Design of cross-directional controllers with optimal steady state performance*

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Abstract

Actuator constraint handling is necessary for many cross-directional controllers. We discuss how optimal steady state performance can be guaranteed by modifying an internal model control structure with a non-linear element. For the simple dynamics associated with most web processes this also gives good closed-loop dynamic behaviour. Thus unconstrained control design techniques may be applied directly to the constrained control problem.

1 Introduction

Cross-directional control design is required for a wide class of industrial web forming processes including paper making, plastic film extrusion, coating processes and steel rolling. In these manufacturing processes a wide strip of material is produced continuously (see Fig 1). The cross-directional problem is to control variations in the profile across the strip, orthogonal to the direction of production. Typically the profile is controlled with an array of evenly spaced actuators with near identical response. The actuators are usually subject to hard physical constraints, and may have a limited dynamic range. The cross-directional profile is usually subject to disturbances which are constant or slowly varying in time. The main aim of the control system is to remove such disturbances; limitations in actuator configuration as well as robustness requirements usually restrict the control action to low spatial bandwidths across the web. Cross-directional control design has received considerable attention in the academic community—see for example [13] and references therein, as well as [9] and associated contributions.

There are two main schools of cross-directional control design (exemplified by Chapters 6 and 7 in [13]): firstly unconstrained control (perhaps with limited anti-windup) based on robust control methodologies and secondly constrained control achieved via MPC (model predictive control). Briefly the former guarantees robust dynamic behaviour while the latter offers improved steady state response (provided the model is sufficiently accurate [37, 21]). In this paper we discuss how the simple model structure common to most web forming processes allows optimal constraint handling to be incorporated as a modification to an IMC (internal model control) structure. Thus robustness is preserved while steady state performance may (in some circumstances) be improved significantly. The resulting structure may be viewed either as an optimal anti-windup scheme for robust control design or as the basis for an MPC design methodology according to taste.

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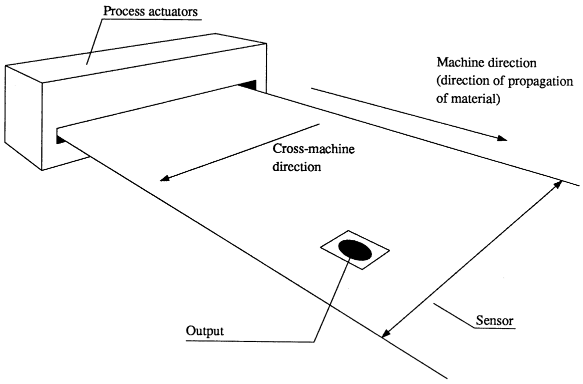

Figure 1: *Generic web forming process.*

We will make the standard assumption that the open loop behaviour of the output profile $y(t)$ may be well approximated by the model

$$y(t) = z^{-k} h(z^{-1}) B u(t) + d(t) \tag{1}$$

Here $y(t) \in \mathbb{R}^n$ represents the measured profile across the web and $u(t) \in \mathbb{R}^m$ represents the array of actuators. Typically $n > m$. We assume the whole profile $y(t)$ is available simultaneously; if raw measurements are obtained from a scanning sensor, then a periodic Kalman filter [15, 27] can be used to estimate $y(t)$. The dynamics are represented as a delay of integer k samples and a biproper transfer function $h(z^{-1})$. Usually $h(z^{-1})$ is low order and often simply a first order response together with a partial delay of $\zeta/(1+\zeta)$ samples, with $\zeta \geq 0$. Approximating the partial delay as an open-loop zero at $z = -\zeta$ we may write

$$h(z^{-1}) = \left(\frac{1-\alpha}{1-\alpha z^{-1}}\right)\left(\frac{1+\zeta z^{-1}}{1+\zeta}\right) \tag{2}$$

We have assumed, without loss of generality, that $h(z^{-1})$ has unit gain. Note that if $\zeta > 1$ the plant model has a non-minimum phase zero.

The $(n \times m)$ interaction matrix B describes the steady state response of the actuators on the profile. Finally $d(t) \in \mathbb{R}^n$ represents disturbances on the plant.

Let B be decomposed as

$$B = \Phi \Sigma \Psi^T \tag{3}$$

with Φ and Ψ orthonormal (the description is rather general as we may allow either Φ or Ψ to be the identity matrix [12]). One possibility is the singular value decomposition [10, 13], in which case Σ is diagnonal (if $n > m$ the upper block of Σ is diagonal, while the lower block is zero).

We can then write $y(t)$ and $u(t)$ in terms of basis functions which are the columns of Φ and Ψ respectively. We will assume that these basis functions are "spectral" [17] in the sense that they are naturally ordered according to some smoothness criterion. With an abuse of terminology, we will classify them as low, medium and high frequency modes. The effect of the interaction matrix is assumed to attenuate for high frequencies. In the case of the singular value decomposition the modes are ordered according to the magnitude of the singular value.

Ultra-high frequency modes of Φ are uncontrollable [10, 17]. With model mismatch the relative uncertainty is greater at high frequencies, and may result in closed-loop instability [13]. Even in the case of closed-loop stability, attempting to control uncertain modes may degrade steady state performance [37]. It is thus generally accepted that the controller should not act on high order modes. It may also be useful to restrict the dimension of the input space [19] — for example the actuators are usually constrained to sum to zero. We will assume the controller is designed to act only on $r \leq \min(n, m)$ modes. It will be useful to define a reduced interaction matrix $B_r \in \mathbb{R}^{(n,m)}$ which can be decomposed as

$$B_r = \Phi_r \Sigma_r \Psi_r^T \tag{4}$$

with $\Phi_r \in \mathbb{R}^{(n,r)}$ representing the modes we wish to control, $\Sigma_r \in \mathbb{R}^{(r,r)}$ and $\Psi_r \in \mathbb{R}^{(m,r)}$. We will write

$$\begin{aligned} \eta(t) &= \Phi_r^T y(t) \\ \mu(t) &= \Psi_r^T u(t) \end{aligned} \tag{5}$$

It will be useful to define $\Psi_r^\perp \in \mathbb{R}^{(m,m-r)}$ as an orthonormal matrix whose columns span the orthogonal complement to the space spanned by the columns of Ψ_r. The restriction on the input space may then be expressed as requiring

$$\left(\Psi_r^\perp\right)^T u(t) = 0 \tag{6}$$

If sufficient modes are excluded it is possible to design robust controllers that do not violate actuator constraints [20, 19, 10, 13, 33]. It has been recommended [19] that r should be chosen to ensure no actuator touches the constraint boundary. The designs of [13] are based on IMC structures. The designs of [19, 33] are modifications of Dahlin controllers, themselves variants of IMC. Furthermore, *any* linear controller of the form

$$u(t) = -\Psi_r C(z^{-1}) \Phi_r^T y(t) \tag{7}$$

can be rearranged as an IMC

$$u(t) = -\Psi_r Q(z^{-1}) \Phi_r^T y(t) + \Psi_r Q(z^{-1}) \Phi_r z^{-k} h(z^{-1}) B_r u(t) \tag{8}$$

with

$$Q(z^{-1}) = C(z^{-1}) \left[I + z^{-k} h(z^{-1}) \Sigma_r C(z^{-1})\right]^{-1} \tag{9}$$

Limited anti-windup schemes have been proposed for such controllers [8, 19, 13, 30]. However there may be mid-frequency modes where the model mismatch is relatively small, but where unconstrained control action would require actuator constraint violation even in steady state. In such cases restricting r as above would result in economic disadvantage [37, 21]. Similarly anti-windup schemes that take no account of actuator directionality can lead to severe performance degradation [22, 32].

