# A sufficient condition for the stability of optimizing controllers with saturating actuators

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SUMMARY

The quadratic programme that must be solved with certain output-feedback model predictive controllers can be expressed as a continuous sector-bounded non-linearity together with two linear transformations. Thus the multivariable circle criterion gives a simple test for stability, with or without model mismatch. In particular it may be applied if the open-loop plant is stable and the actuators are subject to simple saturation constraints. In the case of single horizon model predictive control, it suffices to check for positive realness a transfer function matrix whose dimension corresponds to the number of inputs. For an arbitrary length receding horizon it suffices to check the poles of a low dimension transfer function matrix and the eigenvalues (over an appropriate range of operator values) of a matrix whose dimension is independent of the horizon length. 

KEY WORDS: Anti-windup control, model predictive control, robust control, circle criterion.

## 1. INTRODUCTION

In their recent survey of stability and robustness of model predictive control (MPC), Mayne et al. [15] conclude that it is both possible and straightforward to design a constrained model predictive controller that is guaranteed nominally stable with state feedback. The construction requires a terminal cost, a terminal constraint and a local stabilising controller. It also requires that the steady state does not lie on the constraint boundary, and assumes feasibility. However they conclude that the construction of an implementable controller with guaranteed robustness remains an open problem. For example, attempts to impose robustness via a min-max optimization lead to considerable computational and conceptual difficulties (see also the discussion in [14]).

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This contrasts with industrial practice (see [19]) where there are many examples of implemented controllers which are *apparently* robustly stabilising. Zafiriou [26] proposed that rather than "augmenting the objective functions" (in order to try and satisfy some proposed robustness constraint), "one should study the problem in its nonlinear nature [and] obtain conditions that guarantee nominal and robust stability and performance and tune the parameters of the original optimization problems ... to satisfy them." In particular, he proposed exploiting for analysis the piecewise affine structure of model predictive control that arises with linear plant, quadratic cost and linear inequality constraints. Heise and Maciejowski [10] used exactly such an approach; specifically they observed that if the affine controller corresponding to each possible combination of active sets can be shown to be nominally stabilising, then the controller itself is nominally stabilising. The approach does not seem to have received widespread attention. One reason might be the "curse of dimensionality"—the number of local affine controllers to check can become prohibitively large even for modest horizons. This phenomenon is related to the worst-case complexity for active set quadratic programming [6].

Zheng [27] adopts a similar philosophy to that of [26] and provides a sufficient condition for robust stability of *state* feedback MPC. Sufficient conditions for *nominal* stability of MPC with output feedback are provided in [28] and [5].

In this paper we provide a sufficient condition for the stability of certain output feedback MPC structures with open-loop stable plant and static input constraints. The condition may be applied when there is model mismatch, and hence used to assess the robustness of the controller. Specifically we observe that a certain quadratic programme is sector bounded. Consequently the quadratic programme that occurs in MPC with actuator saturation constraints can be expressed as a continuous sector-bounded non-linearity together with two linear transformations. Thus, in the case of static input constraints, it is not necessary to check the properties of each possible local affine controller. Rather, the multivariable circle criterion gives a simple test for stability (feasibility is guaranteed for such constraints). In the case of single horizon model predictive control, it suffices to check an $(n_u \times n_u)$ transfer function matrix for positive realness, where $n_u$ is the number of actuators. As a first example, we demonstrate the test for a two-input two-output anti-windup scheme proposed in [18]. We further show that for an arbitrary length receding horizon it suffices to check the poles of a low dimension transfer function matrix and the eigenvalues (over an appropriate range of operator values) of a $(2n_u \times 2n_u)$ matrix. We demonstrate the test for a two-input two-output model predictive control problem with a control horizon of 10.

## 2. QUADRATIC PROGRAMMING AS A SECTOR-BOUNDED NONLINEARITY

In this section we demonstrate that certain quadratic programmes may be expressed as sector-bounded nonlinearities. We will exploit these results in Section 3 by applying the multivariable circle criterion to give conditions for stability for a class of optimizing controllers.

