Sampling Zeros and the Euler–Frobenius Polynomials

Steven R. Weller, W. Moran, Brett Ninness, and A. D. Pollington

***Abstract*—In this note, we show that the zeros of sampled-data systems resulting from rapid sampling of continuous-time systems preceded by a zero-order hold (ZOH) are the roots of the Euler–Frobenius polynomials. Using known properties of these polynomials, we prove two conjectures of Hagiwara and coworkers, the first of which concerns the simplicity, negative realness, and interlacing properties of the sampling zeros of ZOH- and first-order hold (FOH-) sampled systems. To prove the second conjecture, we show that in the fast sampling limit, and as the continuous-time relative degree increases, the largest sampling zero for FOH-sampled systems approaches $1/e$, where e is the base of the natural logarithm.**

***Index Terms*—Fast sampling, first-order hold, limiting zeros, sampling zeros, zero-order hold.**

I. Introduction

It is well known that when the input of a continuous-time dynamical system described by a rational transfer functions $G(s)$ is generated by the piecewise constant output of a zero-order hold (ZOH), the system output at instants of time appropriately synchronized with the ZOH can be found using the z-transform [1]. In particular, the discrete-time transfer function (or pulse-transfer function) providing the link between input and output samples with sampling period T is given by

$$G_0(z) = Z\left[\frac{1-e^{-sT}}{s}\,G(s)\right] \tag{1}$$

where $Z[\cdot]$ denotes the z-transform. Likewise, when the system input is generated by a first-order causal extrapolation of sampled values, the sampled inputs and outputs are linked via the first-order hold (FOH)-equivalent transfer function

$$G_1(z) = Z\left[\frac{1+Ts}{Ts^2}\left(1-e^{-sT}\right)^2 G(s)\right]. \tag{2}$$

While the mapping of poles of $G(s)$ under (1) and (2) is readily established, it is difficult to say much about the mapping of finite zeros other than in the limit of fast $(T \to 0)$ or slow $(T \to \infty)$ sampling [2]–[4]. The following two theorems summarize the behavior of the sampled-data models arising from the ZOH- and FOH-sampling of $G(s)$ in the fast sampling limit; see also [5].

Theorem 1.1 (Åström et al. [2]): Suppose that $G(s)$ is a strictly proper rational function

$$G(s) = K\frac{(s-\gamma_1)\cdots(s-\gamma_m)}{(s-\lambda_1)\cdots(s-\lambda_n)}, \qquad n > m \tag{3}$$

where $\lambda_i \in \mathbf{C}$ $(i = 1, 2, \ldots, n)$, $\gamma_i \in \mathbf{C}$ $(i = 1, 2, \ldots, m)$, and $K \neq 0$. Then, for almost every sampling period T, the discrete-time transfer function $G_0(z)$ arising from ZOH-sampling of (3) has $n-1$ zeros. Furthermore, $G_0(z)$ approaches

$$K\,\frac{T^{n-m}}{(n-m)!}\,\frac{(z-1)^m B_{n-m}(z)}{(z-1)^n} \tag{4}$$

as $T \to 0$, where $B_{n-m}(z)$ is the reciprocal polynomial given by

$$B_p(z) = b_1^p z^{p-1} + b_2^p z^{p-2} + \cdots + b_p^p, \qquad p \geq 1 \tag{5}$$

where

$$b_k^p = \sum_{l=1}^{k} (-1)^{k-l} l^p \binom{p+1}{k-l}, \qquad k = 1, \ldots, p. \tag{6}$$

Using (6) and manipulations with binomial identities, it is a straightforward matter to establish that the coefficients of the limiting zero polynomials $\{B_p(z)\}_{p=1}^{\infty}$ can be computed using the following recursive procedure [2]:

$$b_1^p = b_p^p = 1, \tag{7}$$

$$b_k^p = k b_k^{p-1} + (p-k+1) b_{k-1}^{p-1}, \qquad k = 2, \ldots, p-1. \tag{8}$$

