loop transfer function from an additive input disturbance w appearing on the control input to plant output y. That is, the discrete-time controller C_{dt} minimizes the energy in the pulse response of the closed-loop system from w to y, and clearly cancels the sampling zero of $(FPH)_{\text{d}}$.

The second controller we consider is

$$C_{\text{sd}}(z) = \frac{179.5(z - 0.3955)}{(z + 1.7765)(z + 0.9493)}, \tag{23}$$

which minimizes the average of responses to unit intensity impulses applied to w over the period $[0, T)$, and thus captures the intersample behaviour of the closed-loop system more satisfactorily. This controller is designed by solving a purely discrete-time H_2-optimal

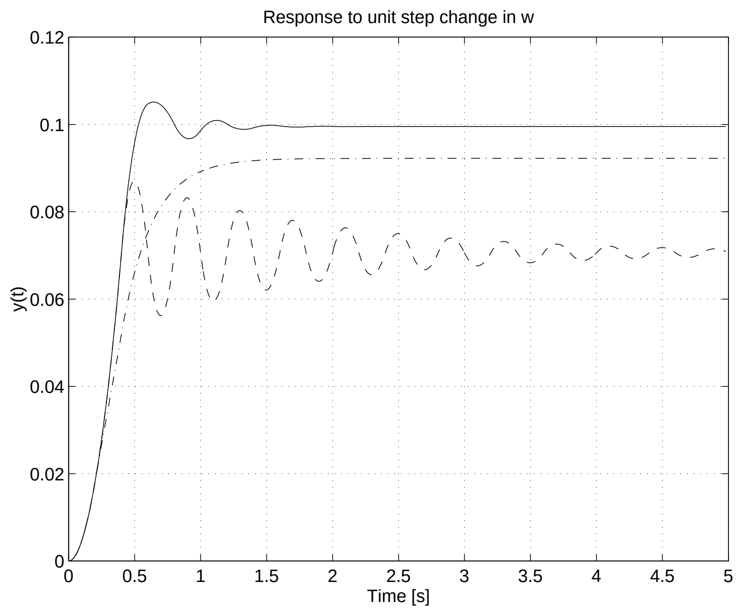

Figure 2: Response to unit step input disturbance applied at time $t = 0$ for three different controllers. Key: $C_{\rm dt}$: - - -, $C_{\rm sd}$: —, $C_{\rm pp}$: -.-.-.

control problem for an appropriately defined auxiliary discrete-time system obtained from the underlying continuous-time plant and the sampling period T [12], [33], and will be referred to as the sampled-data controller.

Figure 2 shows the simulated response to a unit amplitude step in w applied at time $t = 0$ s of the controllers $C_{\rm dt}$ (- - -) and $C_{\rm sd}$ (—). While the discrete-time H_2-optimal controller has a steady-state gain somewhat smaller than the sampled-data controller, a substantial intersample ripple with period 0.4 s (corresponding to the Nyquist frequency $\omega_{\rm N} = 5\pi/$ rad/s) is present in the discrete-time design. Note that the intersample ripple with the sampled-data controller is small, despite the presence of a very lightly damped pole at $z = -0.9493$ in $C_{\rm sd}$.

Also shown as a dash-dotted line (-.-.-.) in Figure 2 is the step response corresponding to the controller

$$C_{\rm pp}(z) = \frac{56.4101(z - 0.4218)}{z + 0.5651} \tag{24}$$

obtained by cancelling the stable pole of the discretized plant $(FPH)_{\rm d}$ at $z = 0.4218$, and placing two poles of the discrete-time transfer function from w to y at the origin.

To assess the robust stability of the feedback systems corresponding to the three controllers $C_{\rm dt}, C_{\rm sd}$ and $C_{\rm pp}$, we evaluate the fundamental complementary sensitivity functions $T_{\rm fun}(j\omega)$ up to a maximum frequency of $2\omega_{\rm N}$ rad/s. From Figure 3, both the discrete-time and sampled-data controllers lead to peaks in $|T_{\rm fun}(j\omega)|$ near $\omega = \omega_{\rm N} = 15.7$ rad/s. In view of the necessary condition in (12), we should therefore expect poor robustness for both of these feedback systems to uncertainty in the continuous-time plant model at frequencies around $\omega_{\rm N}$, where $|T_{\rm fun}(j\omega)|$ is large. By way of comparison, the value of $|T_{\rm fun}(j\omega_{\rm N})|$ for the pole-placement controller $C_{\rm pp}$ is some 20 dB lower than the peak for $C_{\rm sd}$, and almost 50 dB lower than the corresponding value for $C_{\rm dt}$, so that improved stability margins are expected compared with either controller.

