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LDPC Codes For The Classic Bursty Channel

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Abstract

In this paper we consider low-density parity-check (LDPC) codes for the classic bursty channel. We present lower bounds on the maximum resolvable erasure burst length, L_{max}, for arbitrary LDPC codes and then present constructions for LDPC codes with good burst erasure correction performance for channels with and without the presence of noise in the guard band.

1. INTRODUCTION

A low-density parity-check (LDPC) code is a block code defined by a sparse parity-check matrix, H, and decoded iteratively with sum-product decoding. LDPC codes are well known to provide excellent decoding performances on a wide range of memoryless channels (see e.g. [6]), and recent interest has focussed on their performance on channels with memory [4, 5]. We consider in this paper LDPC codes for the simplest case of channels with memory, the classic bursty channel as defined in [2]. The channel has two states; the burst space, in which the channel output carries no information about the inputs, and the guard space, in which channel outputs are erasure free or are randomly corrupted with some small background erasure probability p [2].

An LDPC code can be represented by a bipartite graph, called a Tanner graph, which has a parity-check vertex for every parity-check equation in H and a symbol vertex for every codeword symbol. Each parity-check vertex is connected by an edge to a symbol vertex if that symbol is included in the corresponding parity-check equation. A cycle in a Tanner graph is a sequence of connected vertices which start and end at the same vertex in the graph and contain no other vertices more than once. The length of the cycle is the number edges it contains and the size and distribution of cycles, particularly small cycles, plays an important role in the decoding performance of the code.

A stopping set in an LDPC code is a set of symbol nodes with the property that every check node connected to a symbol node in the stopping set is connected to at least two such nodes. On the binary erasure channel, an erased codeword symbol can be corrected if and only if it is the sole erased symbol in any parity-check equation [1]. Thus if the set of code symbols which are erased contains a stopping set, none of the symbols in the stopping set can be corrected. Conversely, if there is no stopping set in the set of erased symbols they can all be corrected. On a burst erasure channel the distance between the code symbols in a stopping set, rather than just the number of stopping sets, will be the important factor in determining the code performance.

A measure of an LDPC codes performance on a burst erasure channel can be given by its minimum space distance (MSD), which is the minimum number zeros between any two non-zero entries in any row of the parity-check matrix [3]. Song and Cruz observed in [3] that for an LDPC code with a MSD of s, burst erasures of length up to $s+1$ symbols can always be recovered in one iteration of the sum-product decoding algorithm. With this in mind they presented a pseudo-random construction for LDPC codes with large MSD.

Yang and Ryan made a similar observation in [4, 5] and developed an efficient exhaustive search algorithm for finding the longest string of erasures guaranteed to be corrected, called the maximum resolvable erasure burst length, $L_{\max}$, for a given LDPC code.

In this paper we add to these results to show that a 4-cycle free LDPC code with minimum column weight in H of $\gamma > 1$ can correct bursts of length up to $s+2\gamma-1$ if $s \geq \gamma - 1$ and up to size $2s+\gamma$ otherwise, both in at most two iterations of sum-product decoding. Next we show that improved bounds on $L_{\max}$ can be achieved using circulants which enables us to present constructions for binary quasi-cyclic LDPC codes with good

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burst erasure correction performance for channels with and without the presence of noise in the guard band.

2. LOWER BOUNDS ON $L_{\max}$

In this section we prove some lower bounds on the burst erasure correction capability of arbitrary LDPC codes. Lemma 1 was previously stated in [3] but we repeat it here with a proof as it is required for the lemmas which follow.