In such cases optimal steady state performance requires the solution of a constrained optimization. This was recognized for the cross-directional control problem in [3], when limitations in computing power were an impediment to implementation. Subsequently several MPC strategies with a quadratic program have been discussed in the literature, for example [17, 27]. Robustness considerations still require attenuated control action at high modes; this can be achieved either via equality constraints [17], or by penalising such action in the MPC cost function. If the latter strategy is adopted alone, the effects of integral action can lead to severe performance degradation [37, 33]. Strategies for minimising l_1 norms [7] and l_∞ norms [11] have also been discussed. A number of control designs that seek approximate solutions to the quadratic cost have also been discussed—for example [4, 35, 5, 16, 29, 13].

In this paper we discuss the design of cross-directional controllers that preserve the robust properties of unconstrained methods, whilst ensuring, where possible, optimal steady state performance. We will also seek efficient computational implementation and good dynamic response. In particular we propose preserving the IMC structure for both control design and implementation. The constraints can then be satisfied by solving a deadbeat optimization problem. This is, of course, well known as a particular implementation of MPC [28] and may be considered [22] as a generalization of standard anti-windup schemes such as those in [41].

2 Constrained IMC

2.1 Control criteria

Suppose the disturbance $d(t)$ in (1) is fixed to some steady state value d_{ss}. Then the ideal steady state performance criterion is to minimise $||\Phi_r^T y(t)||$ for some norm. The projection term Φ_r^T is included so that higher modes are not penalised. Model mismatch may cause steady state performance degradation [37]. If this degradation is significant, then the number of modes acted upon should be reduced. We will assume the number of modes is well-chosen.

It may not be possible to set $\Phi_r^T y(t)$ to zero since the actuator movement is limited. Typically each actuator has a minimum and maximum value:

$$u_{\min} \leq u_i(t) \leq u_{\max} \tag{10}$$

Often a bending constraint is also placed on adjacent actuators, commonly of the form:

$$\bar{u}_{\min} \leq u_{i-1}(t) - 2u_i(t) + u_{i+1}(t) \leq \bar{u}_{\max} \tag{11}$$

We will represent all such static constraints as requiring

$$u(t) \in \mathbb{U} \tag{12}$$

If a fixed (steady state) estimate $\hat{d}_{ss}$ of d exists, then one method for attempting to minimise $\left|\left|\Phi_r^T y(t)\right|\right|$ is to demand that in steady state $u(t) = u_{ss}$ with

$$u_{ss} = \arg\min_u \left|\left|\Phi_r^T \left(Bu + \hat{d}_{ss}\right)\right|\right| \text{ such that } u \in \mathbb{U} \text{ and } \left(\Psi_r^\perp\right)^T u = 0 \tag{13}$$

In some MPC schemes set points for inputs and outputs (and states) are computed at each sample using such a static optimisation criterion [27]. These set points are then used in the main dynamic optimisation, which is also solved at each sample.

Consider the IMC scheme depicted in Fig 2. A disturbance estimate is obtained as

$$\hat{d}(t) = y(t) - z^{-k} h(z^{-1}) Bu(t) \tag{14}$$

In turn the disturbance estimate is passed through a linear filter (we will assume that the set point is zero) to give

$$\hat{d}_f(t) = Q(z^{-1})\hat{d}(t) \tag{15}$$

Finally the control action $u(t)$ is generated via a static non-linear function

$$u(t) = NL\left[\hat{d}_f(t)\right] \tag{16}$$

If we choose $Q(1) = 1$ and the non-linearity to be

$$u(t) = \arg\min_u \left|\left|\Phi_r^T \left(Bu + \hat{d}_f(t)\right)\right|\right| \text{ such that } u \in \mathbb{U} \text{ and } \left(\Psi_r^\perp\right)^T u = 0 \tag{17}$$

then we have a controller that satisfies (13), provided $\hat{d}_f(t)$ converges to some $\hat{d}_{ss}$. Under such conditions, if $y(t)$ also converges to a steady state value y_{ss} then

$$\hat{d}_{ss} = y_{ss} - Bu_{ss} \tag{18}$$

Furthermore, if no constraints are active, then (under the assumption that Σ_r has full rank),

$$\begin{aligned} u(t) &= -\Psi_r \Sigma_r^{-1} \Phi_r^T Q(z^{-1}) \hat{d}(t) \\ \hat{d}(t) &= y(t) - z^{-k} h(z^{-1}) Bu(t) \end{aligned} \tag{19}$$

Thus we have a control scheme that behaves as a standard IMC when away from the constraints, but satisfies (when stable) the steady state optimality condition (13). This is achieved by including a static non-linearity in the control structure, and may be considered as an example of the anti-windup schemes recommended in [22]. The solution method for the non-linearity will depend on the choice of norm, as discussed below. In the remainder of this section, we discuss how the control structure may be tailored for the cross-directional control problem.

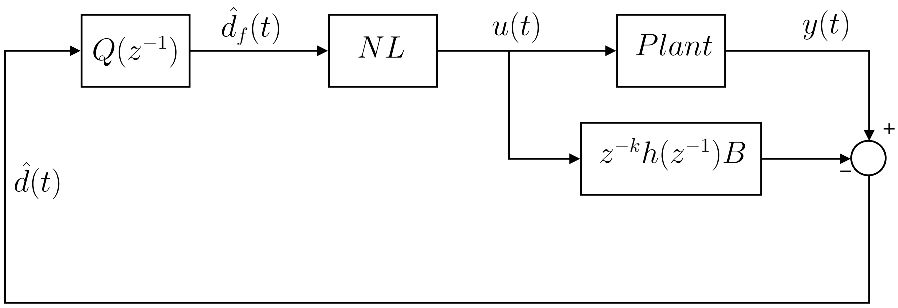

Figure 2: *Basic IMC configuration with non-linear element for anti-windup.*

2.2 Implementation in modal form

Solving (17) is equivalent to finding

$$u(t) = \Psi_r \mu(t) \tag{20}$$

with

$$\mu(t) = \arg\min_{\mu} \left|\left|\Sigma_r \mu + \hat{\delta}_f(t)\right|\right| \text{ such that } \Psi_r \mu \in \mathbb{U} \tag{21}$$

where

$$\hat{\delta}_f(t) = \Phi_r^T \hat{d}_f(t) \tag{22}$$

We may write

$$\hat{\delta}_f(t) = Q_\delta(z^{-1})\hat{\delta}(t) \tag{23}$$

with

$$\begin{aligned} \hat{\delta}(t) &= \Phi_r^T \hat{d}(t) \\ Q_\delta(z^{-1}) &= \Phi_r^T Q(z^{-1})\Phi_r \end{aligned} \tag{24}$$

This leads to the scheme depicted in Fig 3. As stated this is merely a numerical modification. However it is more elegant to design the controller in the context of the modal decomposition [13, 17]. Thus in place of (24), $Q_\delta(z^{-1})$ should be chosen to be diagonal with

$$Q_\delta(1) = I \tag{25}$$

Furthermore it may be seen that careful exploitation of the chosen basis function representation can lead to significant increase in computational efficiency. Nevertheless we stress that the use of basis functions is motivated by the requirement to ensure robust performance rather than computational efficiency.

