Orthonormal Basis Functions for Continuous-Time Systems and L_p Convergence

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Abstract

In this paper, model sets for continuous–time linear time invariant systems that are spanned by fixed pole orthonormal bases are investigated. These bases generalise the well known Laguerre and two–parameter Kautz cases. It is shown that the obtained model sets are norm dense in the Hardy space $H_1(\Pi)$ under the same condition as previously derived by the authors for the norm denseness in the (Π is the open right half plane) Hardy spaces $H_p(\Pi)$, $1 < p < \infty$. As a further extension, the paper shows how orthonormal model sets, that are norm dense in $H_p(\Pi)$, $1 < p < \infty$ and which have a prescribed asymptotic order may be constructed. Finally, it is established that the Fourier series formed by orthonormal basis functions converge in all spaces $H_p(\Pi)$, $1 < p < \infty$. The results in this paper have application in system identification, model reduction and control system synthesis.

1 Notation

$\mathbf{C}$	the field of complex numbers.
$\mathbf{R}$	the field of real numbers.
Π	the open right half plane $\{s \in \mathbf{C} : \mathrm{Re}\{s\} > 0\}$.
$\overline{\Pi}$	the closed right half plane $\{s \in \mathbf{C} : \mathrm{Re}\{s\} \geq 0\}$.
$\mathbf{D}$	the open unit disk $\{z \in \mathbf{C} : \lvert z\rvert < 1\}$.
$\mathbf{T}$	the unit circle $\{z \in \mathbf{C} : \lvert z\rvert = 1\}$.
$H_p(\Pi)$	the Hardy spaces of functions $f(s)$ analytic on Π and such that $\lVert f\rVert_p^p = (1/2\pi)\sup_{x>0}\int_{-\infty}^{\infty} \lvert f(x+jy)\rvert^p \,\mathrm{d}y < \infty$, $0 < p < \infty$ and $\lVert f\rVert_\infty = \sup_{s\in\Pi}\lvert f(s)\rvert < \infty$.
$A(\Pi)$	the right half plane algebra $\{f : f \in H_\infty(\Pi)$ and continuous on $\overline{\Pi}\}$.
$A(\mathbf{D})$	the disk algebra $\{f : f$ analytic on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}\}$.
sp$A$	the linear span of A.
$\overline{a}$	the complex conjugate of a.
$O(\lvert s\rvert^{-m})$	The notation $f(s) = 0(\lvert s\rvert^{-m})$ as $\lvert s\rvert \to \infty$ means that $$\limsup_{\lvert s\rvert\to\infty} \lvert s\rvert^m \lvert f(s)\rvert < \infty$$

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2 Introduction

A fundamental idea in various areas of applied mathematics, control theory, signal processing and system analysis is that of decomposing (perhaps infinite dimensional) descriptions of linear time invariant dynamics in terms of an orthonormal basis. This approach is of greatest utility when accurate system descriptions are achieved with only a small number of basis functions. In recognition of this, there has been much work over the past several decades [18, 5, 10, 27, 28] and, with renewed interest, more recently [32, 31, 30, 12, 8, 20, 4, 19] on the construction, analysis and application of rational orthonormal bases suitable for providing linear system characterisations.

In a system theoretic context, the applications of these orthonormal bases ideas have been manifold, but nevertheless have concentrated mainly on the discrete time setting [30, 31, 25, 21, 22, 8, 19, 4]. Motivated largely by problems of estimation from frequency domain data [1, 17, 24, 7], but also with control system analysis and synthesis in mind [13, 11] this and the companion paper [2] focuses attention on the continuous time scenario by considering the set of basis functions defined by a choice of numbers $\{a_k\} \in \Pi$ as

$$\begin{aligned} B_n(s) &\triangleq \frac{\sqrt{2\mathrm{Re}\{a_n\}}}{s+a_n}\varphi_{n-1}(s), \qquad n \geq 1 \\ \varphi_n(s) &\triangleq \prod_{k=1}^{n} \frac{s-\overline{a_k}}{s+a_k}, \qquad n \geq 1 \end{aligned} \tag{1}$$