Lemma 1 *An LDPC code with H having column weights $\gamma \geq 1$, and an MSD of s can always correct a burst erasure of length up to $s+1$ in one iteration of sum-product decoding.*

Proof: By our requirement for non-zero column weight every one of the $s+1$ erased symbols is checked by at least one parity-check equation, and by the MSD property of the code no parity-check equation includes more than one of the $s+1$ erased symbols. Thus each erased symbol is included in a parity-check equation with all other symbols known and so can be corrected without knowledge of the other erased symbols, thus requiring only one iteration of the decoding algorithm. ■

Lemma 2 *A 4-cycle free LDPC code with minimum column weight $\gamma > 1$ and MSD of s can always correct a burst erasure of length up to $s + 2\gamma - 1$ if $s \geq \gamma - 1$ and up to length $\gamma + 2s$ otherwise, both in at most two iterations of sum-product decoding.*

Proof: First we take the case that $s \geq \gamma - 1$ and suppose that an erasure burst occurred in symbol locations $i, \ldots, i + s + 2\gamma - 2$. By a similar argument to the proof of Lemma 1, the symbols in locations $i+\gamma-1, \ldots, i+s+\gamma-1$ can always be corrected in the first iteration as not all of the γ checks on these symbols can include a second erased symbol without requiring an MSD of less than s or allowing a 4-cycle. Next suppose that symbols $i, \ldots, i + \gamma - 2$ are not corrected in the first iteration. These symbols must be contained in γ checks, however there are at most $\gamma - 1$ symbols (symbols $i + s + \gamma, \ldots, i + s + 2\gamma - 2$) of distance more than s away which were not corrected in the first iteration and thus without allowing a 4-cycle there must be one check on each of the symbols $i, \ldots, i + \gamma - 2$ which can be used to correct them in the second iteration. By symmetry, the same argument applies to the symbols $i + s + \gamma + 1, \ldots, i + s + 2\gamma - 2$.

Secondly we take the case that $s < \gamma - 1$ and suppose that an erasure burst occurred in symbol locations $i, \ldots, i + 2s + \gamma - 1$. Again by a similar argument to the proof of Lemma 1, the symbols in locations $i + s, \ldots, i + \gamma + s - 1$ can always be corrected in the first iteration as not all of the γ checks on these symbols can include a second erased symbol without requiring an MSD of less than s or allowing a 4-cycle. As above it is easy to see that if the remaining symbols (symbols $i, \ldots, i+s-1$ and $i+s+\gamma, \ldots, i+2s+\gamma-1$) are not corrected in the first iteration they can always be corrected in the second. ■

These bounds give us a useful measure of the erasure correcting capability of an arbitrary LDPC code if we know its MSD. However in order to prove that LDPC codes are capable of good burst erasure correction we need to prove the existence of LDPC codes for a given MSD. This is the subject of the next section.

3. LDPC CODES WITH GOOD $L_{\max}$

In order to construct LDPC codes with good MSD, and thus provably good burst erasure correction, we consider codes with parity-check matrix described by a row of circulants, called row circulant LDPC codes. Firstly we prove a stronger result than that of Lemma 2 for the burst erasure correction capability of a column weight 2 circulant with a given MSD.

Lemma 3 *A 4-cycle free column weight 2 circulant with a MSD of s can correct a burst erasure of length up to $2(s+1)$.*

Proof: For a subset of the erasures to remain uncorrected there must exist a stopping set within the set of erased symbols. As 4-cycles are avoided there cannot be stoping sets of size 2. For stopping sets of size t, in a circulant of column weight 2 we require a $2t$-cycle through the points of the stopping set. However since the non-zero entries in any check must be at least $s+1$ symbols apart, the outside entries of any $2t$-cycle must be distance $t(s+1)$ symbols apart, and thus any burst of length $2s+2$ or less cannot include a stopping set. ■

A $v \times v$ circulant A will have a maximum MSD of

$$s = \left\lceil \frac{v}{2} \right\rceil - 2,$$

achieved for

$$a(x) = 1 + x^b, \quad \text{where} \quad b = \left\lceil \frac{v}{2} \right\rceil - 1.$$

For v odd, A will have a girth of $2v$ and rank of $v-1$, and concatenating m copies of A will give a quasi-cyclic code

$$H = [A, A, \ldots, A] \tag{1}$$

with rate

$$R = (vm - v - 1)/vm$$

and optimal burst erasure correcting capability of

$$n - k = v - 1.$$

This can be seen by observing that any subset of v codeword symbols describes some rotated version of the circulant A and then substituting

$$s = (v+1)/2 - 2$$

into Lemma 3.