Suppose $K_2 - K_1$ is positive definite for some $K_1, K_2 \in \mathbb{R}^{N\times N}$. The mapping $f : \mathbb{R}^N \to \mathbb{R}^N$ belongs to the sector $[K_1, K_2]$ (see e.g. [12]) if

$$(f(x) - K_1x)^T(f(x) - K_2x) \leq 0 \text{ for all } x \in \mathbb{R}^N \tag{1}$$



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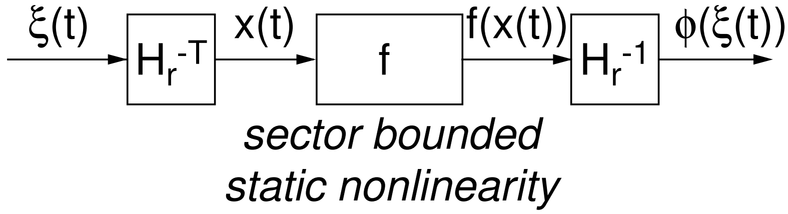


Figure 1. Quadratic programming as the static nonlinearity $\phi$. The static nonlinearity $f$ is continuous in x and sector bounded.

Let the function $f$ be given by the quadratic programme

$$\begin{aligned} f(x) \quad &= \quad \arg\min_{n} \frac{1}{2} n^T n - n^T x \\ & \quad\quad \text{subject to } Ln \preceq b \text{ with } b \succeq 0 \\ & \quad\quad \text{and } Mn = 0 \end{aligned} \tag{2}$$

with "$\preceq$" and "$\succeq$" denoting term-by-term inequality.

**Result 1:** Let $f$ be given by (2). Then $f$ belongs to the sector $[0, I]$.

**Proof:** The Karoush Kuhn Tucker (KKT) conditions [6] for $f$ give

$$\begin{aligned} f(x) + M^T\zeta + L^T\lambda - x &= 0 \\ Mf(x) &= 0 \\ Lf(x) + s &= b \\ s^T\lambda &= 0 \end{aligned} \tag{3}$$

for some $\zeta$ with $s \succeq 0$ and $\lambda \succeq 0$. Pre-multiplying the first KKT condtion by $f(x)^T$ and substituting gives

$$f(x)^T f(x) - f(x)^T x = -b^T\lambda \leq 0 \tag{4}$$

Hence we may say

$$f(x)^T (f(x) - x) \leq 0 \tag{5}$$

□

Consider the function $\phi$, given by the positive definite quadratic programme

$$\begin{aligned} \phi(\xi) \quad &= \quad \arg\min_{\nu} \frac{1}{2} \nu^T H \nu - \nu^T \xi, \\ & \quad\quad \text{subject to } A\nu \preceq b \text{ with } b \succeq 0 \\ & \quad\quad \text{and } C\nu = 0 \end{aligned} \tag{6}$$

Then $\phi$ may be expressed as a sector bounded non-linearity together with two linear transformations (see Fig 1). Specifically:

**Corollary 1:** Let $\phi$ be given by (6) with $H$ positive definite. Let $H_r \in \mathbb{R}^{N\times N}$ satisfy $H_r^T H_r = H$. Then we may express $\phi$ as

$$\begin{aligned} \phi(\xi) &= H_r^{-1} f(x) \\ x &= H_r^{-T}\xi \end{aligned} \tag{7}$$

with $f$ given by (2). Furthermore, $f$ belongs to the sector $[0, I]$.

**Proof:** The result follows immediately from the substitutions

$$\begin{aligned} n &= H_r \nu \\ x &= H_r^{-T}\xi \\ A &= L H_r \\ C &= M H_r \end{aligned} \tag{8}$$

□

If $H_r$ is not already given explicitly we may take it to be either the right Cholesky factor of $H$ or to be $H^{\frac{1}{2}}$.

## 3. A FRAMEWORK FOR OPTIMIZING CONTROL

Many optimizing controllers can be expressed in the form:

$$\begin{aligned} u(t) &= K_\nu(\cdot)\nu^*(t) \\ \nu^*(t) &= \arg\min_\nu \frac{1}{2}\nu^T H \nu - \nu^T x(t) \\ &\quad\ \text{subject to } A\nu \preceq b(t) \text{ and } C\nu = d(t) \\ x(t) &= K_u(\cdot)u(t) + K_y(\cdot)y(t) + K_r(\cdot)r(t) \end{aligned} \tag{9}$$

See Fig 2. Here $u(t) \in \mathbb{R}^{n_u}$ and $y(t) \in \mathbb{R}^{n_y}$ are the plant input and plant output respectively. Assume $x(t) \in \mathbb{R}^N$ and $H \in \mathbb{R}^{N,N}$ for some $N$, with $H$ symmetric and positive definite. The objects $A$, $b(t)$, $C$ and $d(t)$ are matrices (or column vectors) of appropriate dimension. The objects $K_\nu(\cdot)$, $K_u(\cdot)$, $K_y(\cdot)$ and $K_r(\cdot)$ are all proper transfer function matrices of appropriate dimension, with $K_u(\cdot)$ strictly proper. They may be either discrete or continuous. Such controllers are most naturally implemented digitally; however, some examples in the literature (in particular for anti-windup) are expressed in the continuous domain. The optimal value $\nu^*$ can be obtained by standard quadratic programming software—see e.g. [14] or [25] in the context of model predictive control.

