Orthonormal Basis Functions for Modelling Continuous-Time Systems

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Abstract

This paper studies continuous-time system model sets that are spanned by fixed pole orthonormal bases. The nature of these bases is such as to generalise the well known Laguerre and two–parameter Kautz bases. The contribution of the paper is to establish that the obtained model sets are complete in all of the Hardy spaces $H_p(\Pi)$, $1 < p < \infty$ and the right half plane algebra $A(\Pi)$ provided that a mild condition on the choice of basis poles is satisfied. A characterisation of how modelling accuracy is affected by pole choice, as well as an application example of flexible structure modelling are also provided.

Key words: Rational basis functions, orthonormal, completeness, continuous-time systems.

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1 Notation

$\mathbf{C}$	the field of complex numbers.
$\mathbf{R}$	the field of real numbers.
Π	the open right half plane $\{s \in \mathbf{C} : \mathrm{Re}\{s\} > 0\}$.

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$\overline{\Pi}$	the closed right half plane $\{s \in \mathbf{C} : \mathrm{Re}\{s\} \geq 0\}$.
$\mathbf{D}$	the open unit disk $\{z \in \mathbf{C} : \lvert z \rvert < 1\}$.
$\mathbf{T}$	the unit circle $\{z \in \mathbf{C} : \lvert z \rvert = 1\}$.
$H_p(\Pi)$	the Hardy spaces of functions $f(s)$ analytic on Π and such that $\lVert f \rVert_p^p = (1/2\pi) \sup_{x>0} \int_{-\infty}^{\infty} \lvert f(x+jy) \rvert^p \, dy < \infty$, $0 < p < \infty$ and $\lVert f \rVert_\infty = \sup_{s \in \Pi} \lvert f(s) \rvert < \infty$.
$A(\Pi)$	the right half plane algebra $\{f : f \in H_\infty(\Pi)$ and continuous on $\overline{\Pi}\}$.
$A(\mathbf{D})$	the disk algebra $\{f : f$ analytic on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}\}$.
$\mathrm{sp}A$	the linear span of A.
$\overline{a}$	the complex conjugate of a.

2 Introduction

The use of orthonormal bases for the purposes of approximation and analysis is fundamental to many areas of applied mathematics. In particular, in the areas of control theory, signal processing and system identification, there has long been interest in the use of the trigonometric (FIR), 'Laguerre', and 'two-parameter Kautz' bases [20,14,19]. More recently, in a discrete-time setting, this interest has been revived in a string of works [36,37,10,8,29,9,27,39] and this has led workers to consider orthonormal constructions which generalise the Laguerre and two-parameter Kautz cases [11,15,5,7,31]. One of these efforts presented in [28,3] considers the orthonormal basis functions defined on $\mathbf{D} \cup \mathbf{T}$ by

$$\mathcal{B}_{n-1}(z) \triangleq \frac{\sqrt{1-|\xi_n|^2}}{1-\overline{\xi_n}z}\,\phi_{n-1}(z), \qquad \phi_n(z) \triangleq \prod_{k=1}^{n} \frac{z-\xi_k}{1-\overline{\xi_k}z}, \quad \phi_0(z) \triangleq 1. \quad (1)$$

In the special case of $\xi_n = \xi \in \mathbf{R}$ this becomes the discrete-time Laguerre basis, and in the special case of $\xi_n = \xi \in \mathbf{C}$ this provides the discrete-time two-parameter Kautz basis; see [28] for more details on the generalising aspects of the definition (1).

In the case of considering continuous-time model descriptions, several important works have also recently appeared that employ continuous-time Laguerre and two-parameter Kautz bases [38,22,23] and rational wavelet bases [12]. The purpose of this paper is to make some further contribution in this area by considering some issues related to a natural generalisation of continuous-time Laguerre and Kautz bases.

The generalisation to be considered is analogous to the extension presented in (1). Namely, a set of basis functions $\{B_n(s)\}$ are treated which are defined by a choice of numbers $\{a_k\} \in \Pi$, $\forall k$ as

$$B_n(s) \triangleq \frac{\sqrt{2\mathrm{Re}\{a_n\}}}{s+a_n}\varphi_{n-1}(s), \qquad \varphi_n(s) \triangleq \prod_{k=1}^{n} \frac{s-\overline{a_k}}{s+a_k}, \ \ \varphi_0(s) \triangleq 1. \quad (2)$$

We set $B_0(s) \equiv 1$. The rational basis functions $\{B_n\}_{n\geq 1}$ are orthonormal in $H_2(\Pi)$ with respect to the inner-product (see § 4)

$$\langle B_n, B_m \rangle \triangleq \frac{1}{2\pi}\int_{-\infty}^{\infty} B_n(j\omega)\overline{B_m(j\omega)}\,\mathrm{d}\omega = \begin{cases} 1\ ; m = n \\ 0\ ; m \neq n. \end{cases}$$

Analogous to the discrete-time case, the continuous-time Laguerre basis (studied, for example, in [38,21,30]) is obtained as a special case of (2) by the choice $a_n = a \in \mathbf{R}$ and the continuous time two-parameter Kautz basis (studied in [38]) by the choice $a_n = a \in \mathbf{C}$.

It should be acknowledged that both the continuous and discrete-time generalisations shown in (2) and (1) enjoy a long history in both the pure mathematics [35,24,13] and engineering literature [33,25,18].

3 Main Result

As mentioned in the introduction, an important motivation for the consideration of orthonormal parameterisations is for approximation purposes. In this setting, a dominant question must arise as to the quality of the approximation. Pertaining to this, one of the most fundamental properties that might be required is that linear combinations of the basis elements be capable of arbitrarily good approximation.

Put more precisely in the context of systems theory, this involves considering an element $f(s)$ living in a normed linear function space $(X, \|\cdot\|_X)$, and for arbitrary $\epsilon > 0$ and for sufficiently large n being able to find an element $g(s) \in \mathrm{sp}\{B_k(s)\}_{k=1}^n$ such that $\|f-g\|_X \leq \epsilon$. Here X is a complex vector space and the linear span is with respect to the field of complex numbers.