Theorem 1.2 (Hagiwara et al. [4]): Suppose that $G(s)$ is a strictly proper rational function given by (3). Then, for almost every sampling period T, the discrete-time transfer function $G_1(z)$ arising from FOH-sampling of (3) has n zeros. Furthermore, $G_1(z)$ approaches

$$K\,\frac{T^{n-m}}{(n-m+1)!}\,\frac{(z-1)^m C_{n-m}(z)}{z(z-1)^n} \tag{9}$$

as $T \to 0$, where $C_{n-m}(z)$ is given by

$$C_p(z) = B_{p+1}(z) + (p+1)(z-1)B_p(z), \qquad p \geq 1. \tag{10}$$

These theorems suggest that the m so-called *limiting zeros* approaching $z = 1$ correspond to the mapping of the finite zeros $\gamma_1, \gamma_2, \ldots, \gamma_m$, while the remaining $n-m-1$ (or $n-m$) zeros arise via the ZOH (or FOH) sampling process. Hagiwara *et al.* [4] have justified this assertion, and the m limiting zeros approaching $z = 1$ are therefore referred to as the *intrinsic* zeros, while the zeros approaching the roots of $B_p(z)$ or $C_p(z)$, where $p = n - m$ is the continuous-time relative degree, are the *limiting sampling zeros*, also known as *discretization zeros*.

By evaluating the roots of the polynomials $B_p(z)$ and $C_p(z)$ for $p = 1, 2, \ldots, 50$, Hagiwara and coworkers produced compelling numerical evidence to support the following conjecture.

Conjecture 1.1 (Hagiwara et al. [4]):

a) All roots of $B_p(z)$ are single and negative real for any p. Furthermore, the roots of $B_p(z)$ interlace the roots of $B_{p+1}(z)$ on the negative real axis.

b) All roots of $C_p(z)$ are single and real for any p. Furthermore, the kth smallest root of $C_p(z)$ lies between the kth smallest root of $B_p(z)$ and the kth smallest root of $B_{p+1}(z)$.

c) The largest root of $C_p(z)$ approaches $z = 1/e$ (≈ 0.3679) as $p \to \infty$, where e is the base of the natural logarithm.

Hagiwara *et al.* [4] established that property a) implies property b). In Sections II and III, we establish properties a) and c), respectively, thereby completing the proof of the conjecture.

The paper is organized as follows. In Section II we establish a differential recurrence relation satisfied by the polynomials shown by Åström *et al.* [2] to have as roots the sampling zeros of ZOH-sampled systems. We then show that the polynomials satisfying this relation are in fact the Euler–Frobenius polynomials, the properties of which have been studied in the context of cardinal spline interpolation

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[6]–[8]. In Section III, we prove the second component of the conjecture of Hagiwara *et al.*, namely that as the continuous-time relative degree increases without bound, the largest (i.e., most positive) sampling zero of FOH-sampled systems tends to $1/e$, where e is the base of the natural logarithm. An earlier version of this paper appeared as the conference paper [9].

II. The Euler–Frobenius Polynomials

In this section we establish simplicity, negative-realness and interlacing properties of the limiting sampling zeros. These results were also proved independently in the Masters thesis of Mårtensson [10]. However, no mention of these results appears in the paper [2], and it is clear from numerous papers, including [11], [12], and [4], that these properties are not widely recognized in the literature.

The following lemma establishes a differential recurrence relation satisfied by the polynomials $\{B_p(z)\}_{p=1}^{\infty}$ directly, rather than in terms of the individual coefficients, as in (7) and (8).

Lemma 2.1: The polynomials $B_p(z)$, whose coefficients are given by (6), satisfy the following differential recurrence relation:

$$B_1(z) = 1 \tag{11}$$

$$B_p(z) = (1+(p-1)z)B_{p-1}(z) + z(1-z)B'_{p-1}(z), \quad p = 2, 3, \ldots . \tag{12}$$

Proof: The result is true by definition for $p = 1$. For $p \geq 2$, the recursion can be verified by direct substitution of (5) into (12), equating coefficients of powers of z, and simplification using (8); see also [13]. □□□

In the following definition, we recall the Euler–Frobenius polynomials, which arise in the study of cardinal spline interpolation [6]–[8], [14], [18], [15], [16].