We will require the condition that the solution $\nu(t) = 0$ is always feasible. This is equivalent to the condition that $b(t) \succeq 0$ and $d(t) = 0$ for all $t$.

Let the transfer function matrix $P(\cdot)$ represent the plant dynamics from $u$ to $y$, and put

$$K_p(\cdot) = K_u(\cdot) + K_y(\cdot)P(\cdot) \tag{10}$$

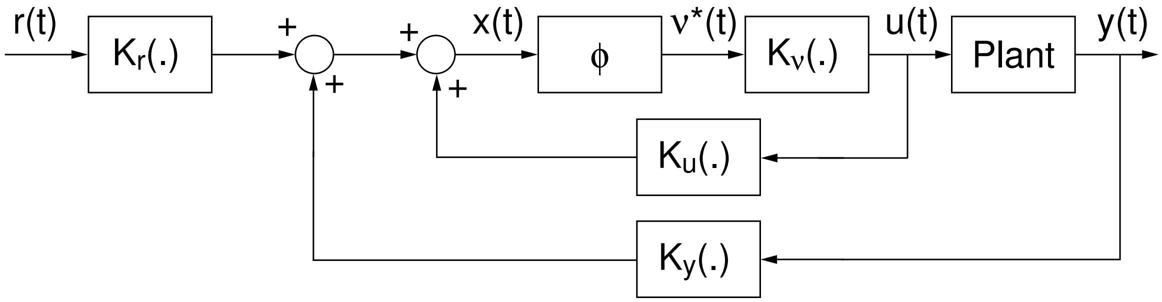


Figure 2. Framework for optimizing control.

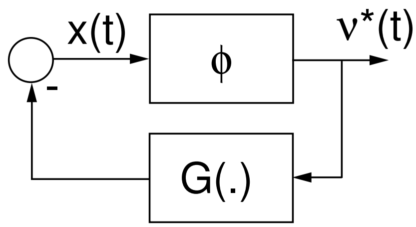


Figure 3. Feedback around the nonlinearity.

We can express the closed-loop system as the negative feedback interconnection of $\phi$ and $G(\cdot)$ (see Fig 3) with $\phi$ given by (6) and

$$G(\cdot) = -K_p(\cdot)K_\nu(\cdot) \tag{11}$$

Equivalently, we can express the closed-loop system as the negative feedback interconnection of $f$ given by (2) and $\tilde{G}(\cdot)$ with

$$\tilde{G}(\cdot) = (H_r^T)^{-1}G(\cdot)H_r^{-1} \tag{12}$$

We may now apply either the discrete or continuous version of the multivariable circle criterion [7, 8] to establish stability. Before we state the result we require the following definition:

**Definition:** A square transfer function matrix $K(z)$ (or $K(s)$) is strongly positive real if:

1. $K(\cdot)$ is asymptotically stable,
2. $K(e^{j\theta}) + K^*(e^{j\theta})$ (or $K(j\omega) + K^*(j\omega)$) is positive definite for all $\theta \in [0, 2\pi]$ (or for all real $\omega$). Here $K^*$ denotes the conjugate transpose of $K$.
3. $D + D^T > 0$ where $D = K(\infty)$.

**Result 2:** Suppose a controller of the form (9) is used with plant $P(\cdot)$. If setting $\nu(t) = 0$ is always feasible then a sufficient condition for closed-loop asymptotic stability is that $H + G(\cdot)$



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is strongly positive real, with $G(\cdot)$ given by (11).

**Proof:** It is well-known that $f$ is continuous in $x$ [4, 2]. Result 1 shows that $f$ is also sector bounded. It follows immediately from the discrete (or continuous) version of the multivariable circle criterion [7, 8] that $I + \tilde{G}(\cdot)$ be strongly positive real is a sufficient condition for closed-loop asymptotic stability. Finally $I + \tilde{G}(\cdot)$ is strongly positive real if and only if $H + G(\cdot)$ is strongly positive real. □

The condition that $\nu(t) = 0$ is always feasible may be fairly restrictive. For example, rate constraints on the actuators or more general state constraints may violate the condition. But the condition is always satisfied if we set $K_\nu(\cdot)$ as the constant gain $K_\nu(\cdot) = I$ or $K_\nu(\cdot) = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}$ and we have only saturation constraints on the actuators with the form

$$u_{\min} \preceq u(t) \preceq u_{\max} \text{ with } u_{\min} \preceq 0 \text{ and } 0 \preceq u_{\max} \tag{13}$$

A wide range of constrained control problems fall into such a class.