If this is in fact possible for arbitrarily small ϵ, then $\mathrm{sp}\{B_k(s)\}_{k\geq 1}$ is said to be 'complete' in X. The choice of the function space depends on the application of the approximate model, but for quadratic optimal control purposes or mean square optimal prediction purposes, the choice $H_2(\Pi)$ would be appropriate, while for robust control or estimation purposes, the choices $A(\Pi)$ or $H_p(\Pi)$ (for large p) would be suitable.

With regard to the discrete-time basis (1), the approximation issues have been addressed in [3] where the following result was obtained.

Theorem 1 *[3, Theorem 6 and Corollary 7], Consider the set of functions* $\{\mathcal{B}_k(z)\}$ *defined by (1). Then the set* $X = \mathrm{sp}\{\mathcal{B}_k(z)\}_{k\geq 0}$ *is complete in* $A(\mathbf{D})$ *and* $H_p(\mathbf{T})$ *for all* $1 \leq p < \infty$ *if and only if*

$$\sum_{k=1}^{\infty}(1 - |\xi_k|) = \infty. \tag{3}$$

The main result of this paper is to establish an analogous result for the continuous-time basis (2) as follows.

Theorem 2 *The model set spanned by the basis functions* $\{B_n(s)\}_{n\geq 0}$ *is complete in all of the spaces* $H_p(\Pi)$, $1 < p < \infty$ *and* $A(\Pi)$ *if and only if*

$$\sum_{n=1}^{\infty}\frac{\mathrm{Re}\{a_n\}}{1+|a_n|^2} = \infty. \tag{4}$$

The condition (4) is satisfied by the Laguerre and two-parameter Kautz bases and the rational wavelets in [12]. In fact, it is a very mild condition, since the only way it could be violated is to choose a sequence of poles with fixed real part, and imaginary part diverging to infinity at a faster than linear rate.

In contrast to the Laguerre and two-parameter Kautz bases, where all the poles are fixed at the same value, the general basis (2) enjoys increased flexibility of pole location. For example, slow and fast modes may coexist in the model structure. As a result, a fewer number of basis functions (and hence, for system identification applications, a fewer number of data) may be used without sacrificing modelling accuracy. This feature is quantified in § 5.

Note that the conclusion of Theorem 2 may be extended to $H_1(\Pi)$. Moreover, it is possible to construct orthonormal model sets that are norm dense in $H_p(\Pi)$, $1 \leq p < \infty$ and have a prescribed asymptotic order [4]. In [3], it is also shown that the Fourier series formed by the general basis functions converge in all spaces $H_p(\Pi)$, $1 \leq p < \infty$.

4 Proof of Theorem 2

Before proceeding to the proof of Theorem 2, we shall demonstrate that the basis functions in (2) are indeed orthonormal. When $n > m$, we have by Cauchy's Integral Theorem [34]

$$
\begin{aligned}
\langle B_n, B_m\rangle &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sqrt{\mathrm{Re}\{a_n\}\mathrm{Re}\{a_m\}}}{(j\omega+a_n)(j\omega+a_m)} \prod_{k=m}^{n-1} \frac{j\omega-\overline{a_k}}{j\omega+a_k}\,\mathrm{d}\omega \\
&= -\frac{1}{2\pi j}\int_{j\mathbf{R}} \frac{2\sqrt{\mathrm{Re}\{a_n\}\mathrm{Re}\{a_m\}}}{(s+a_n)(s+a_m)} \prod_{k=m}^{n-1} \frac{s-\overline{a_k}}{s+a_k}\,\mathrm{d}s \\
&= -\lim_{r\to\infty}\left[\frac{1}{2\pi j}\int_{-jr}^{jr} \frac{2\sqrt{\mathrm{Re}\{a_n\}\mathrm{Re}\{a_m\}}}{(s+a_n)(s+a_m)}\mathrm{d}s + O\left(r^{-1}\right)\right] \\
&= \lim_{r\to\infty}\left[\frac{1}{2\pi j}\oint_{\Gamma_r} \frac{2\sqrt{\mathrm{Re}\{a_n\}\mathrm{Re}\{a_m\}}}{(s+a_n)(s+a_m)}\mathrm{d}s + O\left(r^{-1}\right)\right] \\
&= 0
\end{aligned}
\tag{5}
$$

where $a_0 \triangleq 0$ and the closed path Γ_r consists of a segment of the imaginary axis and an r radius semicircle in Π centred at the origin and is traversed counter clockwise. When $n = m$, we have

$$
\langle B_n, B_n\rangle = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Re}\{a_n\}}{\omega^2 + j\omega(\overline{a_n} - a_n) + |a_n|^2}\,\mathrm{d}\omega = 1
$$

where the second equality follows from the formula 3.3.16 in [1]

$$
\int \frac{1}{ax^2+bx+c}\,\mathrm{dx} = \frac{2}{\sqrt{4\mathrm{ac}-\mathrm{b}^2}}\,\tan^{-1}\frac{2\mathrm{ax}+\mathrm{b}}{\sqrt{4\mathrm{ac}-\mathrm{b}^2}}.
$$

Now we return to the proof of Theorem 2. Consideration is first given to the underlying space being $A(\Pi)$.

Lemma 3 *The linear span of the set $\{B_n(s)\}_{n\geq 0}$ with $B_n(s)$ defined in (2) is complete in $A(\Pi)$ if (4) holds.*

PROOF. This will be established by first addressing the completeness of $\mathrm{sp}\{\varphi_n(s)\}_{n\geq 0}$ in $A(\Pi)$. Notice that since the bilinear map

$$
s = \frac{1-z}{1+z}
$$

preserves the supremum norms between $A(\Pi)$ and $A(\mathbf{D})$, the question of whether $\mathrm{sp}\{\varphi_n(s)\}_{n\geq 0}$ is complete in $A(\Pi)$ is equivalent to the question of the completeness of $\{\varphi_n(\frac{1-z}{1+z})\}_{n\geq 0}$ in $A(\mathbf{D})$. Let

$$
\xi_n \triangleq \frac{1-\overline{a_n}}{1+\overline{a_n}}, \qquad \alpha_n \triangleq (-1)^n \prod_{\ell=1}^{n} \frac{1+\overline{a_\ell}}{1+a_\ell}. \tag{6}
$$