Definition 2.1: The Euler–Frobenius polynomials $E_k(x), k = 1, 2, \ldots$ are defined by the following Rodriguez formula [15]:

$$E_k(x) = \frac{(1-x)^{k+2}}{x}\left(x\frac{d}{dx}\right)^k \frac{x}{(1-x)^2}$$

$$E_0(x) = 1. \tag{13}$$

We are now in a position to state the first key result of the paper:

Theorem 2.1: The limiting sampling zero polynomials are the Euler–Frobenius polynomials.

Proof: The idea of the proof is to show that polynomials satisfying the Rodriguez formula (13) simultaneously satisfy the differential recurrence relation defining the limiting sampling zeros. Due to the inconsistency between the numbering of the initial terms $B_1(z)$ and $E_0(x)$ [cf. (11) and (13)], we do not work with (11) and (12), but rather with the recurrence

$$B_0(z) = 1 \tag{14}$$

$$B_n(z) = (1+nz)B_{n-1}(z) + z(1-z)B'_{n-1}(z), \quad n = 2, 3, \ldots \tag{15}$$

which leads to the same sequence of polynomials as in [2], but with a numbering consistent with (13).

Following Sobolev [15], we introduce the polynomials $K_k(y)$ related to $E_k(x)$ as follows:

$$K_k(y) = (y-1)^k E_k\left(\frac{y+1}{y-1}\right) \tag{16}$$

$$E_k(x) = 2^{-k}(x-1)^k K_k\left(\frac{x+1}{x-1}\right) \tag{17}$$

where the $K_k(y)$ satisfy the recurrence relation [15]

$$K_k(y) = \frac{d}{dy}\left[\left(y^2-1\right)K_{k-1}(y)\right]. \tag{18}$$

From (18), the change of variables $y = (x+1)/(x-1)$ yields

$$\begin{aligned} K_k\left(\frac{x+1}{x-1}\right) &= -\frac{(x-1)^2}{2}\frac{d}{dx}\left(\frac{4x}{(x-1)^2}K_{k-1}\left(\frac{x+1}{x-1}\right)\right) \\ &= -2x\frac{d}{dx}K_{k-1}\left(\frac{x+1}{x-1}\right) + K_{k-1}\left(\frac{x+1}{x-1}\right)2\,\frac{x+1}{x-1} \end{aligned}$$

from which it follows that

$$E_k(x) = 2^{-k}(x-1)^k K_k\left(\frac{x+1}{x-1}\right) \tag{19}$$

$$= -2^{-k+1}x(x-1)^k\frac{d}{dx}K_{k-1}\left(\frac{x+1}{x-1}\right) + \underbrace{2^{-(k-1)}(x-1)^{k-1}K_{k-1}\left(\frac{x+1}{x-1}\right)(x+1)}_{(x+1)E_{k-1}(x)}. \tag{20}$$

From (17)

$$\frac{d}{dx}E_{k-1}(x) = 2^{-k+1}\left((x-1)^{k-1}\frac{d}{dx}K_{k-1}\left(\frac{x+1}{x-1}\right) + K_{k-1}\left(\frac{x+1}{x-1}\right)(k-1)(x-1)^{k-2}\right)$$

so that the first term in (20) is given by

$$\begin{aligned} &-2^{-k+1}x(x-1)^k\frac{d}{dx}K_{k-1}\left(\frac{x+1}{x-1}\right) \\ &= -x(x-1)\underbrace{2^{-k+1}(x-1)^{k-1}\frac{d}{dx}K_{k-1}\left(\frac{x+1}{x-1}\right)}_{E_{k-1}(x)} \\ &= -x(x-1)\left(\frac{d}{dx}E_{k-1}(x) - 2^{-k+1}K_{k-1}\left(\frac{x+1}{x-1}\right)\right. \\ &\qquad \left.\times\,(k-1)(x-1)^{k-2}\right). \end{aligned}$$

