Zames-Falb multipliers for quadratic programming

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Abstract

In constrained linear model predictive control a quadratic program must be solved on-line at each control step, and this constitutes a nonlinearity. If zero is a feasible point for this quadratic program then the resultant nonlinearity is sector bounded. We show that if the nonlinearity is static then it is also monotone and slope restricted; hence we show the existence of Zames-Falb multipliers for such a nonlinearity. We express the results in terms of integral quadratic constraints. The multipliers may be used in a general and versatile analysis of the robust stability of input constrained model predictive control.

I. Introduction

Recently we proposed a new stability test for constrained linear MPC (model predictive control) [8], [7]. If zero is feasible then the nonlinearity arising from the associated on-line quadratic program is sector bounded. Hence the multivariable circle criterion gives a sufficient condition for closed-loop stability. The results may be used to demonstrate stability against both structured and unstructured infinity-norm bounded model uncertainty [6].

In this paper we show the existence of Zames-Falb multipliers when such a nonlinearity is static—for example when the only constraints are fixed input constraints. In particular we show that the nonlinearity is bounded, monotone and slope restricted in the sense of [12] which summarizes the natural generalization of [17] to MIMO (multi-input multi-output) nonlinearities $\phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ (in such analysis it is sufficient, without loss of generality, to consider only square systems). We express the results in terms of IQCs (integral quadratic constraints).

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The multipliers, together with the results in [6], may be used for the analysis of the robust stability of input constrained MPC. The analysis is general and versatile, but we should note a caveat: the Zames-Falb stability criterion does not necessarily guarantee continuity of the input-output map [11].

Notation: Let L_2^N be the space of $\mathbb{R}^N$-valued functions f such that $\int_{-\infty}^{\infty}|f(t)|^2dt$ is finite; let l_2^N be the space of $\mathbb{R}^N$-valued sequences f_k such that $\sum_{k=-\infty}^{\infty}|f_k|^2$ is finite. Write L_2 for L_2^1 and l_2 for l_2^1. Let L_1 be the space of real functions f such that $\int_{-\infty}^{\infty}|f(t)|dt$ is finite; let l_1 be the space of real sequences f_k such that $\sum_{k=-\infty}^{\infty}|f_k|$ is finite. Let $<x,y>$ denote the inner product between x and y, and let $x*y$ denote the convolution of x and y.

II. Zames-Falb multipliers for MIMO nonlinearities

In [17] Zames and Falb derive multipliers in two parts:

ZF1: the first concerns bounded and monotone nondecreasing nonlinearities;

ZF2: the second concerns nonlinearities that are also slope-restricted.

Both results are derived for single-valued nonlinearities, where it is commented that they can be "easily" generalized to MIMO nonlinearities. It is observed in [15] that, in fact, the first result only generalizes to a MIMO (multi-input multi-output) nonlinearity $\phi : \mathbb{R}^N \to \mathbb{R}^N$ under a further condition. Specifically it is shown that a necessary and sufficient further condition for Zames-Falb multipliers (ZF1) to exist is for the line integral $\int_A^B \phi(x)^T dx$ to be independent of path, and this is equivalent to the condition that ϕ is the gradient of some convex potential function. Hence in [12] the term "monotone" is reserved for this case. In this spirit, we say ϕ is bounded (by $c>0$) if $\|\phi(x)\| \leq c\|x\|$ for all $x\in\mathbb{R}^N$; ϕ is incrementally positive if $[\phi(x)-\phi(y)]^T(x-y)\geq 0$ for all $x,y\in\mathbb{R}^N$; ϕ is monotone if it is incrementally positive and may be expressed as the gradient of a convex potential function; ϕ is slope-restricted to the interval $[a,b]$ if $[\phi(x)-\phi(y)-a(x-y)]^T[\phi(x)-\phi(y)-b(x-y)]\leq 0$ for all $x,y\in\mathbb{R}^N$.

The Zames-Falb multipliers are then given as:

Zames-Falb multipliers: Let $\phi : \mathbb{R}^N \to \mathbb{R}^N$ be bounded and monotone with $\phi(0)=0$. Let $h\in L_1$ satisfy $\int_{-\infty}^{\infty}|h(t)|dt<1$ (or let $h\in l_1$ satisfy $\sum_{t=-\infty}^{\infty}|h(t)|<1$) and either let ϕ be odd or let $h(t)\geq 0$ for all t. Then

ZF1: For any $x \in L_2^N$ (or $x \in l_2^N$) we have

$$\langle \phi(x), x \rangle \geq \langle \phi(x), h * x \rangle \quad (1)$$

ZF2: If ϕ is slope-restricted to the interval $[\alpha, \beta - \varepsilon]$ for some real $\alpha > 0$, $\beta > 0$, $\varepsilon > 0$ with $\beta - \varepsilon > \alpha$ then for any $x \in L_2^N$ (or $x \in l_2^N$) we have

$$\langle \phi(x) - \alpha x, \beta x - \phi(x) \rangle \geq \langle \phi(x) - \alpha x, h * (\beta x - \phi(x)) \rangle \quad (2)$$