Then provided $\mathrm{Re}\{a_n\} > 0$ for all n, $\xi_n \in \mathbf{D}$ for all n. Also

$$\varphi_n\left(\frac{1-z}{1+z}\right) = (-1)^n \prod_{\ell=1}^{n} \frac{1+\overline{a_\ell}}{1+a_\ell} \prod_{k=1}^{n} \frac{z-\xi_k}{1-\overline{\xi_k} z} = \alpha_n \phi_n(z).$$

Therefore, it is sufficient to establish the completeness of $\mathrm{sp}\{\phi_n(z)\}_{n\geq 0}$ in $A(\mathbf{D})$. To achieve this, consider the functions $\{\mathcal{B}_k(z)\}$ defined in (1). Then

$$\mathcal{B}_{n-1}(z) = \frac{\overline{\xi_n}\phi_n(z) + \phi_{n-1}(z)}{\sqrt{1-|\xi_n|^2}}; \qquad n \geq 1$$

and hence $\mathrm{sp}\{\mathcal{B}_k(z)\}_{k=0}^{n-1} \subset \mathrm{sp}\{\phi_k(z)\}_{k=0}^{n}$. However by Theorem 1, $\mathrm{sp}\{\mathcal{B}_k(z)\}_{k\geq 0}$ is complete in $A(\mathbf{D})$ if and only if (3) is satisfied.

Now, since

$$1 + |a_n|^2 \geq |1 - a_n^2| = |1 - a_n||1 + a_n|$$

then by the definition in (6)

$$|\xi_n| = \left|\frac{1-\overline{a_n}}{1+\overline{a_n}}\right| \geq \frac{|1-a_n|^2}{1+|a_n|^2}$$

so that

$$\sum_{n=1}^{\infty}(1-|\xi_n|) \leq \sum_{n=1}^{\infty}\left(1 - \frac{|1-a_n|^2}{1+|a_n|^2}\right) = 2\sum_{n=1}^{\infty}\frac{\mathrm{Re}\{a_n\}}{1+|a_n|^2}.$$

Conversely since

$$\left|\frac{1-\overline{a_n}}{1+\overline{a_n}}\right| = \left|\frac{1-a_n}{1+a_n}\right|\left|\frac{1+a_n}{1+a_n}\right| = \frac{|1-a_n^2|}{|1+a_n|^2} \leq \frac{1+|a_n^2|}{|1+a_n|^2}$$

then

$$\sum_{n=1}^{\infty}(1-|\xi_n|) \geq \sum_{n=1}^{\infty}\left(1 - \frac{1+|a_n^2|}{|1+a_n|^2}\right) = 2\sum_{n=1}^{\infty}\frac{\mathrm{Re}\{a_n\}}{|1+a_n|^2}.$$

As well, when $|a_n| < 2$ then

$$|1+a_n|^2 \leq 1 + |a_n|^2 + 2|a_n| \leq 5(1+|a_n|^2)$$

while when $|a_n| \geq 2$ then

$$|1+a_n|^2 \leq 1 + |a_n|^2 + 2|a_n| \leq 1 + 2|a_n|^2 \leq 2(1+|a_n|^2)$$

so that

$$\frac{2}{5}\sum_{n=1}^{\infty}\frac{\text{Re}\{a_n\}}{1+|a_n|^2} \leq \sum_{n=1}^{\infty}(1-|\xi_n|) \leq 2\sum_{n=1}^{\infty}\frac{\text{Re}\{a_n\}}{1+|a_n|^2}.$$

Therefore, (3) holds if and only if (4) holds implying that under the definition (6), then (4) is necessary and sufficient for $\text{sp}\{\mathcal{B}_k(z)\}_{k\geq 0}$ to be complete in $A(\mathbf{D})$ and hence for $\text{sp}\{\varphi_n(s)\}_{n\geq 0}$ to be complete in $A(\Pi)$. Summing the identity

$$\sqrt{2\text{Re}\{a_k\}}B_k(s) = \varphi_{k-1}(s) - \varphi_k(s); \qquad n \geq 1$$

over $k = 1, \cdots, n$ then provides

$$\sum_{k=1}^{n}\sqrt{2\text{Re}\{a_k\}}B_k(s) = 1 - \varphi_n(s) = B_0(s) - \varphi_n(s).$$

Hence $\text{sp}\{\varphi_n(s)\}_{n\geq 0} \subset \text{sp}\{B_n(s)\}_{n\geq 0}$. This completes the proof.

Next, the question of the $H_p(\Pi)$ sufficiency of (4) is considered.

Lemma 4 *Suppose that (4) is satisfied. Then* $\text{sp}\{B_n\}_{n\geq 1}$ *is complete in* $H_p(\Pi)$ *for all* $1 < p < \infty$.

PROOF. Let m denote the multiplicity of a_1. Suppose first that m is finite. Then reorder the basis poles so that $a_1 = a_2 = \cdots = a_m$. We redefine the basis functions in (2) by

$$\widetilde{B}_n(s) \triangleq \begin{cases} B_n(s), & n < m, \\ \left(\dfrac{s+a_1}{s-\overline{a_1}}\right) B_{n+1}(s), & n \geq m. \end{cases} \tag{7}$$

Let Q_n denote the set containing all partial fraction expansion terms of $\widetilde{B}_n$. For example,

$$Q_m \triangleq \left\{1, \frac{1}{s+a_1}, \cdots, \frac{1}{(s+a_1)^{m-1}}, \frac{1}{s+a_{m+1}}\right\}$$

and so on. Let

$$Q \triangleq \bigcup_{n=0}^{\infty} Q_n.$$

Since

$$\sum_{k\neq m} \frac{\mathrm{Re}\{a_k\}}{1+|a_k|^2} = \infty,$$

then $\mathrm{sp}\{\widetilde{B}_n\}_{n\geq 0}$ is a complete set in $A(\Pi)$ by Lemma 3, and so is $\mathrm{sp}\, Q$ since

$$\mathrm{sp}\{\widetilde{B}_n\}_{n\geq 0} \subset \mathrm{sp}\, Q.$$

Let $f \in H_p(\Pi)$ and $\epsilon > 0$. Approximate f by a function $g \in A(\Pi)$ that has the properties

$$\lim_{|s|\to\infty} |s||g(s)| = 0, \;\; s \in \Pi,$$

and

$$\|f - g\|_p < \epsilon.$$

This is possible since such functions form a dense subset of $H_p(\Pi)$ (see for example, Garnett [16, Corollary 3.3]). Let $h(s) = (s+a_1)g(s)$. Then $h \in A(\Pi)$. Since $\mathrm{sp}\, Q$ is a complete set in $A(\Pi)$, there exists a $u \in \mathrm{sp}\, Q$ such that

$$\|h - u\|_\infty < \epsilon$$

or

$$\left| g(s) - \frac{u(s)}{s+a_1} \right| < \frac{\epsilon}{s+a_1}, \;\; s \in \Pi.$$