with $B_0(s) = \varphi_0(s) \equiv 1$. With respect to the usual inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(j\omega)\overline{g(j\omega)}\, \mathrm{d}\omega$$

on $H_2(\Pi)$ these functions are orthonormal. Previous work on continuous time orthonormal bases has concentrated on special cases of the basis (1) wherein all the $\{a_k\}$ are the same real number $a_k = a \in \mathbf{R}$ in which case the ensuing basis is known as the 'Laguerre' basis [16, 15, 7, 23], or the case of all the $\{a_k, a_{k+1}\}$ being the same complex conjugate pair $a_k = a, a_{k+1} = \overline{a}$ [32].

For the general basis (1) studied here the only restriction on the pole choice $\{a_k\}$ is via the following result which was recently established in [2].

Theorem 2.1 *The linear span of the set of basis functions $\{B_n(s)\}_{n\geq 0}$ is norm dense in all of the spaces $H_p(\Pi)$, $1 < p < \infty$ and $A(\Pi)$ if and only if*

$$\sum_{n=1}^{\infty} \frac{Re\{a_n\}}{1+|a_n|^2} = \infty. \tag{2}$$

The first result of this paper is, via Theorem 3.1, to extend this result to the case (which has important applications in a robust control context) of $p = 1$.

A function in $f(s) \in H_p(\Pi)$ is said to have 'asymptotic order' m if $f(s) = O(|s|^{-m})$ as $|s| \to \infty$. Clearly the bases defined by (1) have asymptotic order 1, but as illustrated in other work on continuous time orthonormal bases such as [32], for the purposes of model error approximation and minimisation, there is great utility in being able to construct bases of asymptotic order greater than 1. Accordingly, Theorem 4.1 in § 4 establishes a method to construct an infinite set of orthonormal bases, each of which have arbitrary asymptotic order, and whose linear span is norm dense in $H_p(\Pi)$ for all $1 \leq p < \infty$.

Up until and including § 4 the paper has established that approximants of arbitrary small H_p norm exist, but not what they might be. In § 5 a specific (and obvious) approximant is considered which is the generalised Fourier series approximant. There, via Theorem 5.1, it is established that this approximant is, in fact, of arbitrarily small H_p norm distance from the function being approximated for any $1 < p < \infty$. The paper concludes by showing how this continuous time result may be used to establish an equivalent discrete time one.

3 Complete Model Sets in $H_1(\Pi)$

The paper begins by presenting the following result which extends the result in Theorem 2.1 to include the $H_1(\Pi)$ space.

Theorem 3.1 *The linear span of the set of basis functions $\{B_n\}_{n\geq 1}$ is norm dense in all of the spaces $H_p(\Pi)$, $1 \leq p < \infty$ (and $A(\Pi)$) if and only if (2) holds.*

Proof. The necessity of (2) is proven in [2]. To prove the sufficiency, let $f \in H_1(\Pi)$ be arbitrary. It is known that such an f may be factored as $f = gh$ for some $g, h \in H_2(\Pi)$ (see, for example Garnett [14]). Let $\epsilon > 0$ also be arbitrary and choose a function ϕ and a set of basis elements $\{B_1, \cdots, B_n\}$ such that $\phi \in \text{sp}\{B_k\}_{k=1}^{n}$ and

$$\|g - \phi\|_2 < \epsilon/2\|h\|_2. \tag{3}$$

This is possible since by Theorem 2.1 $\text{sp}\{B_k\}_{k\geq 1}$ is a norm dense in $H_2(\Pi)$. Next choose a sufficiently large number m so that all basis poles in the set $\{-a_1, \cdots, -a_n\}$ with finite multiplicities are not included in the set $\{-a_{m+1}, -a_{m+2}, \cdots\}$. Therefore, $m = n$ when all the elements in $\{-a_1, \cdots, -a_n\}$ are infinitely repeated basis poles. Define a new set of basis functions in $H_2(\Pi)$ by

$$\widetilde{B}_k(s) \triangleq \frac{\sqrt{2\text{Re}\{a_{k+m}\}}}{s + a_{k+m}}\widetilde{\varphi}_{k-1}(s), \qquad \widetilde{\varphi}_k(s) \triangleq \prod_{i=m+1}^{k+m} \frac{s - \overline{a_i}}{s + a_i}, \qquad \widetilde{\varphi}_0 = 1.$$