## 4. APPLICATION TO CONSTRAINED INTERNAL MODEL CONTROL

### *4.1. IMC framework*

The close relation between constrained internal model control (IMC) and MPC has been observed by many authors [20, 10, 9]. We will first illustrate the result using the IMC structure. Here the prediction horizon is one, so $N = n_u$. A detailed account of the analysis applied to a large scale process with many actuators is given in [1].

Suppose we have a model $\hat{P}(\cdot)$ of the plant $P(\cdot)$, with

$$P(\cdot) = \hat{P}(\cdot) + \Delta(\cdot) \tag{14}$$

We will write our constraints as requiring $u(t) \in \mathbb{U}$ for some $\mathbb{U}$. We write the controller as

$$\begin{aligned} \hat{d}(t) &= y(t) - \hat{P}(\cdot)u(t) \\ \tilde{f}(t) &= Q_1(\cdot)\left(r(t) - \hat{d}(t)\right) - Q_2(\cdot)u(t) \\ u(t) &= \arg\min_{v} \left|\left|H_r v - \tilde{f}\right|\right|^2 \\ & \quad\ \text{subject to } v \in \mathbb{U} \end{aligned} \tag{15}$$

with $Q_2(\cdot)$ strictly proper. See Fig 4.

The unconstrained nominal sensitivity $\hat{S}(\cdot)$ is given by

$$\hat{S}(\cdot) = I - \hat{P}(\cdot)\left(H_r + Q_2(\cdot)\right)^{-1} Q_1(\cdot) \tag{16}$$

We suppose the unconstrained nominal sensitivity is required to be $I - \hat{P}(\cdot)Q(\cdot)$ for some $Q(\cdot)$. Thus we require

$$Q(\cdot) = \left(H_r + Q_2(\cdot)\right)^{-1} Q_1(\cdot) \tag{17}$$

Let $Q_1(\cdot)$ be chosen as (compare [29])

$$Q_1(\cdot) = \lambda H_r Q(\cdot) + (1 - \lambda) H_r Q(\infty) \tag{18}$$

 

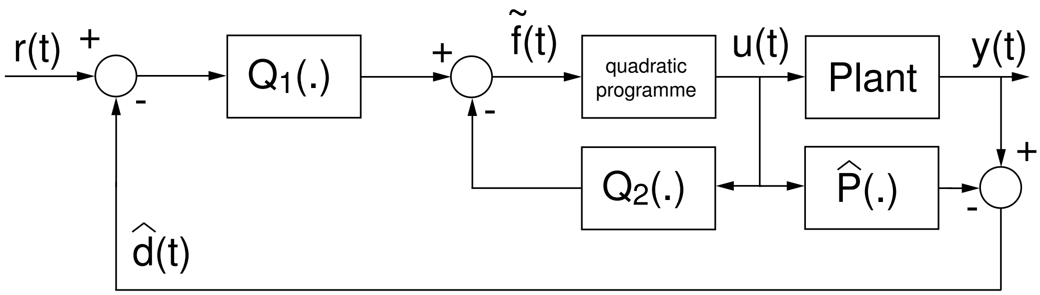


Figure 4. Internal model control with quadratic programme.

for some $\lambda$. In turn, $Q_2(\cdot)$ must be chosen as

$$\begin{aligned} Q_2(\cdot) &= Q_1(\cdot)Q(\cdot)^{-1} - H_r \\ &= (1-\lambda)H_r\left(Q(\infty)Q(\cdot)^{-1} - I\right) \end{aligned} \quad (19)$$

For square plants we can ensure correct steady state behaviour (when away from the constraints) by setting either

$$Q(1) = \hat{P}(1)^{-1} \text{ for } \hat{P} = \hat{P}(z) \text{ or } Q(0) = \hat{P}(0)^{-1} \text{ for } \hat{P} = \hat{P}(s) \quad (20)$$

If constraints are active in steady state, then choosing either

$$H_r = \hat{P}(1) \text{ for } \hat{P} = \hat{P}(z) \text{ or } H_r = \hat{P}(0) \text{ for } \hat{P} = \hat{P}(s) \quad (21)$$

gives nominal optimal steady state performance.