Substituting this expression into (20) gives

$$\begin{aligned} E_k(x) &= x(1-x)\left(\frac{d}{dx}E_{k-1}(x) - 2^{-k+1}K_{k-1}\left(\frac{x+1}{x-1}\right)\right. \\ &\qquad \left.\times\,(k-1)(x-1)^{k-2}\right) + (x+1)E_{k-1}(x) \\ &= x(1-x)\frac{d}{dx}E_{k-1}(x) + x(k-1) \\ &\qquad \times\underbrace{2^{-k+1}(x-1)^{k-1}K_{k-1}\left(\frac{x+1}{x-1}\right)}_{E_{k-1}(x)} \\ &\qquad + E_{k-1}(x) + xE_{k-1}(x) \\ &= (1+kx)E_{k-1}(x) + x(1-x)\frac{d}{dx}E_{k-1}(x) \end{aligned}$$

and the result is proved. □□□

Corollary 2.1: In the fast sampling limit, the sampling zeros arising from the ZOH-sampling of continuous-time systems of relative degree 2 or greater are simple and negative real.

Proof: These are known properties of the Euler–Frobenius polynomials; see [15], for example. □□□

Lemma 2.2: In the fast sampling limit, the sampling zeros arising from the ZOH-sampling of continuous-time systems having progressively higher relative degrees are interlaced on the negative real axis.

Proof: Consider the recurrence relation (12) evaluated at any of the $p-2$ roots z_i^* of $B_{p-1}(z)$

$$B_p(z_i^*) = z_i^*(1-z_i^*)B'_{p-1}(z_i^*).$$

From Corollary 2.1, all roots of $B_{p-1}(z)$ are negative real, so that $z_i^*(1-z_i^*) < 0$ and thus the sign of $B_p(z_i^*)$ is opposite that of $B'_{p-1}(z_i^*)$. Since $B_p(0) = 1$ for all $p \geq 1$, it follows from the simplicity of the z_i^* and the Mean Value Theorem that the ith root of $B_p(z)$ lies strictly to the right of the corresponding root of $B_{p-1}(z)$ for $i = 1, 2, \ldots, p-2$. Since

$$\lim_{z\to-\infty} B_p(z) \begin{matrix} < 0, & p \text{ even} \\ > 0, & p \text{ odd} \end{matrix}$$

$\lim_{z\to-\infty} B_p(z)$ and $\lim_{z\to-\infty} B_{p-1}(z)$ have opposite signs. From the Mean Value Theorem, there must exist a root of $B_p(z)$ to the left of the most negative root of $B_{p-1}(z)$, and the proof is completed. □□□

Taken together, Corollary 2.1 and Lemma 2.2 constitute a proof of Conjecture 1.1 a).

III. The Largest Zero of FOH-Sampled Systems

In this section, we prove the second part of the Hagiwara conjecture, namely that in the fast sampling limit, the largest sampling zero of FOH-sampled systems approaches $1/e$ as the continuous-time relative degree increases. The proof does not rely heavily on the fact that the limiting sampling zeros of ZOH-sampled systems are the roots of the Euler–Frobenius polynomials, but does use a key change of variables and a series expansion introduced by Sobolev in his study of the roots of these polynomials [15].

Theorem 3.1: In the fast sampling limit, the most positive sampling zero arising from the FOH-sampling of continuous-time systems having relative degree p approaches $1/e$ as $p \to \infty$.