Since

$$\sum_{k=m+1}^{\infty} \frac{\text{Re}\{a_k\}}{1 + |a_k|^2} = \infty,$$

then $\text{sp}\{\widetilde{B}_k\}_{k\geq 1}$ is also norm dense in $H_2(\Pi)$. Therefore, there exists a finite set of basis elements $\{\widetilde{B}_k\}_{k=1}^{N}$ and $\psi \in \text{sp}\{\widetilde{B}_k\}_{k=1}^{N}$ such that

$$\|h - \psi\|_2 < \epsilon/2(\epsilon + \|g\|_2). \tag{4}$$

Let $\Phi = \phi\psi$. Applying the Cauchy–Schwarz and triangle inequalities provides, via (3) and (4) that

$$\begin{aligned} \|f - \Phi\|_1 &\leq \|g - \phi\|_2\|h\|_2 + \|\phi\|_2\|h - \psi\|_2 \\ &< \epsilon/2 + (\|g\|_2 + \epsilon)\|h - \psi\|_2 \\ &< \epsilon. \end{aligned} \tag{5}$$

It remains to show that $\Phi \in \text{sp}\{B_1, B_2, \cdots\}$. To this end it is first established that $\Phi \in \text{sp}\, Q$ where

$$Q \triangleq \left\{\frac{1}{s + a_1}, \frac{1}{(s + a_2)^{M(2)}}, \cdots\right\} \tag{6}$$

and $M(k)$ denotes the multiplicity of a_k in the set $\{a_1, \cdots, a_k\}$. Thus Q is precisely the set containing all possible partial fraction expansions of basis functions $B_1, B_2, \cdots$. Since

$$\mathrm{sp}\{B_1, \cdots, B_n\} \subset \mathrm{sp}\left\{\frac{1}{s+a_1}, \frac{1}{(s+a_2)^{M(2)}}, \cdots, \frac{1}{(s+a_n)^{M(n)}}\right\}$$

and

$$\mathrm{sp}\{\widetilde{B}_1, \cdots, \widetilde{B}_N\} \subset \mathrm{sp}\left\{\frac{1}{s+a_{m+1}}, \cdots, \frac{1}{(s+a_{m+N})^{\widetilde{M}(N)}}\right\}$$

where $\widetilde{M}(k)$ is the multiplicity of a_{m+k} in the set $\{a_{m+1}, \cdots, a_{m+k}\}$, then Φ can be written for some coefficients $\{c_k\}, \{d_k\}$ as

$$\Phi = \sum_{k=1}^{n} \sum_{\ell=1}^{N} \frac{c_k d_\ell}{(s+a_k)^{M(k)}(s+a_{m+\ell})^{\widetilde{M}(\ell)}}.$$

If a_k or $a_{\ell+m}$ has finite multiplicity, then by construction $a_k \neq a_{m+\ell}$ and the summand above admits a partial fraction expansion

$$\sum_{i=1}^{M(k)} \frac{e_i}{(s+a_k)^i} + \sum_{i=1}^{M(m+\ell)} \frac{f_i}{(s+a_{m+\ell})^i}$$

for some coefficients e_i, f_i. If a_k and a_ℓ both have infinite multiplicities and $a_k \neq a_\ell$, the partial fraction expansion above still applies. In the last case, we are left with

$$\frac{c_k d_\ell}{(s+a_k)^{M(k)}(s+a_{m+\ell})^{\widetilde{M}(\ell)}} = \frac{c_k d_\ell}{(s+a_k)^{M(k)+\widetilde{M}(\ell)}}.$$