The control takes the form of (9) with

$$\begin{aligned} K_\nu(\cdot) &= I \\ K_u(\cdot) &= H_r^T(Q_1(\cdot)\hat{P}(\cdot) - Q_2(\cdot)) \\ K_y(\cdot) &= -H_r^T Q_1(\cdot) \\ K_r(\cdot) &= H_r^T Q_1(\cdot) \end{aligned} \quad (22)$$

and thus $\tilde{G}(\cdot)$ in (12) becomes

$$\tilde{G}(\cdot) = \left(Q_1(\cdot)\Delta(\cdot) + Q_2(\cdot)\right) H_r^{-1} \quad (23)$$

Sector bounds have been used to demonstrate robust stability of many non-optimizing anti-windup schemes (for example [24, 11, 13]). Our framework allows such methods to be extended to optimizing anti-windup schemes. As an example, we demonstrate below that the framework encompasses that in [18]. The analysis may also be extended to the anti-windup scheme in [21]. Indeed the analysis in [22] and [23] suggests that many anti-windup schemes with algebraic loops are equivalent to implementing single horizon MPC with a quadratic programme.

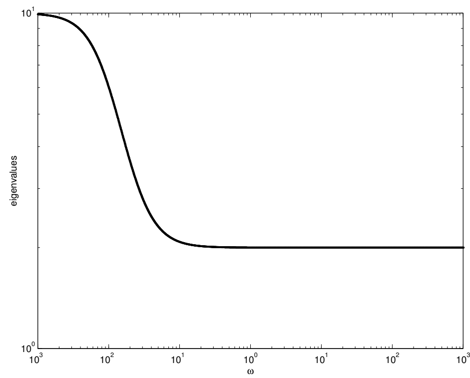


Figure 5. Eigenvalues of $I+\tilde{G}(j\omega)+I+\tilde{G}^*(j\omega)$ for the illustrative IMC example with no model mismatch.

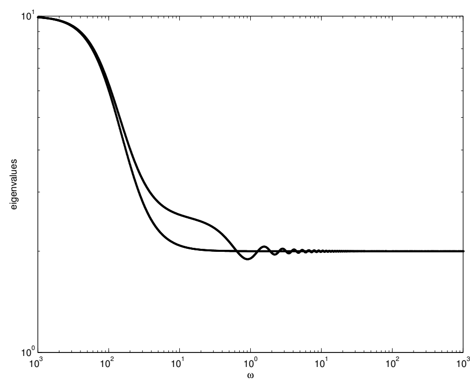


Figure 6. Eigenvalues of $I+\tilde{G}(j\omega)+I+\tilde{G}^*(j\omega)$ for the illustrative IMC example. There is a five second delay mismatch between the plant and plant model on the second output.

### *4.2. Anti-windup example*

An example of such a controller is the anti-windup scheme proposed in [18], which is discussed as a continuous domain controller. Here the reference signal entering the loop is modified to ensure the actuator signal is feasible. If the unconstrained controller is $K(s)$ then their scheme is equivalent to setting

$$u(t) = \arg\min_{v} \left|\left|K_1^{-1}v + (y-r) + K_1^{-1}K_2(s)v\right|\right|_{\Lambda} \tag{24}$$

for some positive definite $\Lambda$ where

$$\begin{aligned} K_1 &= K(\infty) \\ K_2(s) &= K^{-1}(s)K_1 - I \end{aligned} \tag{25}$$

This may be expressed in the form of (9) by setting

$$\begin{aligned} K_\nu &= I \\ H &= (K_1^T)^{-1}\Lambda K_1^{-1} \\ K_u &= -(K_1^T)^{-1}\Lambda K_1^{-1}K_2(s) = HK_2(s) \\ K_y &= -(K_1^T)^{-1}\Lambda = -HK_1 \\ K_r &= (K_1^T)^{-1}\Lambda = HK_1 \end{aligned} \tag{26}$$

If we set $\Lambda = I$ it may be posed as an IMC with

$$\begin{aligned} Q(s) &= \left(K(s)\hat{P}(s) + I\right)^{-1} K(s) \\ \lambda &= 0 \\ H_r &= K^{-1}(\infty) \end{aligned} \tag{27}$$

As a specific example, Peng et al. [18] consider the nominal plant

$$\hat{P}(s) = \frac{10}{1+100s}\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \tag{28}$$

with the unconstrained controller

$$K(s) = \frac{1+100s}{200s}\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} \tag{29}$$

Observe that in this case

$$K^{-1}(\infty) = 2\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} = \frac{1}{5}\hat{P}(0) \tag{30}$$

so the choice of $H_r$ effectively corresponds to the criterion for optimal nominal steady state performance (21).