It is insightful to consider both the unconstrained performance and the steady state performance:

- Without the constraint $u(t) \in \mathbb{U}$ (and under the assumption Σ_r has full rank) this gives the solution

$$\begin{aligned} u(t) &= \Psi_r \mu(t) \\ \mu(t) &= -\Sigma_r^{-1} Q_\delta(z^{-1})\hat{\delta}(t) \\ \hat{\delta}(t) &= \Phi_r^T y(t) - z^{-k}h(z^{-1})\Sigma_r \mu(t) \end{aligned} \tag{26}$$

- In steady state (under the assumption of closed-loop stability)

$$\begin{aligned} u_{ss} &= \Psi_r \mu_{ss} \\ \mu_{ss} &= \arg\min_{u} \left|\left|\Sigma_r \mu + \hat{\delta}_{ss}\right|\right| \text{ such that } \Psi_r \mu \in \mathbb{U} \\ \hat{\delta}_{ss} &= \Phi_r^T y_{ss} - \Sigma_r \mu_{ss} \end{aligned} \tag{27}$$

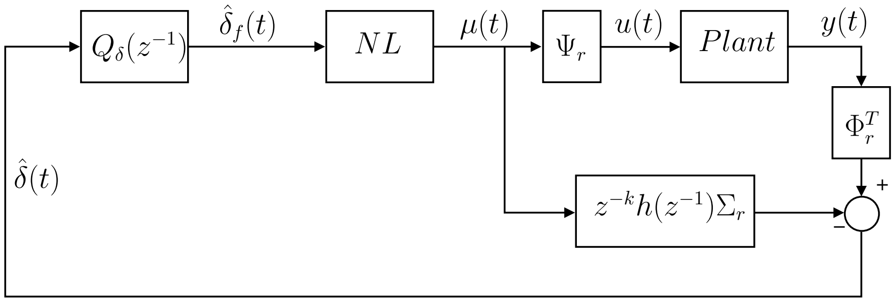

Figure 3: *IMC with anti-windup in modal form.*

2.3 Choice of Q_δ

The most natural choice for $Q_\delta(z^{-1})$ is a scalar transfer function $q(z^{-1})$ times the identity matrix (with dimension $(n \times n)$ for the scheme depicted in Fig 2 and $(r \times r)$ for the scheme depicted in Fig 3). With $h(z^{-1})$ first order (2) we distinguish two cases (see for example [24, 31]):

1. If $\zeta < 1$ a Dahlin controller is appropriate and $q(z^{-1})$ would be chosen as

$$q(z^{-1}) = \left(\frac{1-\beta}{1-\beta z^{-1}}\right)\left(\frac{1-\alpha z^{-1}}{1-\alpha}\right)\left(\frac{1+\zeta}{1+\zeta z^{-1}}\right) \tag{28}$$

   with β a tuning parameter.

2. If $\zeta > 1$ the controller (28) would be unstable and a more appropriate choice would be

$$q(z^{-1}) = \left(\frac{1-\beta}{1-\beta z^{-1}}\right)\left(\frac{1-\alpha z^{-1}}{1-\alpha}\right)\left(\frac{1+\zeta}{\zeta+z^{-1}}\right) \tag{29}$$

Our contention is that if the dynamics of the plant are simple, then designs for constrained systems should be based on design strategies for unconstrained systems. Thus we refer the reader to the modal designs in (for example) [8, 13, 33] for a more detailed discussion on choice of $Q_\delta(z^{-1})$. Where necessary, the transformation (9) should be exploited. However we note the following:

- If an IMC is designed for step output disturbances, it may give unacceptable response for slow output or input disturbances. Generally speaking good design requires a higher order filter $Q_\delta(z^{-1})$ to ensure the appropriate sensitivities are small in closed loop. Most cross-directional control problems are regulator problems (i.e. the output set-point is zero), but if a servo response it required it may be better to incorporate such dynamics in the feedback path. See [31] for a discussion.
- Often such controllers are designed mode by mode, with faster action on the lower order modes (where in general the model is better known), and slower action on the higher order modes. Provided integral action is incorporated, this will have no effect on the implicit steady state cost (13). However in [33] high gain proportional action is used in place of integral action.

2.4 Feedback round the nonlinear block

It is well-known that for anti-windup schemes where the non-linear element is saturation, better response can be obtained by incorporating a feedback term around the non-linearity [41]. The natural generalisation of this to our case is depicted in Fig 4 with $Q_b(z^{-1})$ strictly proper. The control law can be described as:

$$\begin{aligned}
\hat{\delta}(t) &= \Phi_r^T y(t) - z^{-k}h(z^{-1})\Sigma_r\mu(t) \\
\hat{\delta}_q(t) &= Q_f(z^{-1})\hat{\delta}(t) - Q_b(z^{-1})\Sigma_r\mu(t) \\
\mu(t) &= \arg\min_{\mu} \left|\left|\Sigma_r\mu + \hat{\delta}_q(t)\right|\right| \text{ such that } \Psi_r\mu \in \mathbb{U} \\
u(t) &= \Psi_r\mu(t)
\end{aligned} \tag{30}$$

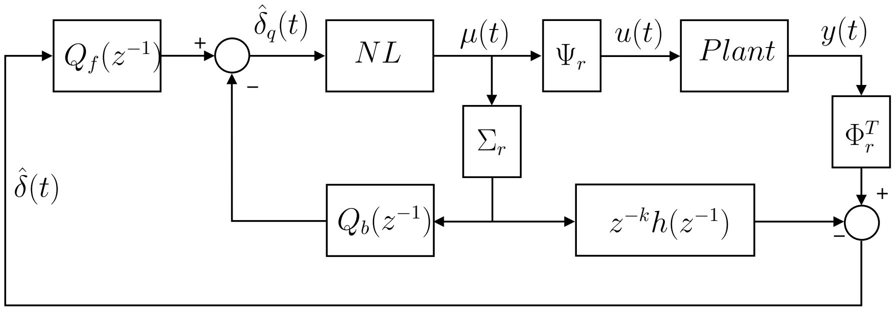

Figure 4: *IMC in modal form with feedback around the non-linear element. This structure was used for the simulations.*

Once again we consider two conditions:

- Without the constraint $u(t) \in \mathbb{U}$ (and under the assumption Σ_r has full rank) this gives the solution

$$\begin{aligned} u(t) &= \Psi_r \mu(t) \\ \mu(t) &= -\Sigma_r^{-1}\left(I - Q_b(z^{-1})\right)^{-1} Q_f(z^{-1})\hat{\delta}(t) \\ \hat{\delta}(t) &= \Phi_r^T y(t) - z^{-k} h(z^{-1})\Sigma_r \mu(t) \end{aligned} \tag{31}$$

- In steady state

$$\begin{aligned} u_{ss} &= \Psi_r \mu_{ss} \\ \mu_{ss} &= \arg\min_u \left|\left| (I - Q_b(1))\, \Sigma_r \mu + Q_f(1)\hat{\delta}_{ss} \right|\right| \text{ such that } \Psi_r \mu \in \mathbb{U} \\ \hat{\delta}_{ss} &= \Phi_r^T y_{ss} - \Sigma_r \mu_{ss} \end{aligned} \tag{32}$$

In order to ensure equivalence with the previous case (Fig 3) under these two conditions, it is sufficient that $Q_f(1)$ is some scalar times the identity and

$$Q_b(z^{-1}) = I - Q_f(z^{-1})\left[Q_\delta(z^{-1})\right]^{-1} \tag{33}$$

For example if we choose

$$Q_f(z^{-1}) = \lambda Q_\delta(z^{-1}) + (1-\lambda) Q_\delta(0) \tag{34}$$

for some scalar λ then $Q_b(z^{-1})$ is guaranteed strictly proper. The choice of λ (in the context of a saturating non-linearity) is discussed in [41].