Hence $\Phi \in \mathrm{sp}\, Q$. Write the partial fraction expansions of $B_1, B_2, \cdots$ in the following linear equation form

$$\begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_n \end{bmatrix} = \begin{bmatrix} \alpha_{11} & 0 & \cdots & 0 \\ \alpha_{21} & \alpha_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix} \begin{bmatrix} \frac{1}{s+a_1} \\ \frac{1}{(s+a_2)^{M(2)}} \\ \vdots \\ \frac{1}{(s+a_2)^{M(n)}} \end{bmatrix}. \tag{7}$$

The degree of B_k is k. Then $\alpha_{kk} \neq 0$ for all $k \leq n$ and the lower triangular matrix above is invertible. Thus

$$\left\{\frac{1}{s+a_1}, \cdots, \frac{1}{(s+a_n)^{M(n)}}\right\} \subset \mathrm{sp}\{B_k, k = 1, \cdots, n\}.$$

It follows that $\Phi \in \mathrm{sp}\{B_n\}_{n\geq 1}$. ♣

4 Orthonormal Basis Functions with Prescribed Asymptotic Order

This section presents a derivation of model sets that are norm dense in in $H_p(\Pi)$ for $1 < p < \infty$ and for which the orthonormal basis functions $B_n(s)$ defining the sets each have a prescribed asymptotic order. That is, $B_n(s) = O(|s|^{-m})$ as $|s| \to \infty$. The basis functions studied in the previous section all have asymptotic order $m = 1$.

The problem of synthesis of bases of arbitrary asymptotic order has been investigated in the literature for various specific cases of choice a_k of pole position or of asymptotic order m [29, 18, 6, 16]. In particular, the generalised Laguerre functions

$$\psi_k(s) = \frac{1}{\sqrt{k^2+k}}\left[\frac{\sqrt{2a}k}{s+a}\left[\frac{s-a}{s+a}\right]^k - \frac{1}{\sqrt{2a}}\left(1-\left[\frac{s-a}{s+a}\right]^k\right)\right]$$

are known be of asymptotic order $m = 2$ and also to be norm dense in $H_2(\Pi)$ [29, 6, 16].

In contrast to these previous specific cases available in the literature, the following result provides a recipe to construct bases of arbitrary asymptotic order m and with arbitrary pole position a_k (that satisfies (2)).

Theorem 4.1 *Suppose that (2) is satisfied. Let*

$$\psi_n(s) \triangleq \frac{B_n(s)}{P(s)} - \sum_{k=1}^{n-1}\left\langle \frac{B_n}{P}, \psi_k\right\rangle \frac{\psi_k(s)}{\|\psi_k\|_2}, \qquad \psi_1(s) \triangleq \frac{B_0(s)}{P(s)} \tag{8}$$

where $P(s)$ is an arbitrary mth order polynomial with roots in the complement of $\overline{\Pi}$. Then the basis functions

$$\phi_n(s) \triangleq \frac{\psi_n(s)}{\|\psi_n\|_2}, \qquad n \geq 1 \tag{9}$$

are orthonormal and have the asymptotic order $\phi_n(s) = O(|s|^{-m})$ as $|s| \to \infty$. Moreover $sp\{\phi_n\}_{n\geq 1}$ is norm dense in $H_p(\Pi)$ for all $1 \leq p < \infty$.

Proof. Let $f \in H_p(\Pi)$ and $\epsilon > 0$ be arbitrary. By [14, Corollary 3.3], it is possible to select a function $g \in A(\Pi)$ such that $\|f-g\|_p < \epsilon$ and

$$\lim_{|s|\to\infty} |s|^m |g(s)| = 0, \quad s \in \Pi.$$

Take $h(s) = P(s)g(s)$. Then $h \in A(\Pi)$ and since by Theorem 2.1sp $\{1, B_1, B_2, \cdots\}$ is norm dense in $A(\Pi)$ there exists a function $H \in \mathrm{sp}\{1, B_1, B_2, \cdots\}$ such that$\|h - H\|_\infty < \epsilon$ which implies that

$$\left|g(s) - \frac{H(s)}{P(s)}\right| < \frac{\epsilon}{P(s)}, \qquad s \in \Pi.$$