Hence

$$\left\| f - \frac{u}{s+a_1} \right\|_p < \epsilon + \left\| \frac{1}{s+a_1} \right\|_p \epsilon.$$

We have shown that the set

$$P \triangleq \left\{ \frac{v(s)}{s+a_1} : v(s) \in \mathrm{sp}\, Q \right\} \tag{8}$$

is a dense subset of $H_p(\Pi)$ for all $1 < p < \infty$. It remains to show that $P \subset \mathrm{sp}\{B_n\}_{n\geq 1}$. This will imply that $\mathrm{sp}\{B_n\}_{n\geq 1}$ is a complete set in $H_p(\Pi)$ for all $1 < p < \infty$. To this end, let $v \in \mathrm{sp}\, Q$. Since $v \in \mathrm{sp}\, Q$, it can be written uniquely as a linear combination of the elements in Q as follows

$$v(s) = c_0 + \sum_{k=1}^{\infty} \frac{c_k}{(s+a_k)^{N(k)}}$$

where $N(k)$ denotes the multiplicity of a_k in the set $\{a_1, a_2, \cdots, a_k\}$ and only a finite number of the coefficients c_k are nonzero. Notice that $c_m = 0$. Then

$$\frac{v(s)}{s+a_1} = \frac{c_0}{s+a_1} + \cdots + \frac{c_{m-1}}{(s+a_1)^m} + \sum_{k=m+1}^{\infty} \frac{c_k}{(s+a_1)(s+a_k)^{N(k)}}.$$

Since $a_k \neq a_1$ for $k > m$, the terms $c_k/(s+a_1)(s+a_k)^{N(k)}$ admit further expansions

$$\frac{c_k}{(s+a_1)(s+a_k)^{N(k)}} = \frac{d_1}{s+a_1} + \frac{d_2}{s+a_k} + \cdots + \frac{d_{N(k)}}{(s+a_k)^{N(k)}}.$$

Hence

$$\frac{v(s)}{s+a_1} \in \mathrm{sp}\left\{\frac{1}{s+a_1}, \cdots, \frac{1}{(s+a_1)^m}, \frac{1}{s+a_{m+1}}, \cdots\right\} = \mathrm{sp}\, F$$

where

$$F \triangleq Q \bigcup \left\{\frac{1}{(s+a_1)^m}\right\}. \tag{9}$$

Thus $P \subset \mathrm{sp}F$.

To complete the proof for $m < \infty$, we need to show that $\mathrm{sp}F \subset \mathrm{sp}\{B_n\}_{n\geq 1}$. Let n be an arbitrary positive integer. Write the partial fraction expansions of the basis elements $B_1, B_2, \cdots, B_n$ in the following linear equation form

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{s+a_1} \\ \frac{1}{(s+a_2)^{N(2)}} \\ \vdots \\ \frac{1}{(s+a_2)^{N(n)}} \end{bmatrix}.$$

The degree of B_k is k, which implies that $\alpha_{kk} \neq 0$ for all $k \leq n$ and thus the lower triangular matrix above is invertible. Hence, for $i = 1, \cdots, n$

$$\frac{1}{(s+a_i)^{N(i)}} \in \mathrm{sp}\,\{B_k,\ k = 1, \cdots, n\} \subset \mathrm{sp}\{B_k\}_{k\geq 1}.$$

Consequently $\mathrm{sp}F \subset \mathrm{sp}\{B_n\}_{n\geq 1}$.

Suppose now that $m = \infty$. Then for each n, define Q_n as the set containing all partial fraction expansion terms of B_n. In this case, the proof above still applies with great simplifications since P defined in (8) equals $\mathrm{sp}\, Q$, and F defined in (9) equals Q for all m.

It only remains to establish the necessity of (4) for completeness in $H_p(\Pi)$ spaces. Suppose then that (4) fails to hold. Then in this case the finite Blaschke products $\lambda_n(s)$ defined by

$$\lambda_n(s) \triangleq \beta_n \varphi_n(s), \qquad \beta_n \triangleq \prod_{k=1}^{n} \frac{|1-\overline{a_n}^2|}{1-\overline{a_n}^2}, \quad \beta_0(s) \triangleq 1$$

converge (as $n \to \infty$) uniformly on Π to a non-zero function $\lambda(s) \in H_\infty(\Pi)$ which has zeros precisely at the points $\overline{a_n}$; see, for example, Garnett [16, Chapter II]. Therefore, the linear functional F defined on $H_p(\Pi)$ for all $1 \leq p < \infty$ and $A(\Pi)$ by

$$F(h) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} h(j\omega) \frac{\overline{\lambda(j\omega)}}{(-j\omega+1)^2} \, \mathrm{d}\omega$$

is nontrivial and bounded. However, by Cauchy's Integral Theorem, it vanishes for any B_n of the form (2) since

$$\begin{aligned}
\overline{F(B_n)} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-\sqrt{2\mathrm{Re}\{a_n\}}}{(j\omega+1)^2(j\omega-\overline{a_n})} \prod_{k=1}^{n-1} \frac{j\omega+a_k}{j\omega-\overline{a_k}} \prod_{i=1}^{\infty} \frac{|1-\overline{a_i}^2|}{1-\overline{a_i}^2} \frac{j\omega-\overline{a_i}}{j\omega+a_i} \, \mathrm{d}\omega \\
&= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Psi(j\omega)}{(j\omega+1)^2(j\omega+a_n)} \, \mathrm{d}\omega \\
&= -\frac{1}{2\pi j} \int_{j\mathbf{R}} \frac{\Psi(s)}{(s+1)^2(s+a_n)} \, \mathrm{d}s \\
&= -\lim_{r\to\infty} \left[\frac{1}{2\pi j} \int_{-jr}^{jr} \frac{\Psi(s)}{(s+1)^2(s+a_n)} \, \mathrm{d}s + O\left(r^{-2}\right) \right] \\
&= \lim_{r\to\infty} \left[\frac{1}{2\pi j} \oint_{\Gamma_r} \frac{\Psi(s)}{(s+1)^2(s+a_n)} \, \mathrm{d}s + O\left(r^{-2}\right) \right] \\
&= 0
\end{aligned}$$