2.5 Choice of non-linear function

So far we have not specified the choice of norm on the non-linear function. The standard choice would be a 2-norm, which results in a quadratic program. But as we have separated the non-linearity from the dynamics it is straightforward to introduce any other choice of norm, without changing the dynamics away from the constraints. In particular both l_1 norm [7] and l_∞ norm [11] criteria have been recommended for certain cross-directional problems. Either may be substituted in this scheme, resulting in a linear program. Such choices should be motivated by the required steady state performance.

Similarly we may choose to weight each mode differently in the cost function. For example, it may be preferred to weight low order modes (where in general the model is better known) more heavily than high order modes.

In [38, 36] the authors propose adding a barrier to the steady state calculation in MPC. The barrier has a fixed weighting and prevents actuator constraint violation. Such a barrier may also be included in the non-linear function. It has the effect of ensuring the inputs lie on the interior of the constraint set, while a limit is put on the associated performance degradation by the duality gap. Such optimization problems may be solved efficiently via modified interior point algorithms [36].

Finally in [18] a modified steady state criterion is proposed that is robust to model mismatch. It may also be incorporated into the non-linear function. The associated optimization is then usually a conic program.

2.6 Rate constraints

So far we have only considered static constraints on the maximum and minimum values of the actuators (10) as well as their (static) bending constraints (11). Sometimes it is desirable to limit the actuator slew rate

$$\tilde{u}_{\min} \leq u_i(t) - u_i(t-1) \leq \tilde{u}_{\max} \tag{35}$$

Such constraints may be introduced to the control by replacing the condition $\Psi_r\mu \in \mathbb{U}$ in, for example, (30) with the condition

$$\Psi_r\mu \in \mathbb{U} \text{ and } \tilde{u}_{\min} + [\Psi_r\mu(t-1)]_i \leq [\Psi_r\mu]_i \leq \tilde{u}_{\max} + [\Psi_r\mu(t-1)]_i \tag{36}$$

The rate constraints may be satisfied more elegantly by including a first order filter after the non-linearity (see Fig 5)

$$\begin{aligned} u(t) &= q_s(z^{-1})\Psi_r\mu(t) \\ q_s(z^{-1}) &= \frac{1-\rho}{1-\rho z^{-1}} \end{aligned} \tag{37}$$

Then by convexity it follows immediately that the rate constraints will be satisfied (and all other constraints preserved) provided

$$1-\rho \quad \leq \quad \frac{\min(\tilde{u}_{\max}, -\tilde{u}_{\min})}{u_{\max} - u_{\min}} \tag{38}$$

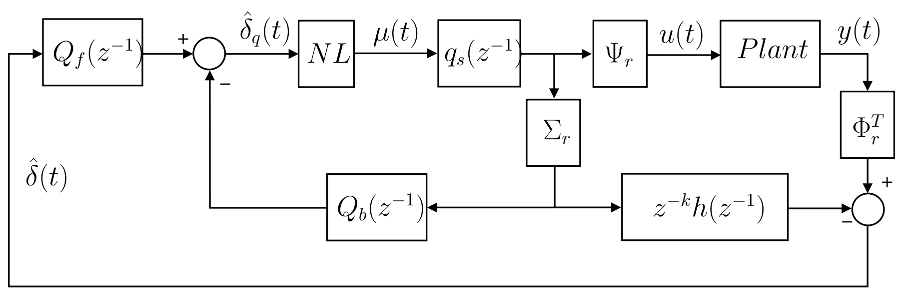

Figure 5: *IMC with automatic rate constraint satisfaction.*

3 Simulation and implementation

3.1 Simulation results

Consider a cross-directional plant model in the form of (1) and (2) with $m = 101$ and $n = 501$, the integer delay term $k = 4$ and parameters $\alpha = 0.95$ and $\zeta = 2$. For the purposes of this simulation, we assume the true interaction matrix is given by B and an estimate of the interaction matrix is denoted by $\hat{B}$ (they are both truncated sinc functions - see [37]). The profiles across the strip for a single actuator u_{51} are shown in Fig 6 for both B and $\hat{B}$. To give some idea of the sensitivity to model mismatch, Fig 7 shows

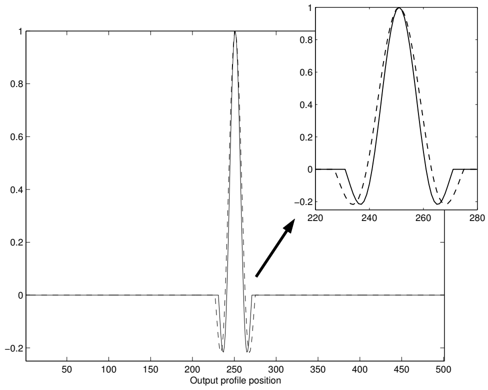

Figure 6: *The cross-directional profiles for both B (solid) and $\hat{B}$ (dashed) with $u_{51} = 1$. Note that $\hat{B}$ is a slightly stretched version of B.*

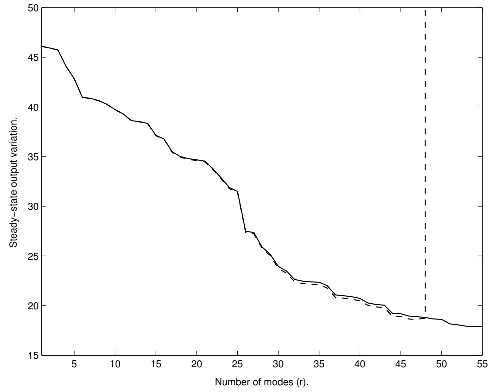

Figure 7: *Steady-state output variation for a step cross-directional disturbance. Both unconstrained (dashed) and constrained (solid) responses are shown. Note that after $r \approx 47$ the variation (with unconstrained response) becomes very large. Tuning parameter values $\beta = 0.2$ and $\lambda = 0.5$ were chosen for both cases.*

the mean output variation for a range of modes (r's) in the unconstrained case. It was found that for $r > 47$, the steady-state variation became very large.

The disturbance term $d(t)$ was constructed as follows. Let $e(t) \in \mathbb{R}^n$ be a zero-mean coloured Gaussian noise sequence with spatial covariance

$$\mathcal{E}\left[e_i(t)e_j(t)\right] = 0.9^{|i-j|}\frac{0.01}{0.81} \tag{39}$$

Then $d(t)$ is given by,

$$d(t) \quad = \quad \frac{0.01}{0.8}\frac{1-0.2z^{-1}}{1-0.99z^{-1}}e(t) \tag{40}$$

Note that once created, this disturbance was fixed and used for each simulation.

Constraints on the actuators are present in the form of (10) and (11) with $u_{\min} = -1$, $u_{\max} = 1$, $\bar{u}_{\min} = -0.5$ and $\bar{u}_{\max} = 0.5$.

We introduce the following measure, which we term the cumulative output variation, to distinguish between different controller designs,

$$\bar{y} \quad = \quad \sum_{i=N_1}^{N_2} \|y(i)\|_2^2 \tag{41}$$

where $N_1 \leq N_2$ are integers. In what follows we fix $N_1 = 200$ and $N_2 = 1000$.