Equation (19) pinpoints the reason for the large peaks in $|T_{\rm fun}(j\omega)|$ for the $C_{\rm dt}$ and

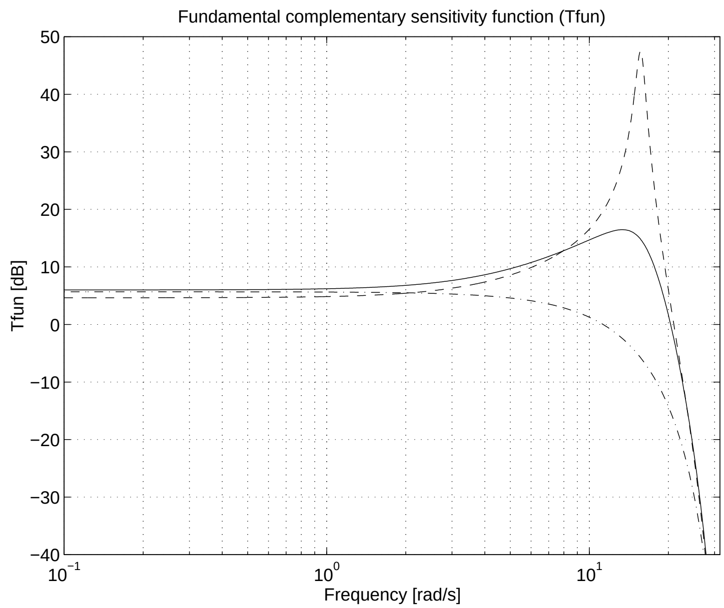

Figure 3: Fundamental complementary sensitivity function magnitude ($|T_{\rm fun}(j\omega)|$) for three different controllers. Key: $C_{\rm dt}$: - - -, $C_{\rm sd}$: —, $C_{\rm pp}$: -.-.-.

$C_{\rm sd}$ controllers. That is, while there is a substantial gap between the continuous-time and discrete-time plant frequency responses at $\omega = \omega_{\rm N}$ due to the sampling zero in $(FPH)_{\rm d}$:

$$\frac{1}{T}\left|\frac{F(j\omega_{\rm N})P(j\omega_{\rm N})H(j\omega_{\rm N})}{(FPH)_{\rm d}(e^{j\omega_{\rm N}T})}\right| \approx 12 \text{ dB},$$

the corresponding value of the discrete-time complementary sensitivity function $|T_{\rm d}(e^{j\omega_{\rm N}T})|$ is comparatively large for controllers $C_{\rm dt}$ and $C_{\rm sd}$, as shown in Figure 4. In contrast, $|T_{\rm d}(e^{j\omega_{\rm N}T})| = -18$ dB for the pole-placement controller, which does not cancel the plant sampling zero at $z = -0.8768$ (as does $C_{\rm dt}$) nor almost cancel it (as does $C_{\rm sd}$).

To give a concrete example of the poor robustness implied by the peak values of $|T_{\rm fun}(j\omega)|$, consider the effect of a single neglected high-frequency pole in the continuous-time plant:

$$P'(s) = \frac{1}{(s^2+2s-10)}\frac{1}{(\tau_p s+1)}, \qquad 0 \le \tau_p \le \tau_{\rm max}. \tag{25}$$

A little experimentation shows that the necessary condition (12) is just barely satisfied for $C_{\rm dt}$ when $\tau_{\rm max} = 0.0003$ s and $W(s)$ is chosen as in (13), indicating extreme sensitivity to neglected high-frequency dynamics in the continuous-time plant. For the sampled-data controller $C_{\rm sd}$ the necessary condition is satsified when $\tau_{\rm max} = 0.01$ s, which is an improvement over $C_{\rm dt}$, but still indicates poor sensitivity to neglected dynamics. For the pole-placement controller, the necessary condition (12) is satisfied for weighting function (13) when $\tau_{\rm max} = 0.1$ s.