where

$$\Psi(s) \triangleq \sqrt{2\mathrm{Re}\{a_n\}} \prod_{k=1}^{n} \frac{|1-\overline{a_k}^2|}{1-\overline{a_k}^2} \prod_{i=n+1}^{\infty} \frac{|1-\overline{a_i}^2|}{1-\overline{a_i}^2} \frac{s-\overline{a_i}}{s+a_i}$$

is analytic on Π and the closed path Γ_r is as in (5). Similarly, $F(B_0) = 0$. (Note that the remainder term above vanishes as $O\left(r^{-1}\right)$ in this case). Hence by an application of the Hahn-Banach Theorem (see, for example [2, § 30]), $\text{sp}\{B_n\}_{n\geq 0}$ defined by (2) is not dense in any of the spaces $A(\Pi)$ and $H_p(\Pi)$, $1 \leq p < \infty$. This concludes the proof.

It should be pointed out that the sufficiency part of Lemma 3 together with Lemma 4 gives the half of the sufficiency condition in Achieser [2, § A.4] for the completeness of the model sets spanned by the Cauchy kernels $1, 1/(s+a_1), 1/(s+a_2), \cdots$ in the Lebesgue spaces L_p, $1 < p < \infty$ and in the space of complex functions continuous on the imaginary axis including ∞. The linear span of the Cauchy kernels does not contain systems which have repeated poles whereas with the bases (2), a greater flexibility is utilised on the choice of basis poles.

4.1 Ensuring Real-Valued Impulse Response

Up until this point, the basis (2) has been considered with complete generality of pole location save for the condition (4). However, in any application involving the modelling of a physical process, it is necessary to ensure that the underlying modelled impulse response is real valued. If complex valued choices for $\{a_k\}$ are made in order to accommodate resonant characteristics, then this realness of impulse response is lost unless some restriction is placed on how linear combination of the basis functions are taken.

The purpose of this section is to illustrate how to use the basis formulation (2) in such a way that imposing realness of the weightings in the linear combination ensures realness of the underlying impulse response. This is achieved by requiring that if a *set* of poles $\{a_1, a_2, \cdots, a_n\}$ used to define bases via (2) contains a complex valued element (say a_k), then it always also includes its conjugate $\overline{a_k}$.

To be more explicit on this point, suppose $n-1$ poles $\{a_1, \cdots, a_{n-1}\}$ have been included in $\{B_1, \cdots, B_{n-1}\}$ and we now wish to include a complex pole at a_n. Then two new basis functions $\mathcal{B}'_n, B''_n$ with real impulse responses should be formed as linear combinations of B_n, B_{n+1} generated by (2). These new functions then replace B_n and B_{n+1} in any modelling applications that require a real valued impulse response.

The linear combination we are suggesting can be expressed as

$$\begin{pmatrix} B'_n \\ B''_n \end{pmatrix} = \begin{pmatrix} c_0 & c_1 \\ c'_0 & c'_1 \end{pmatrix} \begin{pmatrix} B_n \\ B_{n+1} \end{pmatrix}, \qquad c_0, c'_0, c_1, c'_1 \in \mathbf{C}. \tag{10}$$

Therefore, considering only B'_n for the moment, if we choose complex poles in conjugate pairs as $a_{n+1} = \overline{a_n}$ then

$$B_n'(s) = \frac{\sqrt{2\text{Re}\{a_n\}}(\beta s + \mu)}{s^2 + (a_n + \overline{a_n})s + |a_n|^2} \prod_{k=1}^{n-1} \left(\frac{s - \overline{a_k}}{s + a_k}\right) \tag{11}$$

where the co-efficients β, μ are related to the choice of c_0, c_1 by

$$c_0 = \frac{\overline{a_n}\beta + \mu}{2\overline{a_n}}, \qquad c_1 = \frac{\overline{a_n}\beta - \mu}{2\overline{a_n}}.$$

Therefore, to ensure a unit norm for B_n' we must choose β and μ according to the constraint that $|c_0|^2 + |c_1|^2 = 1$ which (constraining β and μ to be real) becomes

$$x^T M x = 2|a_n|^2 \tag{12}$$

where

$$x \triangleq (\beta, \mu)^T, \qquad M \triangleq \begin{pmatrix} |a_n|^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, suppose we make two pairs of choices $x = (\beta, \mu)$ giving a basis function B_n' and $y = (\beta', \mu')$ giving a basis function $\mathcal{B}_n''$. These two choices correspond to two pairs of choices $\{c_0, c_1\}$ and $\{c_0', c_1'\}$. The requirement $c_0\overline{c_0'} + c_1\overline{c_1'} = 0$ ensuring orthogonality of B_n' and $\mathcal{B}_n''$ can be expressed as needing

$$x^T M y = 0 \tag{13}$$

to hold, and in fact many solutions x and y to (12) and (13) will exist.

To formulate them, suppose we begin by choosing any x satisfying (12). Then a y that also satisfies (12) but also satisfies (13) may be found by rotating x by ninety degrees in the normalised eigenspace of M:

$$y = M^{-1/2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} M^{1/2} x$$

or, to be more explicit

$$\begin{pmatrix} \beta' \\ \mu' \end{pmatrix} = \begin{pmatrix} 0 & -1/|a_n| \\ |a_n| & 0 \end{pmatrix} \begin{pmatrix} \beta \\ \mu \end{pmatrix} \tag{14}$$

To summarise this discussion, if we want to include complex modes in a model structure, then we obtain two basis vectors B'_n and $\mathcal{B}''_n$ from two linear combinations of B_{n-1} and B_n that come from the unifying construction (2). The first basis function B'_n is found as

$$B'_n(s) = \frac{\sqrt{2\text{Re}\{a_n\}}(\beta s + \mu)}{s^2 + (a_n + \overline{a_n})s + |a_n|^2} \prod_{k=1}^{n-1} \left(\frac{s - \overline{a_k}}{z + a_k} \right)$$

where $x^T = (\beta, \mu)$ is chosen to lie anywhere on the ellipse (12). A vector $y^T = (\beta', \mu')$ is then found that also lies on the ellipse (12) by using the formula (14). The second basis function B''_n is then obtained as

$$B''_n(s) = \frac{\sqrt{2\text{Re}\{a_n\}}(\beta' s + \mu')}{s^2 + (a_n + \overline{a_n})s + |a_n|^2} \prod_{k=1}^{n-1} \left(\frac{s - \overline{a_k}}{s + a_k} \right)$$

These real valued impulse response basis vectors B'_n and B''_n are then used for modelling instead of B_n and $\mathcal{B}_{n+1}$. If we require further basis functions with complex modes then we repeat the process in (10) by forming $\mathcal{B}'_{n+1}$ and B''_{n+1} from linear combinations of B_{n+2} and B_{n+3} and so on, and in this way arbitrary complex pole configurations (that satisfy the completeness condition (4)) may be accommodated.