Consider the controller design from Fig 4. We choose a control structure for $Q_\delta(z^{-1})$ as given by (29) and consider a range of choices for β - from $\beta = 0$ through to $\beta = 0.6$ in steps of 0.1. The feedback term $Q_b(z^{-1})$ is given by (33) with $Q_f(z^{-1})$ given by (34). We consider a range of choices for λ - namely $\lambda = 0$, $\lambda = 0.5$ and $\lambda = 1$. As a further alternative we also consider the choice given by (71) which we term (with some abuse of notation) the minimum variance structure. The number of modes was chosen at $r = 40$. The cumulative output variation $\bar{y}$ for this range of β's and λ's are shown in Table 1.

λ/β	0	0.1	0.2	0.3	0.4	0.5	0.6
0	1.6069	1.6065	1.6071	1.6090	1.6130	1.6205	1.6334
0.5	1.6118	1.6090	1.6082	1.6093	1.6130	1.6205	1.6333
1	1.9166	1.7769	1.6813	1.6310	1.6154	1.6195	1.6329
MV	1.6069	1.6065	1.6072	1.6091	1.6131	1.6206	1.6334

Table 1: *Cumulative output variation measure $\bar{y}$ for different parameter values and controller structure. The values shown have been scaled by* 10^4.

Fig 8 shows the profile across the strip at $t = 845$. Both open and closed loop responses are shown. For the closed loop response 40 modes were controlled ($r = 40$) and the tuning parameters β and λ were set to 0.2 and 0.5 respectively.

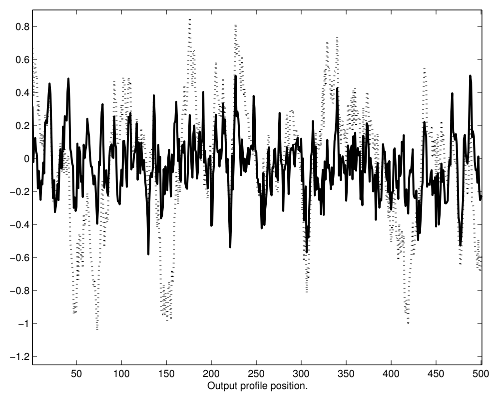

Figure 8: *Cross-directional profile for $t = 845$ in both closed loop (solid) and open loop (dotted). In the closed loop case 40 modes were controlled ($r = 40$) and tuning parameter values $\beta = 0.2$ and $\lambda = 0.5$ were chosen.*

With the same values of β and λ, consider two choices for r - namely $r = 13$ and $r = 40$. The smaller value of $r = 13$ was selected as the largest r such that input constraints are not violated. Fig 9 shows the output variation at each time interval for both choices of r. The cumulative output variation for the case $r = 13$ was $\bar{y} = 3.6785e + 04$ (more than double that for $r = 40$ given in Table 1).

As a final comparison, we considered a related optimisation problem to that in (30) which uses a barrier function with fixed weighting term $\nu = 1$, to represent the constraint set $\mathbb{U}$. The associated optimisation problem appears as,

$$\mu(\nu) \quad = \quad \arg\min_{\mu} \|\Sigma_r \mu + \hat{\delta}_q(t)\| - \nu \sum_{i=1}^{M} \ln(b_i - \{A\Psi_r \mu\}_i) \tag{42}$$

where $\{A\Psi_r\mu\}_i$ is the i'th element of $A\mu$ and A and b represent the constraints on the inputs. In the limit as $\nu \to 0$, the solution to (42) coincides with the solution to that in (30). We chose $\beta = 0.2$ and the minimum variance structure with $r = 40$ and $\nu = 1$. Fig 10 shows a comparison of actuator profiles for $\nu = 1$ and $\nu = 0$, while Fig 11 shows a comparison of the bending actuator profiles for $\nu = 1$ and $\nu = 0$. Note in both cases, the profile for $\nu = 1$ is away from the constraints.

3.2 Discussion of computation

Efficient interior point algorithms [25, 26] and active set algorithms [2] have been tailored for the control of cross-directional variations using model predictive control. Both reported approaches exploit the sparse

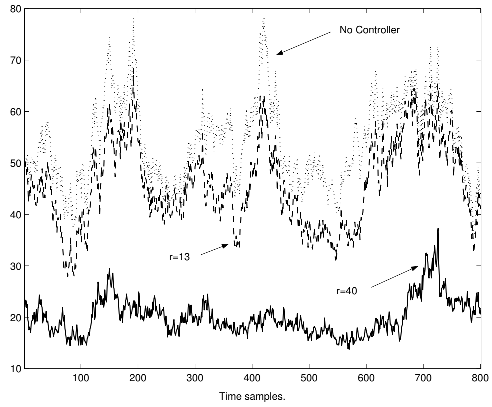

Figure 9: *This figure shows the output variation over the simulation period. Three cases are shown: no controller (dotted),* $r = 13$ *(dashed) and* $r = 40$ *(solid). Tuning parameter values* $\beta = 0.2$ *and* $\lambda = 0.5$ *were chosen.*

structure that arises if robustness considerations are ignored. As we have argued (and others have argued elsewhere) it is necessary to attenuate control action at higher modes. This may be achieved either using equality constraints (setting action on high modes to zero) or including appropriate weighting in the cost function (and removing integral action at high modes). The former approach may reduce the dimension of the Hessian, but with either approach the Hessian loses sparseness. Nevertheless, it is quite practical to solve the optimisation problems required for the control structures proposed in this paper in real time; in particular, as there is no receding horizon, the dimension of the Hessian is no greater than the number of actuators.

To solve the optimisation problem in the simulations we used K. Schittkowski's dual-active-set method written in Fortran (from `http://64.238.116.66/aemdesign/download-ffsqp/qld.f`, see also [23] and references therein). The software was implemented via the Matlab interface for the simulation example in Section 3.1, which includes a maximum of 40 modes and has 400 constraints. The worst case computation time on a Pentium IV, 1.8GHz machine was 9.5ms with a mean computation time of 3.8ms.

As a comparison, we also used a modified version of the OOQP package from Gertz and Wright, which uses a primal-dual interior-point algorithm (see `http://www.cs.wisc.edu/ swright/ooqp/`). For this application the algorithm was somewhat slower, with a worst case computation time of 95ms and a mean computation time of 78ms. Note that by changing a few lines of code, we could also solve the weighted barrier alternative (42) for any value of $\nu > 0$.

These results are an indicator of what might be achievable on an industrial problem. Performance is dictated by a number of factors including choice of hardware and software, the size and density of the interaction matrix B, the form of the constraints (including equality constraints) and the number of active constraints. Since the resultant Hessian is dense for this case, we would usually expect active set methods to be faster than interior point methods (see for example [34] for a discussion). Both active set and interior point methods can be made considerably more efficient by exploiting the specific problem structure. The following discussion is restricted to interior point algorithms (see also [39]).

Interior-point methods typically involve the solution of a system of linear equations at each iteration. For problem (30) this system appears as,

$$\begin{bmatrix} \Sigma_r^T \Sigma_r & L^T \\ L & -D \end{bmatrix} \begin{bmatrix} \Delta\mu & \Delta z \end{bmatrix} = \begin{bmatrix} g_1 & g_2 \end{bmatrix} \tag{43}$$

where D is a positive definite diagonal matrix and z is associated with the Lagrange multipliers for the inequality constraints $L\mu \leq b$. Equation (43) can be reduced to the normal equation form by eliminating Δz, resulting in a positive definite system,

$$\left[\Sigma_r^T \Sigma_r + L^T D^{-1} L\right] \Delta\mu = g_1 + L^T D^{-1} g_2 \tag{44}$$

Equation (43) can be factored using a sparse LU-factorisation - see e.g. the MA27, MA47 and MA57 packages from the Harwell library (`http://hsl.rl.ac.uk/`).