Therefore

$$\left\|f - \frac{H}{P}\right\|_p < \epsilon + \left\|\frac{1}{P}\right\|_p \epsilon$$

and hence the linear span

$$Q \triangleq \mathrm{sp}\left\{\frac{1}{P}, \frac{B_1}{P}, \frac{B_2}{P}, \cdots\right\}$$

is norm dense in $H_p(\Pi)$ for all $1 \leq p < \infty$. Finally, observe that (8) is the Gram-Schmidt orthogonalisation procedure applied to Q. ♣

Note that the terms B_n/P in (8) admit further partial fraction expansions unless any finite multiplicity pole of $\{B_n\}_{n\geq 1}$ coincide with the root of $P(s)$. In this case, $\mathrm{sp}\{\phi_n\} \subset \mathrm{sp}\{B_n\}$.

5 Convergence of Generalised Fourier Series in $H_p(\Pi)$

Let $\{B_k\}_{k\geq 1}$ be a set of basis functions which satisfy (2). Then $\{B_k\}_{k\geq 1}$ is a norm dense set of basis functions for $H_2(\Pi)$ and every $f \in H_2(\Pi)$ has a Fourier series expansion

$$\widehat{f}_n(s) \triangleq \sum_{k=1}^{n} \langle f, B_k \rangle B_k(s) \tag{10}$$

that converges to f in the $L_2(j\mathbf{R})$–norm. When $B_k = z^k$, $k = 1, 2, \cdots$ and the underlying space is $H_p(\mathbf{D})$, it is well known that every $f \in H_p(\mathbf{D})$ has a Fourier series which also converges in the $L_p(\mathbf{T})$–norm for all $1 < p < \infty$. In this section it is shown that the same is true for the basis functions in (1). First it is necessary to derive an upper bound on $|\varphi_n(s)|$.

Lemma 5.1 *Let φ_n be as in (1). Then for each $s \in \Pi$*

$$|\varphi_n(s)| \leq \exp\left(-\frac{1}{5}\left[1 - \left|\frac{1-s}{1+s}\right|\right] \sum_{k=1}^{n} \frac{Re\{a_k\}}{1+|a_k|^2} \right). \tag{11}$$

Proof. The bilinear transformation

$$s = \psi(z) \triangleq \frac{1-z}{1+z}$$

maps analytic functions on Π to analytic functions on $\mathbf{D}$. In [3, Lemma 3], it is shown that for each $z \in \mathbf{D}$ and for each sequence of complex numbers $\{\xi_k\}_{k\geq 1}$ in $\mathbf{D}$, the following inequality holds

$$\left| \prod_{k=1}^{n} \frac{z - \xi_k}{1 - \overline{\xi_k} z} \right| \leq \exp\left(-\frac{1}{2}(1-|z|) \sum_{k=1}^{n} (1 - |\xi_k|) \right). \tag{12}$$

Let

$$\xi_n = \frac{1-\overline{a_n}}{1+\overline{a_n}}.$$

Then

$$|\xi_k| = \left|\frac{1-a_k}{1+a_k}\right| \left|\frac{1+a_k}{1+a_k}\right| = \frac{|1-a_k^2|}{|1+a_k|^2} \leq \frac{1+|a_k^2|}{|1+a_k|^2}.$$