While these codes are interesting in that the have the optimal burst erasure correction performance they will perform poorly in the more useful channel model which includes the presence of erasures in the guard band (the repeated columns ensure stopping sets of size 2). In order to produce LDPC codes which also perform well in the presence of random erasures we require that the codes are 4-cycle free, which in this case is equivalent to requiring that the codes have no repeated columns.

3.1. 4-cycle Free Erasure Correcting Codes

Construction 1 *Choose any integer v and integer $m < v/2$ and construct a length mv, rate $\approx (m-1)/m$, 4-cycle free LDPC code,*

$$H = [A_1, A_2, \ldots, A_m], \qquad (2)$$

using m $v \times v$ circulants $A_1, \ldots A_m$, where the circulant A_i is described by the polynomial $a_i(x) = 1 + x^{b_i}$, with

$$b_i = \left\lceil \frac{v}{2} \right\rceil - i.$$

Lemma 4 *The codes of Construction 1 can always correct burst erasures of length up to*

$$L_{\max} \geq 2\left\lceil \frac{v}{2} \right\rceil - 2m.$$

Proof: The MSD of the ith circulant is $s_i = \lceil \frac{v}{2} \rceil - i - 1$ and so by Lemma 3 any burst of length up to

$$L_{\max} = 2(s_m) + 2 = 2\left\lceil \frac{v}{2} \right\rceil - 2m,$$

that occurs entirely within a circulant can be corrected.

Suppose that a length $2\lceil \frac{v}{2} \rceil - 2i$ erasure occurs across two circulants, A_i and A_{i+1}, with k erased symbols in the last k positions of the left circulant, A_i, and the remaining $2\lceil \frac{v}{2} \rceil - 2i - k$ erasures in the first $2\lceil \frac{v}{2} \rceil - 2i - k$ positions of the right circulant, A_{i+1}.

First assume $k = 1$, then symbols $1 \ldots 2\lceil \frac{v}{2} \rceil - 2i - 1$ in the right circulant are erased. The first column of A_i has non-zero entries in positions 1 and $s_i + 2$, while the first column of A_{i+1} has non zero entries in positions 1 and $s_i + 1$ where s_i is the MSD of circulant A_i,

$$s_m + 1 \leq s_i \leq \left\lceil \frac{v}{2} \right\rceil - 3.$$

Since each following column is a cyclic shift of the preceding one, the last symbol in A_i is included in the vth and the $(v - s_i - 1)$th rows of H and these rows contain the $(s_{i+1} = s_i + 1)$th, $(v - s_i - 1)$th, $(v - 1)$th, and vth symbols in A_{i+1}. Thus the single erased bit in A_i appears, to A_{i+1}, as erasures in symbols v and $v - 1$, (the $(s_i + 1)$th and $(v - s_i - 1)$th symbols are erased already) and so the effect on the circulant A_{i+1} is equivalent to a burst of length $2\lceil \frac{v}{2} \rceil - 2i + 1$ solely within A_{i+1}. Actually it is equivalent to a burst in the circulant given by shifting the rows of A_{i+1} to the right by 2 positions, but as long as it is a burst in some cyclic shift of A Lemma 3 will apply.

The same is true as k is increased, since the $(v-k)$th symbol in A_i is included in a row with the $(s_i + 1)$th, $(v - s_i - 1)$th, $(v - k)$th, and $(v - k + 1)$th symbols in A_{i+1}, and thus the effect on the circulant A_{i+1} is always equivalent to a burst of length $2\lceil \frac{v}{2} \rceil - 2i + 1$ solely within A_{i+1}, which we know from Lemma 3 is always correctable. Thus an erasure across circulants of length $2\lceil \frac{v}{2} \rceil - 2m$ is always correctable and the proof follows. ■

Construction 1 provides 4-cycle free, column weight 2 LDPC codes with close to optimal L_{max}. However, on channels with random errors, the decoding performance of LDPC codes can be significantly improved by using column weight 3 codes [6], that is $(3, r)$-regular LDPC codes, where r is the row weight of H, and so we extend the above development to these codes.