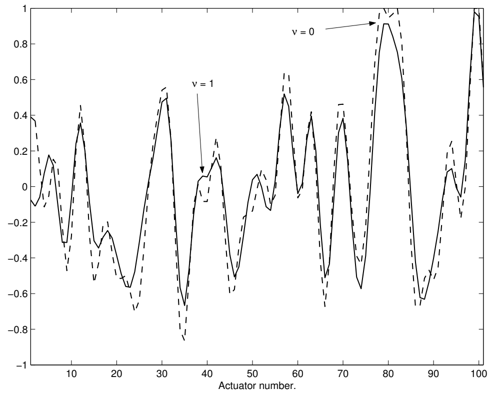

Figure 10: *Input profiles for* $\nu = 0$ *and* $\nu = 1$. *40 modes were controlled* ($r = 40$) *and the minimum variance structure was chosen.*

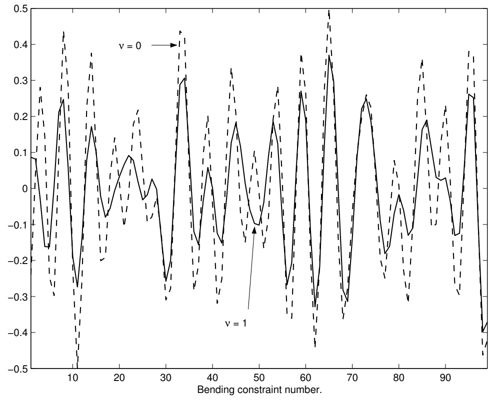

Figure 11: *Bending input profiles for* $\nu = 0$ *and* $\nu = 1$. *40 modes were controlled* ($r = 40$) *and the minimum variance structure was chosen.*

Equation (44) may be factored using a Cholesky factorisation, but care must be taken near the constraint boundaries since D will include large terms in this case. Wright [40] offers a modified Cholesky factorisation which remains stable under these circumstances. In terms of floating-point-operations (flops), the Cholesky factorisation costs $r^3/3$ flops, where r is the number of modes. However, the matrix multiplication to obtain $L^T D^{-1} L$ is $O(rM^2)$ flops, where M is the number of constraints. Hence the normal equation form (44) may well be expensive even when r is small.

4 Relation to MPC

We have argued here and elsewhere [17, 37] that when the plant behaviour is well-known it may be advantageous to design a constrained controller. The usual approach for cross-directional control is to design an MPC (e.g. [17, 27, 7, 21]). In this paper we have proposed a modification to an unconstrained IMC structure, based on [22]. In particular we propose that IMC design considerations should be applied to the constrained case. But it is possible to view the resultant controller as an MPC. The relation between IMC and one-step horizon MPC has been explored in the unconstrained case (e.g. [14]). In [32] it is argued that one-step horizon MPC is equivalent to IMC with saturating anti-windup in only a limited number of cases. This becomes a matter of definition: if the anti-windup is allowed to have a more general non-linearity then the equivalence class becomes larger. In the same paper [32] it is argued that single step horizon MPC is adequate for plants with simple dynamics (in particular where the plant without delay is non-minimum phase). This was also argued in [17], but may equally be viewed as an argument for the control structure proposed in the current paper. In what follows we draw some connections between the proposed scheme and MPC.

4.1 One step horizon MPC

Suppose we wish to implement a control law that minimises the one-step horizon cost (c.f. [6])

$$u(t) = \arg\min_{u(t)} \mathcal{E}\left\{ \left|\left| p_1(z^{-1})\Phi_r^T \hat{y}(t+k) \right|\right|_2^2 + \left|\left| p_2(z^{-1})\Phi_r^T Bu(t) \right|\right|_2^2 \right\} \text{ such that } u(t) \in \mathbb{U} \tag{45}$$

This is equivalent to finding

$$\mu(t) = \arg\min_{\mu}(t)\mathcal{E}\left\{ \left|\left| p_1(z^{-1})\eta(t+k) \right|\right|_2^2 + \left|\left| p_2(z^{-1})\Sigma_r \mu(t) \right|\right|_2^2 \right\} \text{ such that } \Psi_r \mu(t) \in \mathbb{U} \tag{46}$$

Suppose further we may partition

$$p_1(z^{-1})y(t+k) = f_1(z^{-1})e(t+k) + f_2(z^{-1})y(t) + f_3(z^{-1})Bu(t) \tag{47}$$

with $f_1(z^{-1})$ a polynomial of order $k-1$. Then our control is equivalent to

$$\begin{aligned} \mu(t) = {} & \arg\min_{\mu(t)} \left\{ \left|\left| f_2(z^{-1})\eta(t) + f_3(z^{-1})\Sigma_r \mu(t) \right|\right|_2^2 + \left|\left| p_2(z^{-1})\Sigma_r \mu(t) \right|\right|_2^2 \right\} \\ & \text{such that } \Psi_r \mu(t) \in \mathbb{U} \end{aligned} \tag{48}$$

This is a quadratic program. Specifically

$$\mu(t) = \arg\min_{\mu} \left\{ \mu^T \Sigma_r^T \Sigma_r \mu + 2g_1^T(t)\Sigma_r \mu \right\} \text{ such that } \Psi_r \mu(t) \in \mathbb{U} \tag{49}$$

with

$$g_1(t) = \frac{f_2(z^{-1})}{f_3(0) + p_2(0)}\eta(t) + \frac{f_3(z^{-1}) - f_3(0) + p_2(z^{-1}) - p_2(0)}{f_3(0) + p_2(0)}\Sigma_r \mu(t) \tag{50}$$

With scalar filters

$$\begin{aligned} Q_f(z^{-1}) &= q_f(z^{-1})I \\ Q_b(z^{-1}) &= q_b(z^{-1})I \end{aligned} \tag{51}$$

the control of Fig 4 becomes

$$\mu(t) = \arg\min_{\mu} \left|\left| \Sigma_r \mu + q_f(z^{-1})\eta(t) - z^{-k} q_f(z^{-1})h(z^{-1})\Sigma_r \mu(t) - q_b(z^{-1})\Sigma_r \mu(t) \right|\right|_2^2 \tag{52}$$

This is likewise a quadratic program

$$\mu(t) = \arg\min_{\mu} \left\{ \mu^T \Sigma_r^T \Sigma_r \mu + 2g_2^T(t)\Sigma_r \mu \right\} \text{ such that } \Psi_r \mu(t) \in \mathbb{U} \tag{53}$$

with

$$g_2(t) = q_f(z^{-1})\eta(t) - \left[z^{-k} q_f(z^{-1})h(z^{-1}) + q_b(z^{-1}) \right] \Sigma_r \mu(t) \tag{54}$$

Equivalence follows immediately by equating the coefficients of $g_1(t)$ and $g_2(t)$. Specifically set

$$\begin{aligned} q_f(z^{-1}) &= \frac{f_2(z^{-1})}{f_3(0) + p_2(0)} \\ q_b(z^{-1}) &= -\left(\frac{f_3(z^{-1}) - f_3(0) + p_2(z^{-1}) - p_2(0) + z^{-k}h(z^{-1})f_2(z^{-1})}{f_3(0) + p_2(0)} \right) \end{aligned} \tag{55}$$

Note that more general structures (directions) for u in (45) would require non-diagonal filters and changing the non-linear element in Fig 4.