Hence

$$\begin{aligned} 1 - |\xi_k| &\geq 1 - \frac{1+|a_k^2|}{|1+a_k|^2} = 2\,\frac{\mathrm{Re}\{a_k\}}{|1+a_k|^2} && (13)\\ &\geq \frac{2}{5}\,\frac{\mathrm{Re}\{a_k\}}{1+|a_k|^2} && (14) \end{aligned}$$

where the last inequality follows from the chain of inequalities

$$|1+a_k|^2 \leq 1 + |a_k|^2 + 2|a_k| \leq 5(1+|a_k|^2)$$

when $|a_k| < 2$ while

$$|1+a_k|^2 \leq 1 + |a_k|^2 + 2|a_k| \leq 1 + 2|a_k|^2 \leq 2(1+|a_k|^2)$$

when $|a_k| \geq 2$. The inequalities (12), (14), and the inverse transformation $z = \psi^{-1}(s)$ yields the right hand side of (11). The left hand side follows from

$$\varphi_n\left(\frac{1-z}{1+z}\right) = (-1)^n \prod_{\ell=1}^{n} \frac{1+\overline{a_\ell}}{1+a_\ell} \prod_{k=1}^{n} \frac{z-\xi_k}{1-\overline{\xi_k} z}.$$

♣

Use of this result allows the calculation of the $L_p(j\mathbf{R})$ distance from a first order system to the model set spanned by $B_k,\ k = 1, 2, \cdots, n$.

Lemma 5.2 *Let $\widehat{f}_n$ be as in (10) and suppose that $f(s) = 1/(s + \gamma)$ is analytic on Π. Then*

$$\left\| f - \widehat{f}_n \right\|_p \leq \|f\|_p \ \exp\left(-\frac{2Re\{\gamma\}}{5(1+|\gamma|^2)} \sum_{k=1}^{n} \frac{Re\{a_k\}}{1+|a_k|^2} \right). \tag{15}$$

Proof. This lemma follows from Lemma 5.1, the inequality (13) evaluated at $a_k = \overline{\gamma}$, and Lemma 6 in [2] which in the special case considered here is

$$|f(j\omega) - \widehat{f}_n(j\omega)| \leq |f(j\omega)|\, |\varphi_n(\gamma)|.$$

♣

Next it is established that the maps $f \mapsto \widehat{f}_n$ are bounded.

Lemma 5.3 *Let $\widehat{f}_n$ be as in (10). Then there exists a constant $C_p < \infty$, which depends only on p, such that for all $1 < p < \infty$*

$$\|\widehat{f}_n\|_p \leq (1 + C_p)\|f\|_p. \tag{16}$$

Proof. Let $\psi(s) = f(-js)$. Then ψ is analytic on the upper half plane $\text{Im}\{s\} > 0$ and $(1/2\pi)\int_{-\infty}^{\infty} |\psi(t)|^p dt = \|f\|_p$. Hence $\psi(s)$ can be represented by a Cauchy integral [9, Theorem 11.8] as

$$\psi(s) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{\psi(t)}{t-s}\, \mathrm{d}t, \qquad \text{Im}\{s\} = y > 0$$

and consequently

$$f(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(j\omega)}{s - j\omega}\, \mathrm{d}\omega, \qquad s \in \Pi. \tag{17}$$

For the basis functions $\{B_k\}_{k\geq 1}$ defined by (1), the following Christoffel-Darboux formula was derived in [2]

$$\sum_{k=1}^{n} \overline{B_k(j\omega)} B_k(s) = \frac{1 - \overline{\varphi_n(j\omega)}\varphi_n(s)}{s - j\omega}, \qquad s \in \Pi. \tag{18}$$

Therefore from (17)–(18),

$$\begin{aligned} \widehat{f}_n(s) &= f(s) - \frac{\varphi_n(s)}{2\pi}\int_{-\infty}^{\infty} \frac{f(j\omega)\overline{\varphi_n(j\omega)}}{s - j\omega}\, \mathrm{d}\omega, \qquad s \in \Pi. \\ &= f(s) - \varphi_n(s)\frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{f(j\omega)\overline{\varphi_n(j\omega)}}{\omega - js}\, \mathrm{d}\omega \\ &\triangleq f(s) - \varphi_n(s)\widetilde{H}_n(s). \end{aligned} \tag{19}$$