3.2. $(3, r)$-regular Erasure Correcting Codes

Again we consider codes from circulants in order to guarantee codes with good L_{max}. For any column weight 3 circulant we label the three possible lengths for runs of zeros between non-zero entries, by s_1, s_2 and s_3, where $s_1 \leq s_2 \leq s_3$. It can be shown, simply by choosing any code symbol and ensuring that every check on it is erased twice, that

$$L_{max} \geq \min\{s_3 + 2, s_1 + s_2 + 3\}.$$

However, a better bound of $L_{max} \geq v - 3$, can be proven for $v \times v$ column weight 3 circulants which are, counterintuitively, given by $a(x) = 1 + x + x^2$, and this result can be generalized to any column weight 3 circulant:

Lemma 5 *A column weight 3 circulant with space distances of $s_1 \leq s_2 \leq s_3$ can always correct a burst erasure of length up to $s_3 + 1$ symbols.*

Proof: Suppose that a burst of length $L = s_3 + 1$ occurs across any subset of the symbols in a column weight 3 circulant. Take the parity-check equation which includes the Lth erased symbol and no other erased symbols. By the definition of the circulant such a parity-check equation must exist. The other two symbols in this parity-check equation are the last symbol before the burst and a symbol distance $s + 1$ past the burst, with $s \in \{s_1, s_2\}$. Thus this parity-check equation, and the parity-check equations corresponding to the s rows above it in the circulant, contain only one erased symbol, and so the $(L - s + 1, \ldots L)$th erased symbols can be corrected in the first decoding iteration. Since the $(L - s + 1, \ldots L)$th symbols are now known the next s equations above the previous s in the circulant now only contain one erased symbol and thus the $(L - 2s + 1, \ldots L - s + 1)$th erased symbols can be corrected. This process can be repeated until all the erased symbols in a burst of length up to $s_3 + 1$ symbols are corrected, using at most

$$\left\lceil \frac{s_3 + 1}{s + 1} \right\rceil$$

iterations. ■

Using the intuition from Lemma 5 we present the following construction for column weight 3 burst erasure correcting codes which are 4-cycle free.

Construction 2 *Choose any integer v and integer $m < v/8$ and describe m $v \times v$ circulants $A_1, \ldots A_m$, with A_i described by the polynomial $a_i(x) = 1 + x^{b_i} + x^{c_i}$, where*

$$b_i = 2i, \qquad c_i = \left\lceil \frac{3v}{8} \right\rceil + i.$$

Construct the LDPC code as in (2) to give a length mv, rate $\approx (m-1)/m$, quasi-cyclic LDPC code.

Although the codes from Constructions 1 and 2 produce $L_{\max}$ values closer to the optimal at lower rates, we consider higher rate codes to simulate as they have parameters closer to those traditionally considered for a bursty channel. Fig. 1 shows the erasure correction performance on a bursty channel of length 768, rate ≈ 0.875, binary codes from Constructions 1 and 2 compared to a randomly constructed column weight 3 LDPC code. Fig. 2 shows the erasure correction performance of binary, length 4158, rate 0.833, codes from Constructions 1 and 2 on a classic bursty channel compared to a randomly constructed LDPC code with column weight 3 and no 4-cycles. The codes from Construction 2 give promising performances in bursty erasure channels, however as the erasures become more random their performance is worse than that of a randomly constructed LDPC code. Indeed, as the code length increases the codes of Constructions 1 and 2 are weaker in a random erasure channel due to the minimum distance disadvantages inherent in row circulant LDPC codes, which is independent of how they are constructed. We thus require modifications to the previous constructions to produce codes which still have good burst erasure correction but which are more robust in a random erasure channel.