4.2 Minimum variance control

Suppose our plant is given by (1) and (2) with

$$d(t) = \frac{1-\gamma z^{-1}}{1-z^{-1}} e(t) \tag{56}$$

Here $e(t)$ is some zero mean noise with $\mathcal{E}\left\{e(t)e^T(t-k)\right\} = 0$ for $k \geq 1$. Suppose we want the minimum variance control

$$u(t) = \arg\min_u \mathcal{E}\left\{\left|\left|\Phi_r^T \hat{y}(t+k)\right|\right|_2^2\right\} \text{ such that } u(t) \in \mathbb{U} \tag{57}$$

We will distinguish the cases when the open-loop plant is minimum phase ($\zeta < 1$) and when the open-loop plant is non-minimum phase ($\zeta > 1$). In both cases the analysis follows from the standard results for unconstrained systems (for example [1]):

1. If $\zeta < 1$ the unconstrained solution is the Dahlin controller [31]. With constraints, this gives the control

$$\begin{aligned} \mu(t) &= \arg\min_\mu \left\{\left|\left|\frac{g(z^{-1})}{c(z^{-1})}\eta(t) + \frac{b(z^{-1})f(z^{-1})}{c(z^{-1})}\Sigma_r\mu\right|\right|_2^2\right\} \\ &\quad \text{such that } \Psi_r\mu(t) \in \mathbb{U} \end{aligned} \tag{58}$$

where the polynomials $a(z^{-1})$, $b(z^{-1})$ and $c(z^{-1})$ are given by

$$\begin{aligned} a(z^{-1}) &= (1-\alpha z^{-1})(1-z^{-1}) \\ b(z^{-1}) &= (1+\zeta z^{-1})(1-z^{-1})\left(\frac{1-\alpha}{1+\zeta}\right) \\ c(z^{-1}) &= (1-\alpha z^{-1})(1-\gamma z^{-1}) \end{aligned} \tag{59}$$

and the polynomials $f(z^{-1})$ and $g(z^{-1})$ satisfy the Diophantine equation

$$c(z^{-1}) = a(z^{-1})f(z^{-1}) + z^{-k}g(z^{-1}) \tag{60}$$

where $f(z^{-1})$ has degree $k-1$. This yields

$$\begin{aligned} (1-z^{-1})f(z^{-1}) &= 1-\gamma z^{-1} - (1-\gamma)z^{-k} \\ g(z^{-1}) &= (1-\gamma)(1-\alpha z^{-1}) \end{aligned} \tag{61}$$

and

$$\begin{aligned} \frac{g(z^{-1})}{c(z^{-1})} &= \left(\frac{1-\gamma}{1-\gamma z^{-1}}\right) \\ \frac{b(z^{-1})f(z^{-1})}{c(z^{-1})} &= \left(\frac{1-\alpha}{1-\alpha z^{-1}}\right)\left(\frac{1+\zeta z^{-1}}{1+\zeta}\right)\left[1 - z^{-k}\left(\frac{1-\gamma}{1-\gamma z^{-1}}\right)\right] \end{aligned} \tag{62}$$

This in turn gives

$$\begin{aligned} Q_f(z^{-1}) &= \left(\frac{1+\zeta}{1-\alpha}\right)\left(\frac{1-\gamma}{1-\gamma z^{-1}}\right) I \\ Q_b(z^{-1}) &= \left(\frac{-(\alpha+\zeta)z^{-1}}{1-\alpha z^{-1}}\right) I \end{aligned} \tag{63}$$

It can be observed that this corresponds to

$$Q_\delta(z^{-1}) = \left(\frac{1-\alpha z^{-1}}{1-\alpha}\right)\left(\frac{1-\gamma}{1-\gamma z^{-1}}\right)\left(\frac{1+\zeta}{1+\zeta z^{-1}}\right) I \tag{64}$$

as in (28) with $\beta = \gamma$. The choice of $Q_f(z^{-1})$ and $Q_b(z^{-1})$ corresponds to $\lambda = \frac{\beta}{\beta-\alpha}$ in (34).

2. If $\zeta > 1$ the controller (64) would be unstable. Instead, if we split $b(z^{-1})$ into

$$\begin{aligned} b^-(z^{-1}) &= 1 + \zeta z^{-1} \\ b^+(z^{-1}) &= (1 - z^{-1})\left(\frac{1-\alpha}{1+\zeta}\right) \end{aligned} \tag{65}$$

the solution to (57) is given by

$$\begin{aligned} \mu(t) &= \arg\min_{\mu}\left\{\left\|\frac{\tilde{g}(z^{-1})}{c(z^{-1})}\eta(t) + \frac{b^+(z^{-1})\tilde{f}(z^{-1})}{c(z^{-1})}\Sigma_r\mu\right\|_2^2\right\} \\ &\text{such that } \Psi_r\mu(t) \in \mathbb{U} \end{aligned} \tag{66}$$

where $\tilde{f}(z^{-1})$ and $\tilde{g}(z^{-1})$ satisfy the Diophantine equation

$$c(z^{-1})b^{-*}(z^{-1}) = a(z^{-1})\tilde{f}(z^{-1}) + z^{-k}b^-(z^{-1})\tilde{g}(z^{-1}) \tag{67}$$

Here the degree of $\tilde{f}(z^{-1})$ is the sum of $k-1$ and the degree of $b^{-*}(z^{-1})$. In our case

$$b^{-*}(z^{-1}) = \zeta + z^{-1} \tag{68}$$

This yields

$$\begin{aligned} (1 - z^{-1})\tilde{f}(z^{-1}) &= (1 - \gamma z^{-1})(\zeta + z^{-1}) - (1-\gamma)(1 + \zeta z^{-1})z^{-k} \\ \tilde{g}(z^{-1}) &= (1-\gamma)(1 - \alpha z^{-1}) \end{aligned} \tag{69}$$

and

$$\begin{aligned} \frac{\tilde{g}(z^{-1})}{c(z^{-1})} &= \left(\frac{1-\gamma}{1-\gamma z^{-1}}\right) \\ \frac{b^+(z^{-1})\tilde{f}(z^{-1})}{c(z^{-1})} &= \left(\frac{1-\alpha}{1-\alpha z^{-1}}\right)\left[\left(\frac{\zeta + z^{-1}}{1+\zeta}\right) - z^{-k}\left(\frac{1+\zeta z^{-1}}{1+\zeta}\right)\left(\frac{1-\gamma}{1-\gamma z^{-1}}\right)\right] \end{aligned} \tag{70}$$

This in turn gives

$$\begin{aligned} Q_f(z^{-1}) &= \left(\frac{1+\zeta^{-1}}{1-\alpha}\right)\left(\frac{1-\gamma}{1-\gamma z^{-1}}\right) I \\ Q_b(z^{-1}) &= \left(\frac{-(\alpha + \zeta^{-1})z^{-1}}{1-\alpha z^{-1}}\right) I \end{aligned} \tag{71}$$

As before, it can be observed that this corresponds to

$$Q_\delta(z^{-1}) = \left(\frac{1-\alpha z^{-1}}{1-\alpha}\right)\left(\frac{1-\gamma}{1-\gamma z^{-1}}\right)\left(\frac{1+\zeta}{\zeta + z^{-1}}\right) I \tag{72}$$

as in (29) with $\beta = \gamma$.

5 Conclusion

It has become standard to address the cross-directional control problem by decomposing the inputs and outputs into modes. For robust stability and performance it is necessary that the controller acts only on a reduced number of these modes.