Put $\widetilde{f}_n(\omega) = f(j\omega)\overline{\varphi_n(j\omega)}$ and $\widetilde{F}_n(js) = \widetilde{H}_n(s),\ s \in \Pi$. Then the following representation for $\widetilde{F}_n$ is possible

$$\widetilde{F}_n(s) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{\widetilde{f}_n(\omega)}{\omega - s}\, \mathrm{d}\omega, \qquad \text{Im}\{s\} = y > 0. \tag{20}$$

The map $\widetilde{f} \mapsto \widetilde{F}$ defined by (20) is a bounded linear operator from $L_p(\mathbf{R})$ onto $H_p(\{y>0\})$ (see [26, Theorem 5.32]). Hence

$$\|\widetilde{H}_n\|_p = \|\widetilde{F}_n\|_p \le C_p \|\widetilde{f}_n\|_p = C_p \|f\|_p \tag{21}$$

where C_p is a constant which depends only on p. An application of the triangle inequality to (19) and (21) then completes the proof. ♣

For the precise value of C_p, the reader is referred to Chapter III in Garnett [14]. Using Lemmata 5.2–5.3, we prove the following main result of this section can now be established.

Theorem 5.1 *Consider the partial sums of the Fourier series defined by (10). Suppose that (2) holds. Then for all* $1 < p < \infty$

$$\lim_{n\to\infty} \|f - \widehat{f}_n\|_p = 0, \qquad f \in H_p(\Pi). \tag{22}$$

Proof. Choose a sequence of complex numbers $\{\gamma_k\}_{k\ge1}$ satisfying (2) and let $\phi_k = 1/(s+\gamma_k)$. Then by Theorem 2.1, the linear span of $\{\phi_k\}_{k\ge1}$ is norm dense in $H_p(\Pi)$ for all $1<p<\infty$. Let $f \in H_p(\Pi)$ and $\epsilon > 0$. Then there exists a $g = \sum_{k=1}^m c_k\phi_k$ such that

$$\|f-g\|_p < \epsilon. \tag{23}$$

Let $\widehat{g}_n$ denote the nth partial sum of the Fourier series of g. Then from Lemma 5.2,

$$\|g - \widehat{g}_n\|_p < \sum_{l=1}^m |c_l|\,\|\phi_l\|_p \exp\left(-\frac{2\mathrm{Re}\{\gamma_l\}}{5(1+|\gamma_l|^2)}\sum_{k=1}^n \frac{\mathrm{Re}\{a_k\}}{1+|a_k|^2}\right).$$

Hence for sufficiently large n

$$\|g-\widehat{g}_n\|_p < \epsilon. \tag{24}$$

Let $\psi = f-g$ and denote by $\widehat{\psi}_n$ the nth partial sum of its Fourier series. Due to the linearity of Fourier series expansion, notice that $\widehat{\psi}_n = \widehat{f}_n - \widehat{g}_n$. Hence from repeated application of the triangle inequality, the expressions (23), (24) and Lemma 5.3

$$\begin{aligned}\|f-\widehat{f}_n\|_p &\le \|f-g\|_p + \|g-\widehat{f}_n\|_p \\ &< \epsilon + \|g-\widehat{g}_n\|_p + \|\widehat{g}_n - \widehat{f}_n\|_p \\ &< 2\epsilon + (1+C_p)\|\psi\|_p \\ &< (3+C_p)\epsilon. \end{aligned} \tag{25}$$

♣

The remainder of this section will be consumed with the extension of Theorem 5.1 to the discrete–time orthonormal basis functions studied in [19, 3, 20] defined on $\mathbf{D}\cup\mathbf{T}$ by

$$\mathcal{B}_n(z) \triangleq \frac{\sqrt{1-|\xi_n|^2}}{1-\overline{\xi_n}z}\phi_{n-1}(z), \qquad \phi_n(z) \triangleq \prod_{k=1}^n \frac{z-\xi_k}{1-\overline{\xi_k}z}, \qquad \phi_0(z)\triangleq 1. \tag{26}$$