Figure 5 shows the simulated response to a unit amplitude step in w applied at time $t = 0$ s for each of the controllers $C_{\rm dt}$, $C_{\rm sd}$ and $C_{\rm pp}$ applied to the plant (25), where $\tau_p = 0.01$ s. For both controllers $C_{\rm dt}$ and $C_{\rm sd}$ with large peak values of $|T_{\rm fun}(j\omega)|$ at the Nyquist frequency, the robustness is quite poor. Indeed, neither feedback system maintains stability when the true plant includes a pole at $s = -100$ rad/s. In contrast, the

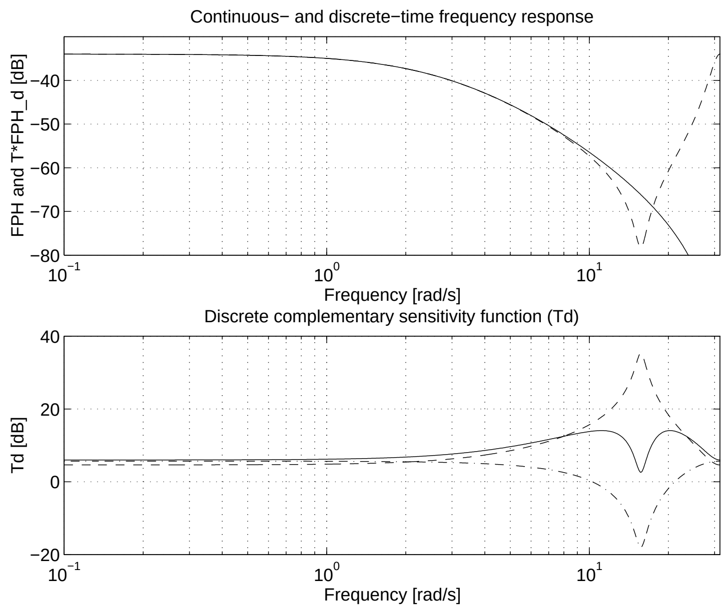

Figure 4: (a) Continuous- and discrete-time plant frequency response. Key: continuous-time ($|F(j\omega)P(j\omega)H(j\omega)|$): —, discrete-time ($T|(FPH)_{\rm d}(e^{j\omega T})|$): - - - (b) Discrete complementary sensitivity function magnitude ($|T_{\rm d}(e^{j\omega T}|$) for three different controllers. Key: $C_{\rm dt}$: - - -, $C_{\rm sd}$: —, $C_{\rm pp}$: -.-.-.

responses of the controller $C_{\rm pp}$ for plants $P(s)$ and $P'(s)$ are virtually identical, and repeated simulation experiments with the controller $C_{\rm pp}$ indicate that stability is maintained even when the bandwidth of the neglected pole is somewhat less than 10 rad/s.

5 Conclusions

In this paper, we have investigated the role of sampling zeros on the robustness of sampled-data control systems to uncertainty in the underlying continuous-time plant model. It has been argued that very lightly damped controller poles (which may arise from attempting to cancel, or almost cancel, minimum phase zeros of the discretized plant near -1) are likely to introduce peaks into the fundamental complementary sensitivity function near the Nyquist frequency. In turn, excessively large peaks in $|T_{\rm fun}(j\omega)|$ make the satisfaction of the necessary condition (12) difficult for all but the most modest amounts of modeling uncertainty in the continuous-time plant. Expressed another way, controllers which cancel, or almost cancel, minimum-phase sampling zeros of the discretized plant close to -1 are virtually assured of violating design guidelines recommending the closed-loop bandwidth be no more than around one-fifth the Nyquist frequency. This may therefore restrict the suitability of discrete-time H_2- and H_∞-optimal design procedures (or their sampled-data counterparts) which lead to (near-) cancellation of all minimum-phase plant zeros.