Having now illustrated how the constraint of realness of impulse response may be easily accommodated via constraining realness of linear combination weights, it remains to establish that this latter restriction does not destroy completeness properties. For this purpose, note that via Theorem 2, for any $G \in H_p(\Pi)$ and any $\epsilon > 0$ there exists a $G_n \in H_p(\Pi)$ given by

$$G_n = \sum_{k=1}^{n} \theta_k B_k(s), \qquad \theta_k = a_k + jb_k; \;\; a_k, b_k \in \mathbf{R}$$

such that $\|G - G_n\|_p \leq \epsilon/2$, where here the bases $\{B_k(s)\}$ refer to a set that may include elements B'_k, B''_k of the form described above so that all the elements of $\{B_k(s)\}$ have real valued impulse responses.

Now, assume that $G(s)$ has a real valued impulse response so that $\overline{G(j\omega)} = G(-j\omega)$, and write $\widetilde{G}_n = G - \sum_{k=1}^{n} a_k B_k$, $G_n^I = j \sum_{k=1}^{n} b_k B_k$. Then, using the fact that $\|f\| = \|\overline{f}\|$ for any $f \in H_p(\Pi)$ provides

$$\|G - G_n\|_p \leq \frac{\epsilon}{2} \;\Rightarrow\; \|\widetilde{G}_n - G_n^I\|_p \leq \frac{\epsilon}{2} \;\Rightarrow\; \|\widetilde{G}_n + G_n^I\|_p \leq \frac{\epsilon}{2}$$

so that via the triangle inequality

$$2\|G_n^I\|_p \le \|G_n^I + \widetilde{G}_n\|_p + \|G_n^I - \widetilde{G}_n\|_p \le \epsilon$$

and hence

$$\left\| G - \sum_{k=1}^{n} a_k B_k \right\|_p \le \|G - G_n\|_p + \|G_n^I\|_p \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so that indeed, arbitrarily accurate modelling of real impulse $G(s)$ is possible by taking real linear combinations of real impulse versions of the bases $\{B_k\}$.

5 Approximation of finite-dimensional systems

While the completeness result of Theorem 2 provides a theoretical pedigree for considering the bases (2) for system approximation purposes, it leaves open the question of the quality of approximation for a finite number of bases. Addressing this will be the concern of this section, where a central tool is to use the so-called 'reproducing kernel' $K_n(s,\mu)$ associated with the linear space $\mathrm{sp}\{B_k(s)\}_{k=1}^n$.

Lemma 5 *Consider the basis functions $\{B_k\}_{k=1}^n$ defined by (2). Then*

$$K_n(s,\mu) \triangleq \sum_{k=1}^{n} \overline{B_k(\mu)} B_k(s) = \frac{1 - \overline{\varphi_n(\mu)}\varphi_n(s)}{s + \overline{\mu}}.$$

PROOF. The proof will be by induction. First, when n=1

$$\overline{B_1(\mu)} B_1(s) = \frac{a_1 + \overline{a_1}}{(\overline{\mu} + \overline{a_1})(s + a_1)}$$

while

$$\frac{1 - \overline{\varphi_1(\mu)}\varphi_1(s)}{s + \overline{\mu}} = \left[1 - \frac{(\overline{\mu} - a_1)(s - \overline{a_1})}{(\overline{\mu} + \overline{a_1})(s + a_1)}\right] \frac{1}{s + \overline{\mu}} = \frac{a_1 + \overline{a_1}}{(\overline{\mu} + \overline{a_1})(s + a_1)}$$

so that the result holds for $n = 1$. Now suppose it holds for $n > 1$. Then

$$K_n(s,\mu) = K_{n-1}(s,\mu) + \overline{B_n(\mu)} B_n(s)$$

$$= \frac{1-\overline{\varphi_{n-1}(\mu)}\varphi_{n-1}(s)}{s+\overline{\mu}} + \frac{a_n+\overline{a_n}}{(\overline{\mu}+\overline{a_n})(s+a_n)}\overline{\varphi_{n-1}(\mu)}\varphi_{n-1}(s)$$
$$= \frac{1}{s+\overline{\mu}}\left\{1-\left[\frac{(\overline{\mu}+\overline{a_n})(s+a_n)-(a_n+\overline{a_n})(s+\overline{\mu})}{(\overline{\mu}+\overline{a_n})(s+a_n)}\right]\overline{\varphi_{n-1}(\mu)}\varphi_{n-1}(s)\right\}$$
$$= \frac{1-\overline{\varphi_n(\mu)}\varphi_n(s)}{s+\overline{\mu}}.$$

Therefore, by induction the result holds for all n.

The utility of this result becomes apparent in the derivation of the following expression for the finite order approximation error.