These basis functions were considered in [3] for the purpose of robust estimation. In particular, it was shown that model sets spanned by (26) are complete in $H_p(\mathbf{D})$ for all $1\le p<\infty$ and $A(\mathbf{D})$ if and only if the sequence of complex numbers $\xi_k \in \mathbf{D}$ satisfies

$$\sum_{n=1}^\infty (1-|\xi_n|) = \infty. \tag{27}$$

The following is the discrete-time version of Lemma 5.2. In what follows, the notation $\|\cdot\|_p$ refers to the L_p–norms on the unit circle and the inner product for two functions $f, g \in H_2(\mathbf{T})$ is defined as

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{j\omega})\overline{g(e^{j\omega})}\,\mathrm{d}\omega.$$

Lemma 5.4 *Define $\widehat{f}_n$ as an approximation to $f \in H_1(\mathbf{D})$, obtained by projection onto $\{\mathcal{B}_k\}_{k=1}^n$:*

$$\widehat{f}_n(z) = \sum_{k=1}^{n}\langle f, \mathcal{B}_k\rangle \mathcal{B}_k(z). \tag{28}$$

Suppose f is analytic and magnitude bounded by K on the disk $\{z : |z| < R\}$ for some $R > 1$. Then

$$\|f - \widehat{f}_n\|_\infty \le \frac{KR}{R-1}\exp\left(-\frac{R-1}{2R}\sum_{k=1}^{n}(1-|\xi_k|)\right). \tag{29}$$

Proof. See Lemma 4 in [3]. ♣

Lemma 5.5 *Let $\widehat{f}_n$ be as in (28). Then there exists a constant $C_p < \infty$, which depends only on p, such that for all $1 < p < \infty$*

$$\|\widehat{f}_n\|_p \le (1 + C_p)\|f\|_p. \tag{30}$$

Proof. The proof is very similar to that of Lemma 5.3. Accordingly, it is omitted, and instead the required modifications of Lemma 5.3 are specified. The Cauchy formula for $f \in H_1(\mathbf{D})$ is

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi}\frac{f(e^{j\theta})}{e^{j\theta} - z}\,\mathrm{d}\theta$$

and the Christoffel-Darboux formula [3] is

$$\sum_{k=1}^{n}\mathcal{B}_k(\zeta)B_k(z) = \frac{1 - \overline{\phi_{n+1}(\zeta)}\phi_{n+1}(z)}{1 - \overline{\zeta}z}, \qquad z, \zeta \in \mathbf{D}\cup\mathbf{T}.$$

♣

Theorem 5.2 *Consider the partial sums of the Fourier series defined by (28). Suppose that (27) holds. Then for all $1 < p < \infty$*

$$\lim_{n\to\infty}\|f - \widehat{f}_n\|_p = 0, \qquad f \in H_p(\mathbf{D}). \tag{31}$$

Proof. Choose a sequence of distinct complex numbers $\{\gamma_k\}$ satisfying $|\gamma_k| < 1/R$ for some $R > 1$ and hence also satisfying the criterion (27). Then the linear span of the functions $1/(1 - \overline{\gamma_k}z)$, $k = 1, 2, \cdots$ is dense in $H_p(\mathbf{D})$ for all $1 \le p < \infty$. The rest of the proof is the same as the proof of Theorem 5.1. ♣

The cases $H_1(\mathbf{D})$ and $A(\mathbf{D})$ can not be included in Theorem 5.2. For example when $B_k = z^k$, $k = 1, 2, \cdots$, it is well known that every integrable function does not necessarily admit a Fourier series converging in any of these spaces.

6 Conclusions

This paper has provided analysis of the approximation properties of certain general classes of rational orthonormal basis functions. The nature of the results was such as to establish that for continuous time linear time invariant system modelling, arbitrarily small H_p norm approximation error was possible for any $p \in [1, \infty)$ and furthermore, for the case of $p \in (1, \infty)$, this may be provided while at the same time using bases with arbitrary asymptotic order. Finally, a specific construction of the system approximant via Fourier decomposition was shown to be one in which the H_p norm error is arbitrarily small for any $p \in (1, \infty)$. The results have application in the analysis and design of robust estimation and control strategies.

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