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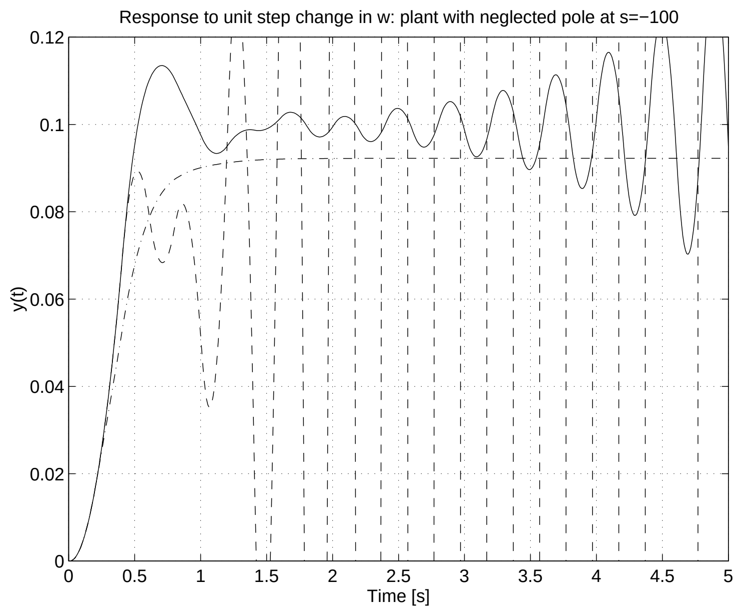

Figure 5: Response to unit step input disturbance applied at time $t = 0$ for three different controllers, with plant including unmodeled pole at $s = -100$ rad/s. Key: C_{dt}: - - -, C_{sd}: —, C_{pp}: -.-.-.

References

[1] K.J. Åström, P. Hagander, and J. Sternby, "Zeros of sampled systems", *Automatica*, vol. 20, no. 1, pp. 21–38, 1984.

[2] T. Hagiwara, T. Yuasa, and M. Araki, "Stability of the limiting zeros of sampled-data systems with zero- and first-order holds", *Int. J. Control*, vol. 58, no. 6, pp. 1325–1346, 1993.

[3] K.J. Åström and B. Wittenmark, *Computer Controlled Systems: Theory and Design*, Prentice-Hall, Englewood Cliffs, NJ, 1984.

[4] M. Morari and E. Zafiriou, *Robust Process Control*, Prentice-Hall, Englewood Cliffs, NJ, 1989.

[5] T. Chen and B. Francis, *Optimal Sampled-Data Control Systems*, Communications and Control Engineering Series. Springer-Verlag, London, 1995.

[6] G. Stephanopoulos, *Chemical Process Control: An Introduction to Theory and Practice*, Prentice-Hall, Englewood Cliffs, NJ, 1984.

[7] G.C. Goodwin, R. Lozano Leal, D.Q. Mayne, and R.H. Middleton, "Rapprochement between continuous and discrete model reference adaptive control", *Automatica*, vol. 22, no. 2, pp. 199–207, March 1986.

[8] A. Tesfaye and M. Tomizuka, "Robust digital tracking with perturbation estimation via the Euler operator", in *Proc. Winter Ann. Meet. ASME DSC*, 1993, pp. 49–54, volume 50.

[9] B.A. Ogunnaike and W.H. Ray, *Process Dynamics, Modeling, and Control*, Oxford University Press, New York, 1994.

[10] R. Sanchis and P. Albertos, "Design of ripple-free controllers", in *Proc. 3rd European Control Conf.*, Rome, Italy, September 1995, pp. 3660–3664.

[11] T. Chen and B.A. Francis, "$\mathcal{H}_2$-optimal sampled-data control", *IEEE Trans. Automat. Control*, vol. 36, no. 4, pp. 387–397, April 1991.

[12] P.P. Khargonekar and N. Sivashankar, "$\mathcal{H}_2$ optimal control for sampled-data systems", *Systems & Control Letters*, vol. 17, no. 6, pp. 425–436, 1991.

[13] B.A. Bamieh and J.B. Pearson, Jr., "A general framework for linear periodic systems with applications to $\mathcal{H}^\infty$ sampled-data control", *IEEE Trans. Automat. Control*, vol. 37, no. 4, pp. 418–435, April 1992.

[14] G. E. Dullerud, *Control of Uncertain Sampled-Data Systems*, Birkhäuser, Boston, 1996.