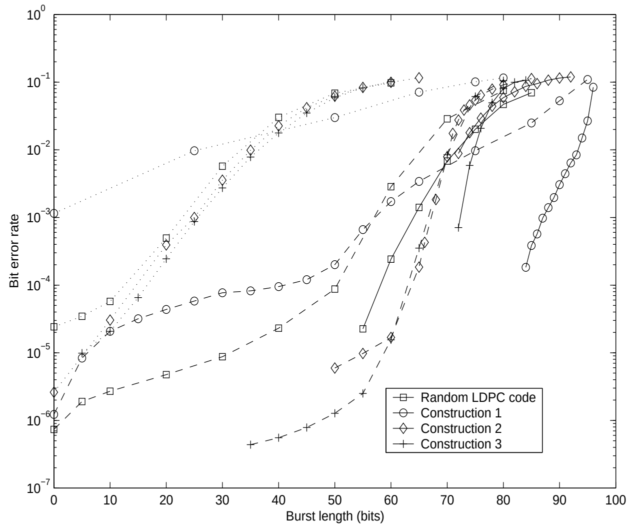

Figure 1: The performance of LDPC codes on a classic bursty channel with guard band erasure probabilities of $p = 0$ - solid curves, $p = 0.01$ - dashed curves, and $p = 0.05$ - dotted curves. All four codes are length 768, while the column weight 3 LDPC codes are rate 0.875, and the column weight 2 LDPC code is rate 0.876.

3.3. Codes for the Classic Bursty Channel

In order to adapt our codes to perform well in both random and bursty channels we construct codes which contain both deterministic and random elements, choosing the deterministic elements to give the best possible burst erasure correcting properties using insights from the codes previously presented.

Construction 3 *Choose any integer v and integer $m < v/2$ and construct a length mv, rate $\approx (m-2)/m$, 4-cycle free LDPC code,*

$$H = \begin{bmatrix} A_1 & \Pi_2(I) & A_3 & \cdots \\ \Pi_1(I) & A_2 & \Pi_3(I) & \cdots \end{bmatrix}$$

using m $v \times v$ circulants $A_1, \ldots A_m$, where the circulant A_i is described by the polynomial $a_i(x) = 1 + x^{b_i}$, with

$$b_i = \left\lceil \frac{v}{2} \right\rceil - i,$$

and $\Pi_i(I)$ is some random permutation matrix.

Lemma 6 *The codes of Construction 3 can always correct burst erasures of length up to*

$$L_{max} \geq 3(\left\lceil \frac{v}{2} \right\rceil - m).$$

Proof: (sketch) Following the proof of Lemma 4, for a burst of symbols across one of the column weight two circulants, A_i, to form a stopping set requires the burst to be of length at least $2(\lceil \frac{v}{2} \rceil - i)$. However, for the third check on each of these symbols, the check in $\Pi_i(I)$, to also be uncorrectable requires erasures on the bits in an adjoining circulant, A_{i+1}. Which, if $\Pi_i(I)$ and $\Pi_{i+1}(I)$ are suitably arranged, can be achieved with only $s_i + 1$ symbols. If fewer symbols in A_i are erased a stopping set can still be formed but at least as many erased symbols will need to be added across A_{i+1}. Taking a burst across the circulant with the smallest MSD gives L_{max}. ∎

Fig. 1 shows the erasure correction performance of a binary, length 768, rate 0.875, code from Construction 3 on the classic bursty channel. As well, Fig. 2 shows the erasure correction performance of a binary, length 4158, rate 0.818, code from Construction 3 on a classic bursty channel. Note that the randomly constructed codes simulated here are not truly random as they have been optimized to avoid repeated columns and cycles of length 4 when possible. The same process has also been applied to the codes of Construction 3, by discarding permutations that lead to cycles, and the code from Construction 3 in Fig. 2 is also 4-cycle free.

The new erasure correcting codes of Construction 3 outperform both the structured codes the randomly constructed codes in channels with random erasures. Thus Construction 3 provides codes with both provably good burst erasure correction capability and excellent performance in the presence of random erasures.

As the codes of Construction 3 are no longer quasi-cyclic, their encoding may be more computational complex. However a simple modification restricting the permutation matrices to be circulant permutations will produce block circulant codes and allow simple shift register encoding circuits as in [7]. For high rate codes another option is to modify the structure of H to be

$$H = \begin{bmatrix} A_1 & \Pi_2(I) & A_3 & \cdots & I & 0 \\ \Pi_1(I) & A_2 & \Pi_3(I) & \cdots & A_{m-1} & B \end{bmatrix},$$

where B has a '1' in all the diagonal entries and, except for in the final column, a '1' in all the entries one below the diagonal. In this case the code can be encoded using back-substitution as in [8] with complexity $\bigcirc(n)$ [8].