We find

$$\begin{aligned} Q_1(s) &= I \\ Q_2(s) &= \frac{8}{100s+1}\begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} \end{aligned} \tag{31}$$

and hence

$$\tilde{G}(s) = \frac{1}{2}\Delta(s)\begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} + \frac{4}{100s+1}I \tag{32}$$

If there is no model mismatch it follows immediately that $I + \tilde{G}(s)$ is strongly positive real. Fig 5 illustrates the eigenvalues of $I + \tilde{G}(j\omega) + I + \tilde{G}^*(j\omega)$ in this case (note that at any one frequency the two eigenvalues are equal). Suppose instead (for example) the plant and plant model have a five second delay mismatch on the second output. Criteria (i) and (iii) for $I + \tilde{G}(s)$ to be strongly positive real still follow immediately. Fig 6 shows the eigenvalues of $I + \tilde{G}(j\omega) + I + \tilde{G}^*(j\omega)$. We see that the scheme is guaranteed robust to such mismatch.

## 5. APPLICATION TO RECEDING HORIZON MODEL PREDICTIVE CONTROL

The analysis may also be applied to discrete receding horizon control problems. In such cases $\nu(t)$ represents predicted inputs, while only the $n_u$ first elements (corresponding to $n_u$ actuators) are implemented. Thus if we predict over a control horizon of $N_u$ steps, we have $u(t) \in \mathbb{R}^{n_u}$ but $\nu(t) \in \mathbb{R}^N$ with $N = N_u n_u >> n_u$. Let us assume the controller can be expressed in the form of (11). Since $G(z)$ is an $N \times N$ transfer function matrix, it might appear at first sight to be a formidable task to check $I + \tilde{G}$ for positive realness. However we can readily exploit structure to simplify the problem.

Usually $K_\nu(z)$ is a linear gain given by

$$K_\nu(z) = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{n_u \times N_u n_u} \tag{33}$$

Correspondingly $K_p(z)$ is an $(N \times n_u)$ transfer function matrix; it is usually straightforward to guarantee that $K_p(z)$ is stable. In the case of an observer based controller, for example, its poles are those of the (unknown) plant and the observer (see below). It remains to find the eigenvalues of $(I + \tilde{G}) + (I + \tilde{G})^*$, which may be obtained directly from those of $\tilde{G} + \tilde{G}^*$.

Let $e(M)$ denote the non-zero eigenvalues of matrix $M$. We may say

$$\begin{aligned} e\left(\tilde{G} + \tilde{G}^*\right) &= e\left((H_r^T)^{-1} K_p K_\nu H_r^{-1} + (H_r^T)^{-1} K_\nu^T K_p^* H_r^{-1}\right) \\ &= e\left((H_r^T)^{-1} \begin{bmatrix} K_p & K_\nu^T \end{bmatrix} \begin{bmatrix} K_\nu \\ K_p^* \end{bmatrix} H_r^{-1}\right) \\ &= e\left(\begin{bmatrix} K_\nu \\ K_p^* \end{bmatrix} H^{-1} \begin{bmatrix} K_p & K_\nu^T \end{bmatrix}\right) \\ &= e\left(\Xi\right) \end{aligned} \tag{34}$$

with

$$\Xi = \left[\begin{array}{c|c} K_\nu H^{-1} K_p & K_\nu H^{-1} K_\nu^T \\ \hline K_p^* H^{-1} K_p & K_p^* H^{-1} K_\nu^T \end{array}\right] \tag{35}$$

Thus the eigenvalues of $(I + \tilde{G}) + (I + \tilde{G})^*$ are given by either 2 or $2 + e(\Xi)$, where $\Xi(z) \in \mathbb{C}^{2n_u \times 2n_u}$.