Proof: The sequence of polynomials of interest is generated by (10), where (from Theorem 2.1), the polynomials $B_p(z)$ satisfy the Rodriguez formula (13). Following Sobolev [15], a key ingredient is to make the substitution $z = e^{\pi\theta}$, leading to

$$B_k(e^{\pi\theta}) = \left(\frac{2}{\pi}\right)^k e^{\pi\theta k/2} \sinh^{k+2}\left(\frac{\pi\theta}{2}\right) \frac{d^k}{d\theta^k} \frac{(\pi/2)^2}{\sinh^2 \pi\theta/2}. \quad (21)$$

Define

$$S_k(\theta) = \frac{d^k}{d\theta^k} \frac{(\pi/2)^2}{\sinh^2 \pi\theta/2} \quad (22)$$

so that the roots of C_k other than 0 correspond to solutions of

$$\overline{S}_k(\theta) = 0 \quad (23)$$

where

$$\overline{S}_k(\theta) = (k+1)S_k(\theta) - \frac{1}{\pi} S_{k+1}(\theta). \quad (24)$$

We note that the Cauchy expansion of $1/\sinh^2 \pi\theta$ gives

$$S_k(\theta) = -(k+1)!(-j)^k \sum_{n=-\infty}^{\infty} \frac{1}{(j\theta - 2n)^{k+2}}. \quad (25)$$

The intuition behind the proof is that for k sufficiently large, the central term

$$h_k(\theta) = -(k+1)!(-j)^k \frac{1}{(j\theta)^{k+2}} \quad (26)$$

dominates the infinite sum (25), so that the solutions of (23) are approximately given by the roots of

$$\begin{aligned} f_k(\theta) &= (k+1)h_k(\theta) - \frac{1}{\pi} h_{k+1}(\theta) \\ &= \frac{(-1)^k(k+1)!}{\theta^{k+2}} \left((k+1) + \frac{1}{\pi}\,\frac{k+2}{\theta}\right). \end{aligned} \quad (27)$$

For k sufficiently large, the single root of $f_k(\theta)$ approaches $-1/\pi$, and we are done.

To make the argument rigorous, we use Rouché's Theorem [17, p. 300] to show that for k sufficiently large, the contribution to $\overline{S}_k(\theta)$ from the neglected (noncentral) terms

$$c_k(\theta) = S_k(\theta) - h_k(\theta) \quad (28)$$

is vanishingly small in the sense that $f_k(\theta)$ and $\overline{S}_k(\theta)$ have the same number of zeros as $k \to \infty$.

Consider the remainder term

$$g_k(\theta) = (k+1)c_k(\theta) - \frac{1}{\pi} c_{k+1}(\theta).$$

Using (25) we have

$$|g_k(\theta)| \leq 4(k+1)(k+1)! \sum_{n=1}^{\infty} \frac{1}{(x^2 + (2n-y)^2)^{(k+2)/2}}.$$

Now use an Integral Test estimate for the right-hand side to obtain

$$\begin{aligned} |g_k(\theta)| &\leq 4(k+2)\,\frac{1}{(x^2+(2-y)^2)^{(k+2)/2}} \\ &\quad + \int_1^\infty \frac{1}{(x^2+(2u-y)^2)^{(k+2)/2}}\,du \\ &\leq 4(k+2)\,\frac{1}{(x^2+(2-y)^2)^{(k+2)/2}} \\ &\quad + \frac{1}{2}\int_{2-y}^\infty \frac{1}{(x^2+u^2)^{(k+2)/2}}\,du \\ &\leq 4(k+2)\,\frac{1}{(x^2+(2-y)^2)^{(k)/2}} \\ &\quad \times \left(1 + \int_{2-y}^\infty \frac{1}{(x^2+u^2)}\,du\right) \\ &\leq \frac{C(k+2)!}{(x^2+(2-\epsilon)^2)^{k/2}} \end{aligned}$$

for some constant C. Here we have $\theta = x + j\xi$ where $|\xi| = y \leq \epsilon < 2$.