Lemma 6 *Suppose $f(s)$ is analytic on Π and has a partial fraction expansion*

$$f(s) = \sum_{k=1}^{m} \frac{c_k}{s+\gamma_k}.$$

Define $f_n(s)$ as an approximation to $f(s)$ obtained by projection onto $\text{sp}\{B_k(s)\}_{k=1}^{n}$*:*

$$f_n(s) \triangleq \sum_{k=1}^{n} \langle f, B_k\rangle B_k(s).$$

Then

$$|f(j\omega)-f_n(j\omega)| \leq \sum_{k=1}^{m}\left|\frac{c_k}{j\omega+\gamma_k}\right|\prod_{\ell=1}^{n}\left|\frac{\gamma_k-a_\ell}{\gamma_k+\overline{a_\ell}}\right|. \tag{15}$$

PROOF. By the definition of $f_n(s)$ and for any $\mu \in \Pi$

$$f_n(\mu) = \sum_{k=1}^{n}\left(\frac{1}{2\pi j}\int_{j\mathbf{R}} f(s)\overline{B_k(s)}\,\mathrm{d}s\right)B_k(\mu) = \frac{1}{2\pi j}\oint_{\Gamma} f(s)\overline{K_n(s,\mu)}\,\mathrm{d}s$$

where Γ consists of the imaginary axis and an infinite radius semi-circle in the open right half plane; it is traversed clockwise. Using this definition and Cauchy's integral formula gives, for $\mu \in \Pi$ arbitrary

$$f(\mu) = \frac{1}{2\pi j}\oint_{\Gamma}\frac{f(s)}{\mu-s}\,\mathrm{d}s.$$

Therefore, by Lemma 5 and using the fact that $\overline{s} = -s$ for $s \in j\mathbf{R}$

$$
\begin{aligned}
|f(\mu) - f_n(\mu)| &= \left| \frac{1}{2\pi j} \oint_\Gamma \frac{f(s)}{\mu - s} \varphi_n(\mu) \overline{\varphi_n(s)} \, \mathrm{d}s \right| \\
&= \left| \frac{\varphi_n(\mu)}{2\pi j} \sum_{k=1}^{m} c_k \oint_\Gamma \frac{1}{(s+\gamma_k)(\mu - s)} \prod_{\ell=1}^{n} \frac{s + a_\ell}{s - \overline{a_\ell}} \mathrm{d}s \right| \\
&= |\varphi_n(\mu)| \left| \sum_{k=1}^{m} c_k \frac{1}{(\mu + \gamma_k)} \prod_{\ell=1}^{n} \frac{\gamma_k - a_\ell}{\gamma_k + \overline{a_\ell}} \right|
\end{aligned}
$$

Where in moving to the last line Cauchy's residue theorem was used to evaluate the integral after performing the change of variable $s \mapsto -s$. Taking the limit as $\mathrm{Re}\{\mu\} \to 0$ then gives the result.

The result exposes the dependence of the approximation error on the choice of poles $\{a_n\}$ in the base $B_n(s)$. Namely, the closer the poles $\{a_n\}$ are chosen to the poles $\{\gamma_k\}$ of the function $f(s)$ being approximated then the more accurate the approximation of $f(s)$ will be, and in such a way as to decrease exponentially with increasing n.

Certainly the error bound (15) gives strong motivation for the consideration of the general basis (2), since (in contrast to the Laguerre and Kautz cases where all the poles are fixed at the same value) the increased flexibility of pole location $\{a_n\}$ will increase the possibility of making $|\gamma_k - a_\ell|$ small (for some ℓ) for every k, and hence making the total product $\prod_{\ell=1}^{n} |\gamma_k - a_\ell||\gamma_k + \overline{a_\ell}|^{-1}$ as small as possible.

6 Application Example

We conclude the paper by presenting an application example that illustrates the utility of the basis (2) for modelling purposes. The example involves measurements of the frequency response of a 58cm long, 5mm wide cantilevered piezo-electric laminate beam (for further details, see [26]). These measurements are shown as dots in figure 1. For the purposes of control (stiffness compensation using the piezo-electric actuators) a transfer function model that explains this frequency response is required. There are many ways in which this may be achieved [32], but for the purposes of illustrating the efficacy of the basis (2), the simple least-squares method of fitting a model

$$
G_n(s) = \frac{N(s)}{D(s)} = \frac{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}{s^n + d_{n-1} s^{n-1} + \cdots + d_1 s + d_0}
$$

to the measured frequency response $\{G(j\omega_1), \cdots, G(j\omega_N)\}$ by means of minimising the cost

$$V_N = \sum_{k=1}^{N} |D(j\omega_k)G(\omega_k) - N(\omega_k)|^2$$

will be studied. As is well known [32], finding this estimate involves solving the so-called 'Normal Equations'

$$\begin{bmatrix} G(j\omega_1) \\ \vdots \\ G(j\omega_N) \end{bmatrix} = \underbrace{\begin{bmatrix} (j\omega_1)^{n-1}, & \cdots 1, & (j\omega_1)^n, & \cdots 1 \\ \vdots & & & \vdots \\ (j\omega_N)^{n-1}, & \cdots 1, & (j\omega_N)^n, & \cdots 1 \end{bmatrix}}_{\Phi} \begin{bmatrix} d_{n-1} \\ \vdots \\ d_0 \\ b_n \\ \vdots \\ b_0 \end{bmatrix}.$$

for which the numerical stability of the solution is highly dependent [17], on the conditioning of the matrix $\Phi^T\Phi$. However this can be altered via re-parameterisations of the model $G(s)$. For example, in [6] the parameterisation

$$N(s) = b_0 + \sum_{k=1}^{n} b_k p_k(s), \qquad D(s) = s^n + \sum_{k=0}^{n-1} d_k p_k(s) \tag{16}$$

where each $p_k(s)$ is an order k 'modified Tchebychev' polynomial ($p_0 = 1$) is suggested as a means of improving numerical conditioning.

In figure 1, the dash-dot line shows the results of using the above Tchebychev parameterisation to fit an $n = 18$'th order model to the observed frequency response. Note that the second resonance peak is completely missed, and that there are three close pole-zero cancellation 'spikes' from the 4'th resonant mode onwards.

However, if the model is parameterised using the orthonormal basis (2) as

$$N(s) = b_0 + \sum_{k=1}^{n} b_k B_k(s), \qquad D(s) = s^n + \sum_{k=1}^{n} d_k B_k(s) \tag{17}$$

with the pole choice choice $a_k = a = 2\omega_N$, then the ensuing 18'th order least squares estimate is the solid line shown in figure 1, which now captures the second resonance peak, and does not have high frequency near pole-zero cancellations.

Since the model structures (16) and (17) both span the same manifold of rational models, the only explanation for the difference in results is that of differences in numerical conditioning. Figure 2 shows the singular values of Φ for three model parameterisation choices. Considering the log-scale employed, the parameterisation using the basis (17) enjoys a two order of magnitude better conditioning (ratio of largest to smallest singular value) than either a Tchebychev polynomial, or conventional polynomial parameterisation.