### 5.1. MPC framework

Suppose for example we have a simple input constrained control with integration introduced via an observer. Let the plant model be given in state space form as

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{36}$$

Let us construct

$$\nu(t) = \begin{bmatrix} \upsilon^T(t) & \upsilon^T(t+1) & \cdots & \upsilon^T(t+N_u-1) \end{bmatrix}^T \tag{37}$$

and let the MPC algorithm be given as

$$\begin{aligned} \nu^*(t) &= \arg\min_\nu \sum_{k=1}^{N_u} ||\hat{y}(t+k) - r||_Q^2 + \sum_{k=1}^{N_u} \left|\left|(1 - z^{-1})\upsilon(t+k-1)\right|\right|_R^2 \\ &\qquad \text{subject to } \upsilon(t+k) \in \mathbb{U} \text{ for } k = 0, \ldots, N_u - 1 \\ u(t) &= K_\nu \nu^*(t) \end{aligned} \tag{38}$$

with $\upsilon(t-1) = u(t-1)$, where $K_\nu$ is given by (33) and where the predicted outputs $\hat{y}(t+k)$ are formed via current and predicted state estimates as follows.

The current state is estimated using an observer that includes a disturbance estimator (see for example [3]). Specifically form

$$A_o = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix},\ B_o = \begin{bmatrix} B \\ 0 \end{bmatrix},\ C_o = \begin{bmatrix} C & 0 \end{bmatrix} \tag{39}$$

and update the observed state vector as

$$\tilde{x}(t+1) = (I - JC_o)A_o\tilde{x}(t) + (I - JC_o)B_o u(t) + Jy(t) \tag{40}$$

for some observer gain $J$. We will write the observer as

$$\tilde{x}(t) = E_y(z)y(t) + E_u(z)u(t) \tag{41}$$

for some transfer function matrices $E_y(z)$ and $E_u(z)$.

The predicted state estimates are formed recursively as

$$\hat{x}(t+k+1) = A_o\hat{x}(t+k) + B_o\upsilon(t+k) \text{ with } \hat{x}(t) = \tilde{x}(t) \tag{42}$$

The predicted outputs $\hat{y}$ are then given by $\hat{y}(t+k) = C_o\hat{x}(t+k)$.

Let $\xi(t)$ be formed as

$$\xi(t) = \begin{bmatrix} \hat{x}^T(t+1) & \hat{x}^T(t+2) & \dots & \hat{x}^T(t+N_u) \end{bmatrix}^T \tag{43}$$

and put

$$M = \begin{bmatrix} I & 0 & \cdots & 0 \\ -I & I & \cdots & 0 \\ & \ddots & \ddots & \\ 0 & \cdots & -I & I \end{bmatrix},\ \Lambda = \begin{bmatrix} A_o \\ A_o^2 \\ \vdots \\ A_o^{N_u} \end{bmatrix},\ \Phi = \begin{bmatrix} B_o & & \\ A_oB_o & B_o & \\ \vdots & & \ddots \\ A_o^{N_u-1}B_o & \cdots & B_o \end{bmatrix} \tag{44}$$

Then

$$\xi(t) = \Lambda\tilde{x}(t) + \Phi\nu(t) \tag{45}$$

Let $\bar{C}_o$, $\bar{Q}$ and $\bar{R}$ be block diagonal matrices with entries $C_o$, $Q$ and $R$ respectively, and let

$$L = \begin{bmatrix} I & I & \cdots & I \end{bmatrix}^T \tag{46}$$

Then $\nu^*(t)$ is given by

$$\begin{aligned} \nu^*(t) \quad = \quad & \arg\min_{\nu} \left|\left|\bar{C}_0\Lambda\tilde{x}(t) + \bar{C}_o\Phi\nu(t) - Lr\right|\right|_{\bar{Q}}^2 + ||M\nu(t)||_{\bar{R}}^2 - 2\upsilon(t)^T Ru(t-1) \\ & \text{subject to constraints} \end{aligned} \tag{47}$$

Then we find

$$\begin{aligned} H &= \Phi^T\bar{C}^T\bar{Q}\bar{C}\Phi + M^T\bar{R}M \\ K_y(z) &= -\Phi^T\bar{C}^T\bar{Q}\bar{C}\Lambda E_y(z) \\ K_u(z) &= -\Phi^T\bar{C}^T\bar{Q}\bar{C}\Lambda E_u(z) + z^{-1}M^T\bar{R}K_\nu^T \\ K_r(z) &= \Phi^T\bar{C}^T\bar{Q}L \end{aligned} \tag{48}$$

with $K_\nu(z)$ given by (33).

In this case the poles of $K_p(z)$ are those of the plant and the observer, which by assumption and design are all stable. Note that while the disturbance estimator in the observer ensures integral action when the constraints are not active, there is no integrating pole in $K_p(z)$.