Consider the value of f_k on a contour Ω defined as the boundary of a square, centered on the point $-1/\pi$, having sides of length 2ϵ, and taken in the counterclockwise direction

$$\begin{aligned} |f_k(\theta)| &= \frac{(k+1)!}{|\theta|^{k+2}} \left|k+1+\frac{1}{\pi}\,\frac{k+2}{\theta}\right| \\ &\geq \frac{(k+2)!}{(x^2+y^2)^{(k+2)/2}} \left(\left(\epsilon^2+x^2\right)^{1/2} - \frac{1}{k+2}\right). \end{aligned}$$

Thus for any given ϵ we can find K such that if $k \geq K$,

$$|f_k(\theta)| > |g_k(\theta)|, \qquad \theta \text{ on } \Omega.$$

By Rouché's theorem, $f_k(\theta)$ and $f_k(\theta) + g_k(\theta) = \overline{S}_k(\theta)$ have the same number of zeros inside Ω. Since f_k has exactly one zero for k sufficiently large, so does (23), and by taking ϵ small we can show that for large k the zero is close to $-1/\pi$. Thus as $k \to \infty$, there are no other real roots of $C_k(z)$ larger than $1/e$, and the result is proved. □□□

IV. Conclusion

In this paper, we have established that the sequence of polynomials whose roots are the limiting sampling zeros of ZOH-sampled systems are in fact the Euler–Frobenius polynomials. Several conjectured properties of the limiting sampling zeros of ZOH- and FOH-sampled systems then follow immediately, or can be established from a differential recurrence formula satisfied by the Euler–Frobenius polynomials. Finally, a conjecture by Hagiwara and coworkers that the largest limiting sampling zero of FOH-sampled systems approaches $1/e$ as the continuous-time relative degree increases has been proved.

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Robust Nonfragile Kalman Filtering for Uncertain Linear Systems with Estimator Gain Uncertainty

Guang-Hong Yang and Jian Liang Wang

***Abstract*—This note is concerned with the problem of a robust nonfragile Kalman filter design for a class of uncertain linear systems with norm-bounded uncertainties. The designed state estimator can tolerate multiplicative uncertainties in the state estimator gain matrix. The robust nonfragile state estimator designs are given in terms of solutions to algebraic Riccati equations. The designs guarantee known upper bounds on the steady-state error covariance. A numerical example is given to illustrate the results.**

***Index Terms*—Fragility, Kalman filter, linear systems, Riccati equations, robustness.**

I. INTRODUCTION

Kalman filtering is a very popular approach for estimating the states of a nominal system by using past measurements [2]. In [3], Bertsekas and Rhodes addressed the problem of estimating the state of a linear system when the initial condition of the system and the input and observation noise vectors are uncertain but lie in given sets, and the designed filter is robust with respect to these considered uncertainties. The robust Kalman filtering problem for uncertain linear systems is addressed in [5], [9], [11], [16]–[18], and the corresponding state estimation results extend the Kalman filter for a nominal system to the case in which the underlying system is uncertain. Based on the use of a fixed quadratic Lyapunov function to establish an upper bound on the state estimation error covariance, Petersen and McFarlane [16] proposed a Riccati equation approach to the construction of an optimal quadratic guaranteed cost state estimator, which minimizes the upper bound on the state estimation error covariance. A suboptimal quadratic guaranteed cost state estimator is given in [18], which was extended by Shaked and de Souza [17] to the case in which the process noise and the measurement noise are assumed to be white signals with known statistics. The optimal robust filtering problem for stochastic linear systems subject to time-varying parameter uncertainties affecting both system dynamics and noise statistics is addressed in [5]. An alternative solution of the robust filtering problem can also be derived by the linear matrix inequality (LMI) approach proposed by Boyd *et al.* [6], where the LMI conditions for state mean and covariance bounds with unit-energy inputs are presented. More recently, an LMI-based formulation of a discrete-time predictive filter for computing ellipsoids of confidence for the state of an uncertain discrete-time system is given in [14].

While robustness relates to uncertainties in the plant, fragility relates to the inaccuracies or uncertainties in the implementation of a designed filter or controller. Such uncertainties in the filter or controller implementation can be due to, among other things, round-off errors in numerical computation during the controller implementation and the need of providing practicing engineers with safe-tuning margins. It has been pointed out in a recent paper (by means of numerical examples) [12] that the controllers designed by using weighted H_∞, μ, and l_1 synthesis techniques [19] may be very sensitive, or fragile, with respect

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