[15] J.S. Freudenberg, R.H. Middleton, and J.H. Braslavsky, "Inherent design limitaions for linear sampled-data feedback systems", *Int. J. Control*, vol. 61, no. 6, pp. 1387–1421, 1995.

[16] J.H. Braslavsky, *Frequency domain analysis of sampled-data control systems*, PhD thesis, Dept. of Electrical and Computer Engineering, University of Newcastle, Newcastle, New South Wales, Australia, October 1995.

[17] J.S. Freudenberg, R.H. Middleton, and J.H. Braslavsky, "Robustness of zero shifting via generalized sampled-data hold functions", *IEEE Trans. Automat. Control*, vol. 42, no. 12, pp. 1681–1692, December 1997.

[18] Y. Yamamoto and P.P. Khargonekar, "Frequency response of sampled-data systems", *IEEE Trans. Automat. Control*, vol. 41, no. 2, pp. 166–176, February 1996.

[19] K.-C. Goh and M.G. Safonov, "H^∞ control: Inverse free formulae for $D_{11} \neq 0$ and eliminating pole-zero cancellations via interpolation", in *Proc. 32nd IEEE Conf. Decision Contr.*, San Antonio, TX, December 1993, pp. 1152–1157.

[20] G. Dullerud and K. Glover, "Robust stabilization of sampled-data systems to structured LTI perturbations", *IEEE Trans. Automat. Control*, vol. 38, no. 10, pp. 1497–1508, October 1993.

[21] G.E. Dullerud and K. Glover, "Analysis of structured LTI uncertainty in sampled-data systems", *Automatica*, vol. 31, no. 1, pp. 99–113, January 1995.

[22] J.H. Braslavsky, R.H. Middleton, and J.S. Freudenberg, "L_2-induced norms and frequency gains of sampled-data sensitivity operators", *IEEE Trans. Automat. Control*, vol. 43, no. 2, pp. 252–258, February 1998.

[23] Y. Yamamoto, "A retrospective view on sampled-data control systems", *CWI Quarterly*, vol. 9, no. 3, pp. 261–276, September 1996.

[24] G.C. Goodwin and M. Salgado, "Frequency domain sensitivity functions for continuous time systems under sampled data control", *Automatica*, vol. 30, no. 8, pp. 1263–1270, August 1994.

[25] R.H. Middleton and J.S. Freudenberg, "Zeros and non-pathological sampling for generalized sampled-data hold functions", *Automatica*, vol. 31, no. 2, pp. 315–319, February 1995.

[26] R. Kalman, Y.C. Ho, and K. Narendra, "Controllability of linear dynamical systems", in *Contributions to Differential Equations*, pp. 189–213. John Wiley, New York, 1963, Volume 1.

[27] B.A. Francis and T.T. Georgiou, "Stability theory for linear time-invariant plants with periodic digital controllers", *IEEE Trans. Automat. Control*, vol. 33, no. 9, pp. 820–832, September 1988.

[28] T. Chen and B.A. Francis, "Input–output stability of sampled-data systems", *IEEE Trans. Automat. Control*, vol. 36, no. 1, pp. 50–58, January 1991.

[29] A. Saberi, P. Sannuti, and B.M. Chen, *H_2 Optimal Control*, Prentice Hall Int., Hemel Hempstead, Hertsfordshire, 1995.

[30] M.C. Tsai, E.J.M. Geddes, and I. Postlethwaite, "Pole-zero cancellations and closed-loop properties of an H^∞ mixed sensitivity design problem", *Automatica*, vol. 28, no. 3, pp. 519–530, May 1992.

[31] S. Skogestad and I. Postlethwaite, *Multivariable Feedback Control—Analysis and Design*, John Wiley & Sons, Chichester, 1996.

[32] S. Hara, Y. Yamamoto, and H. Fujioka, "Modern and classical analysis/synthesis methods in sampled-data control—A brief overview with numerical examples", in *Proc. 35th IEEE Conf. Decision Contr.*, Kobe, Japan, December 1996, pp. 1251–1256.

[33] B. Bamieh and J.B. Pearson, "The $\mathcal{H}^2$ problem for sampled-data systems", *Systems & Control Letters*, vol. 19, no. 1, pp. 1–12, 1992.