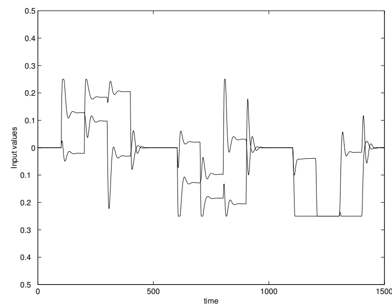


Figure 7. Inputs for the MPC example.

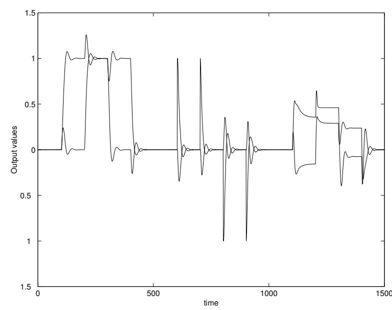


Figure 8. Outputs for the MPC example.

*5.2. Illustrative example*

As an example, consider a two-input two-output plant with nominal model

$$P(z) = \begin{bmatrix} \frac{0.4}{z-0.95} & \frac{0.6}{z-0.5} \\ \frac{0.4}{z-0.5} & \frac{0.5}{z-0.9} \end{bmatrix} \tag{49}$$

Note that despite its simple dynamics $P(z)$ has a transmission zero outside the unit circle. A controller was chosen with a control horizon $N_u = 10$, and diagonal weighting matrices were used for $Q$, $R$ etc. Figs 7 and 8 show the input and output responses to various step reference changes as well as step input and output disturbances. The actuators were constrained to have magnitude less than or equal to 0.25. In some cases the step disturbances were sufficiently large that the actuators saturate in steady state. A single delay mismatch was introduced to the first output. Fig 9 shows $e(\Xi) + 2$ over the frequency range $0 \leq \theta \leq 2\pi$ when there is no model mismatch. Fig 10 shows $e(\Xi) + 2$ over the same frequency range $0 \leq \theta \leq 2\pi$ with the mismatch between plant and model. Finally we observe that $\tilde{G}(\infty) = 0$ so

$$(I + \tilde{G}(\infty)) + (I + \tilde{G}(\infty))^T = 2I > 0 \tag{50}$$

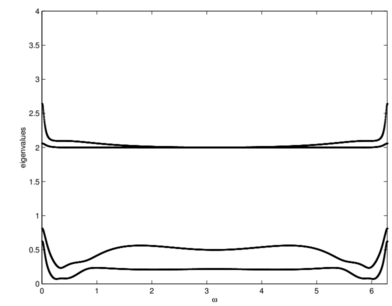


Figure 9. Values of $e(\Xi) + 2$ for the MPC example with no mismatch between plant and model.

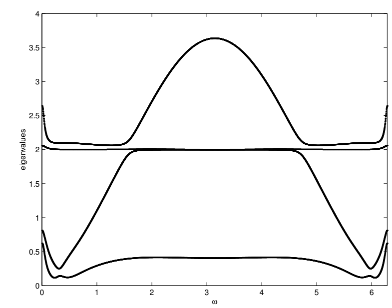


Figure 10. Values of $e(\Xi) + 2$ for the MPC example. There is a single delay mismatch between plant and model in the first output.

## 6. CONCLUSION

We have shown that it is possible to test certain predictive controllers (with either single step or receding horizon control) for stability. The test may be applied in the presence of model mismatch. The test is based on the circle criterion, and stems from the observation that a certain quadratic programme is sector bounded provided the origin is a feasible solution. Furthermore the specific structure that arises in certain receding horizon controllers ensures that it suffices to test relatively low-dimensioned objects.

The test is conservative on many counts. Even with the examples we have discussed, it is straightforward to choose parameters such that while the nominal closed-loop system fails the criterion, good dynamic properties can nevertheless be observed in simulation. We require that the open-loop plant be stable, and the criterion cannot be applied to controllers with rate constraints, state constraints or output constraints. Furthermore the criterion cannot be applied to controllers which have a separate optimization to determine the steady state values (see for example [17]).

The conservativeness of the test is not surprising as the analysis is independent of signal norms. Furthermore we have only considered input constraints (so feasibility is guaranteed). More generally we would expect only local stability results to be attainable, and thus signal norms should be taken into account. Furthermore we have applied only a conservative test (the circle criterion), and ignored any possible generalisation using multipliers. A natural framework for such a generalisation might be the use of integral quadratic constraints [16].

Nevertheless we believe the result of this paper is useful, and the approach might be a useful direction for the further analysis of MPC robustness.

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