As a consequence, for such applications of modelling resonant structures over large bandwidths, we suggest that the basis (2) should be employed in the interests of the resultant frequency response having no artifacts due to poor numerical conditioning.

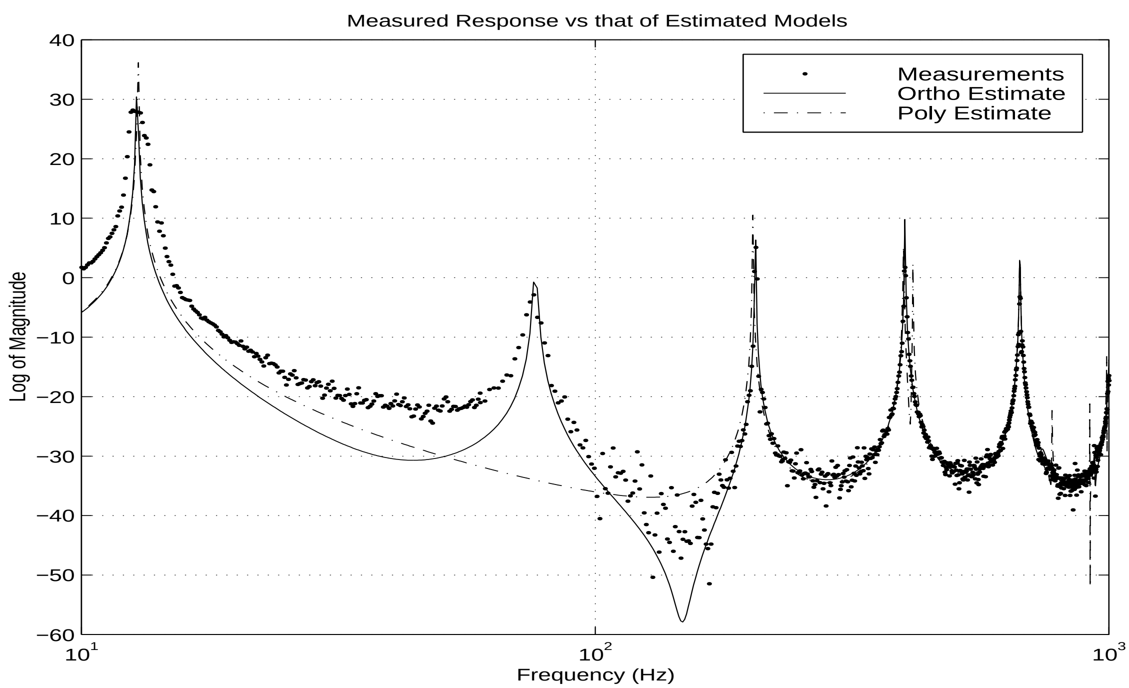

Fig. 1. *Estimation using polynomial and orthonormal basis. Dots are measurements, solid line is estimate using bases (2) to parameterise the model, dash dot line is estimate using Tchebychev polynomials to parameterise the model.*

7 Conclusion

This paper has provided a preliminary study of the approximation properties of a particular class of rational orthonormal bases that are suitable for continuous time system modelling. The main result was to establish that the bases

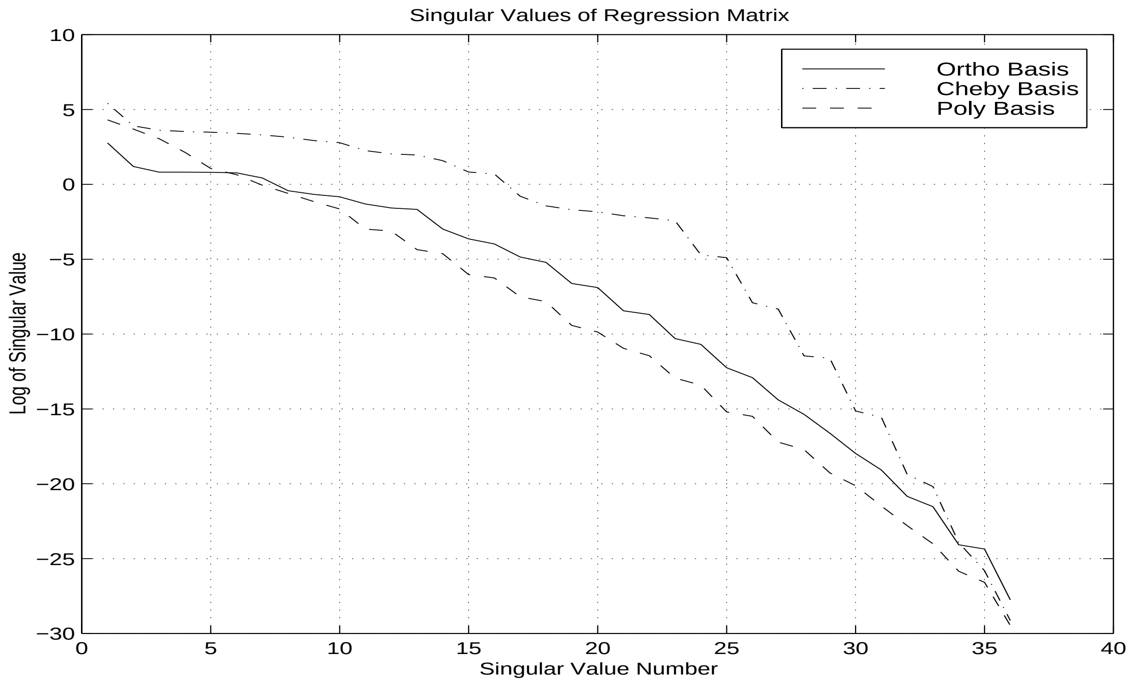

Fig. 2. *Singular Values of Φ using polynomial (natural and Tchebychev) and rational orthonormal basis (2).*

were capable of arbitrarily good approximation with respect to a wide variety of norms employed in the system theoretic analysis of stable systems. The utility of the generalising nature of the particular bases considered here was also exposed by establishing that significantly improved finite order approximation accuracy was possible by exploiting the flexibility in allowed pole position. This is in contrast to the more well known Laguerre and two-parameter Kautz bases, which are obtained as special cases of the bases considered here by choosing all the poles fixed at one location.

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