3.4. Comparisons with Existing Codes

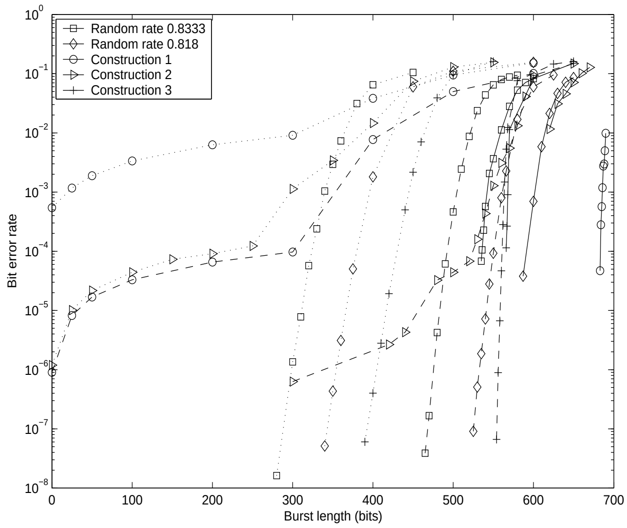

Figure 2: The performance of length 4158 LDPC codes on a classic bursty channel with guard band erasure probabilities of $p = 0$ - solid curves, $p = 0.01$ - dashed curves, and $p = 0.05$ - dotted curves.

Table 1 compares the burst erasure correction properties of the LDPC codes constructed in this paper with previous results. The results shown in the first six rows of the table are from [4, 5]. For the randomly constructed LDPC codes the best we can bound $L_{\max}$ by is the minimum stopping set size, which is $\gamma + 1$ if the code is free of 4-cycles, since there is no guarantee that the columns of the stoping set will not be adjacent. However, this is extremely unlikely, and it is possible to randomly construct LDPC codes with reasonable $L_{\max}$, as evidenced by the results in Table 1.

Table 1: Burst Correction Properties of Selected Codes.

Code type	Length	Rate	$L_{\max}$
MacKay col weight 3	4161	0.82	563
MacKay col weight 4	4161	0.82	515
PG (col weight 65)	4161	0.82	303
EG (col weight 64)	4095	0.82	376
eIRA (col weights 2,5)	4095	0.82	507
Array (col weights 4,5)	4095	0.82	500
Construction 1	4158	0.833	682
Construction 2	4158	0.833	615
Construction 3	4158	0.818	564
MacKay col weight 3	4158	0.833	532
Trad. quasi-cyclic code	4158	0.833	502

Due to the implementation advantages they provide, quasi-cyclic constructions for LDPC codes have been extensively investigated in previous work. For the sake of comparison the parameters of a row circulant quasi-cyclic LDPC code, with length 4158 and rate 0.833, constructed using difference families [9] and the traditional approach of choosing those circulants which minimize the second largest eigenvalue of HH^T, is also shown in Table 1. The extended irregular repeat-accumulate codes of [10], though not quasi-cyclic, are efficiently encodable as the parity-check matrix of these codes contain all but one column of a weight 2 circulant. As well many constructions for block circulant LDPC codes, including the array codes of [11], and cyclic LDPC codes, such as the Euclidean (EG) and projective geometry (PG) codes of [12], have been presented. The burst erasure correction properties of these codes with sum-product decoding, as given in [5], is also shown in Table 1.

4. CONCLUSION

In this paper we have presented lower bounds for the maximum resolvable burst erasure length of arbitrary LDPC codes and have shown that this bound can be dramatically improved by considering codes constructed using circulants. While we show that the minimum space distance of a code can be used to lower bound its maximum resolvable burst erasure length, we also show that for structured codes even better bounds on the maximum resolvable burst erasure length can be achieved for codes with very poor MSD.

In this paper we suggest also how to add this structure to produce codes with good maximum resolvable erasure burst length, and show that by sacrificing some burst erasure correcting capability codes which are also robust to random erasures can be constructed. Simulation results confirm what would be intuitively surmised, that is for channels with a large number of random erasures a pseudo-randomly constructed LDPC code will be the best choice while for channels with most of the erasures in the form of bursts a structured LDPC code can use the memory in the channel to give the best erasure correction performances.

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