If the controller acts on a sufficiently small number of modes the actuators will automatically lie within their constraints. But to maximise economic performance it may be necessary to include a greater number of modes, requiring actuator constraints to be taken explicitly into account. We have shown that in this case it is possible to guarantee optimal steady state performance with a modified IMC structure. For the simple dynamics associated with most web processes this also gives good closed-loop dynamic behaviour (with less simple dynamics introducing a receding horizon may have significant benefit). Thus unconstrained control design techniques may be applied directly to the constrained control problem.

We have demonstrated such a control design with a simulation example. It illustrates that such a controller may be easily implemented in real time, even on systems with fast sampling.

References

[1] K. J. Åström and B. Wittenmark. *Computer-Controlled Systems (2nd edition).* Prentice-Hall, Englewood Cliffs, 1990.

[2] R. A. Bartlett, L. T. Biegler, J. Backstrom, and V. Gopal. Quadratic programming algorithms for large-scale model predictive control. *J. of Process Control*, 12:775–795, 2002.

[3] T. J. Boyle. Control of cross-directional variations in web forming machines. *Can. J. of Chem. Eng.*, 55:457–461, 1977.

[4] T. J. Boyle. Practical algorithms for cross-direction control. *TAPPI*, 61:77–80, 1978.

[5] R. A. Braatz, M. L. Tyler, M. Morari, F. R. Pranckh, and L. Sartor. Identification, estimation, and control of sheet and film processes. *AIChE Journal*, 38:1329–1339, 1992.

[6] D. W. Clarke and P. J. Gawthrop. Self tuning controller. *Proc. IEE*, 122:929–934, 1975.

[7] P. Dave, D. A. Willig, G. K Kudva, J. F Pekny, and F. J. Doyle. LP methods in MPC of large scale systems- application to paper machine CD control. *AIChE Journal*, 43:1016–1031, 1997.

[8] S. R. Duncan. *The cross-directional control of web-forming processes.* PhD thesis, University of London, 1989.

[9] S. R. Duncan. Editorial. Special section: Cross directional control. *IEE Proc. Control Theory Appl.,*, 149:412–413, 2002.

[10] S. R. Duncan and G.F. Bryant. The spatial bandwidth of cross-directional control systems for web processes. *Automatica*, 33(2):139–153, 1997.

[11] S. R. Duncan and K. Corscadden. Mini-max control of cross-directional variation in a paper machine. *IEE Proc. Control Theory Appl.,*, 145:189–195, 1998.

[12] S. R. Duncan, W. P. Heath, A. Halousková, and M. Karný. Application of basis functions to the cross-directional control of web processes. UKACC International Conference on CONTROL '96, 2-5 Sept., 1996.

[13] A. P. Featherstone, J. G. VanAntwerp, and R. D. Braatz. *Identification and Control of Sheet and Film Processes.* Springer-Verlag, London, 2000.

[14] P. J. Gawthrop, R. W. Jones, and D. S. Sbarbaro. Emulator-based control and internal model control: complementary approaches to robust control design. *Automatica*, 32:1223–1227, 1996.

[15] G. C. Goodwin, S. J. Lee, A. Carlton, and G. Wallace. Application of Kalman filtering to zinc coating mass estimation. Proc. of the IEEE Conf. on Control Applications, Piscataway, New Jersey, 1994.

[16] A. Halousková, M. Kárný, and I. Nagy. Adaptive cross-direction control of paper basis weight. *Automatica*, 29:425–429, 1993.

[17] W. P. Heath. Orthogonal functions for cross-directional control of web-forming processes. *Automatica*, 32:183–198, 1996.

[18] D. E. Kassmann, T. A. Badgwell, and R. B. Hawkins. Robust steady-state target calculation for model predictive control. *AIChE*, 46(5):1007–1024, May 2000.

[19] K. Kristinsson and G. A. Dumont. Cross directional control on paper machines using gram polynomials. *Automatica*, 32:533–548, 1996.

[20] D. Laughlin, M. Morari, and R. D Braatz. Robust performance of cross-directional basis-weight control in paper machines. *Automatica*, 29:1395–1410, 1993.

[21] D. L. Ma, J. G. VanAntwerp, M. Hovd, and R. D. Braatz. Quantifying the potential benefits of constrained control for a large-scale system. *IEE Proc. -Control Theory Appl.*, 149:423–432, 2002.

[22] Y. Peng, D. Vrančić, R. Hanus, and S. R. Weller. Anti-windup designs for multivariable controllers. *Automatica*, 34:1559–1565, 1998.

[23] M. J. D. Powell. On the quadratic programming algorithm of Goldfarb and Idnani. *Mathematical Programming Study*, 25:46–61, 1985.

[24] D. M. Prett and C. E. García. *Fundamental Process Control*. Butterworth-Heinemann, Stoneham, 1988.

[25] C. V. Rao, J. C. Campbell, J. B. Rawlings, and S. J. Wright. Efficient implementation of model predictive control for sheet and film forming processes. Proc. American Control Conference, Albuquerque, New Mexico, 1997.

[26] C. V. Rao, J. B. Rawlings, and S. J. Wright. Application of interior point methods to model predictive control. *J. Opt. Theo. Applics.*, 99:723–757, 1998.

[27] J. B. Rawlings and I.-L. Chien. Gage control of film and sheet-forming processes. *AIChE Journal*, 42:753–766, 1996.

[28] N. L. Ricker. Use of quadratic programming for constrained internal model control. *Ind. Eng. Chem. Process Pres. Dev.*, 24:925–936, 1985.

[29] A. Rigopoulos and Y. Arkun. Model predictive control of CD profiles in sheet forming processes using full profile disturbance models identified by adaptive pca. Proc. American Control Conference, Albuquerque, 1997.

[30] O. J. Rojas, G. C. Goodwin, and G. V. Johnston. A spatial frequency anti-windup strategy for cross directional control problems. *IEE Proc. -Control Theory Appl.*, 149:414–422, 2002.

[31] D. E. Seborg, T. F. Edgar, and D. A. Mellichamp. *Process Dynamics and Control*. Wiley, New York, 1989.

[32] M. Soroush and S. Valluri. Optimal directionality compensation in processes with input saturation non-linearities. *Int. J. Control*, 72:1555–1564, 1999.

[33] G. E. Stewart, D. M. Gorinevsky, and G. A. Dumont. Two-dimensional loop shaping. *Automatica*, 39:779–792, 2003.

[34] M. J. Todd. The many facets of linear programming. *Mathematical Programming*, 91:417–436, 2002.

[35] R. G. Wilhelm and M. Fjeld. Control algorithms for cross directional control: the state of the art. pages 163–174. Proc. 5th Conf. on Instrumentation and Automation in the Paper, Rubber, Plastics and Polymerisation Industries, Antwerp, 1996.

[36] A. G. Wills. *Barrier Function Based Model Predicitve Control*. PhD thesis, University of Newcastle, 2003.

[37] A. G. Wills and W. P. Heath. Analysis of steady-state performance for cross-directional control. *IEE Proc. -Control Theory Appl.*, 149:433–440, 2002.

[38] A. G. Wills and W. P. Heath. A recentred barrier for constrained receding horizon control. American Control Conference, Anchorage, May 8–10, 2002.

[39] A. G. Wills and W. P. Heath. Interior-point methods for linear model predictive control. Report EE03016, University of Newcastle, NSW 2308, Australia, 2003.

[40] S. J. Wright. Modified Cholesky factorizations in interior-point algorithms for linear programming. *SIAM Journal on Optimization*, 9:1159–1191, 1999.

[41] A. Zheng, M. V. Kothare, and M. Morari. Anti-windup design for internal model control. *Int. J. Control*, 60:1015–1024, 1994.