Remark: The multipliers can be constructed in the manner of [17], [3], [2]. Note that in [2] a restricted case is considered, and thus more general multipliers are obtained (see also [10]). When the condition ZF2 is derived in this manner, it is necessary to demonstrate that $\tilde{\phi} = \hat{\phi}_1 \circ \hat{\phi}_2^{-1}$ is the gradient of a convex potential function with $\hat{\phi}_1 = (\phi - \alpha I)$ and $\hat{\phi}_2 = (\beta I - \phi)$. Since $\hat{\phi}_2$ is a one-to-one mapping and the gradient of a convex function, it follows from Theorem 26.6 of [14] that $\hat{\phi}_2^{-1}$ is also the gradient of a convex function. We may express $\tilde{\phi}$ as $\tilde{\phi} = (\beta - \alpha)\hat{\phi}_2^{-1} - I$. Hence $\tilde{\phi}$ is the gradient of a potential function, and since $\tilde{\phi}$ is non-decreasing it is the gradient of a convex potential: see [15].

III. Zames-Falb multipliers for quadratic programming

Let the function $f : \mathbb{R}^N \to \mathbb{R}^N$ be given by the quadratic program

$$\begin{aligned} f(x) \quad = \quad & \arg\min_n \frac{1}{2} n^T n - n^T x \\ & \text{subject to } Ln \preceq b \text{ with } b \succeq 0 \\ & \text{and } Mn = 0 \end{aligned} \quad (3)$$

with "$\preceq$" and "$\succeq$" denoting term-by-term inequality. We assume L, M and b are fixed so that f is a static nonlinearity.

In [8] we showed that the nonlinearity that occurs in constrained linear MPC (model predictive control) can be expressed in this manner, provided 0 is feasible. Furthermore f lies in the sector $[0, I]$ in the sense that

$$f(x)^T [f(x) - x] \leq 0 \text{ for all } x \in \mathbb{R}^N \quad (4)$$

Thus the multivariable circle criterion can be used to ascertain the closed-loop stability of MPC.

Here we show that Zames-Falb multipliers may be constructed for f. In particular, we show in the following two lemmas that f satisfies the conditions of [17] and [15] respectively.

Lemma 1: Let f be given by (3). Then $f(0)=0$, f is bounded by 1 and f is incrementally positive. Furthermore f is slope-restricted to the interval $[0,1]$.

Proof: See Appendix. □

Lemma 2: Let f be given by (3). Then f is monotone.

Proof: See Appendix. □

Hence our main result:

Theorem 1 (Zames-Falb multipliers for quadratic programming): Let f be given by (3). Let $h \in L_1$ satisfy $\int_{-\infty}^{\infty} |h(t)|dt < 1$ (or let $h \in l_1$ satisfy $\sum_{t=-\infty}^{\infty} |h(t)| < 1$) and either let f be odd or let $h(t) \geq 0$ for all t. Then

1) For any $x \in L_2^N$ (or $x \in l_2^N$) we have

$$\langle f(x), x \rangle \geq \langle f(x), h * x \rangle \tag{5}$$

2) For any $x \in L_2^N$ (or $x \in l_2^N$) and for any $\varepsilon > 0$ we have

$$\langle f(x), (1+\varepsilon)x - f(x) \rangle \geq \langle f(x), h * [(1+\varepsilon)x - f(x)] \rangle \tag{6}$$

Proof:

1) This follows immediately from applying ZF1 to f.
2) This follows from applying the ZF2 to $f(x)+\alpha x$ with $\beta = 1+\alpha+\varepsilon$.

□

The results can be extended to quadratic programs with more general positive definite Hessians via straightforward substitution. Let the function $\tilde{f}: \mathbb{R}^N \to \mathbb{R}^N$ be given by the quadratic program

$$\begin{aligned} \tilde{f}(\tilde{x}) &= \arg\min_{\tilde{n}} \frac{1}{2}\tilde{n}^T G \tilde{n} - \tilde{n}^T \tilde{x} \\ & \quad \text{subject to } \tilde{L}\tilde{n} \preceq \tilde{b} \text{ with } \tilde{b} \succeq 0 \\ & \quad \text{and } \tilde{M}\tilde{n} = 0 \end{aligned} \tag{7}$$

with $G = G^T > 0$. This can be transformed to the form of (3) by the substitutions

$$\begin{aligned} f &= G_r \tilde{f} \\ x &= G_r^{-T} \tilde{x} \end{aligned} \tag{8}$$

where $G_r^T G_r = G$. Hence we may say:

Corollary 1 (Zames-Falb multipliers for quadratic programming with generalized Hessian): Let $\tilde{f}$ be given by (7). Let $h \in L_1$ satisfy $\int_{-\infty}^{\infty} |h(t)| dt < 1$ (or let $h \in l_1$ satisfy $\sum_{t=-\infty}^{\infty} |h(t)| dt < 1$) and either let $\tilde{f}$ be odd or let $h(t) \geq 0$ for all t. Then

1) For any $\tilde{x} \in L_2^N$ (or $x \in l_2^N$) we have

$$\langle \tilde{f}(\tilde{x}), \tilde{x} \rangle \geq \langle \tilde{f}(\tilde{x}), h * \tilde{x} \rangle \tag{9}$$

2) For any $\tilde{x} \in L_2^N$ (or $x \in l_2^N$) and for any $\varepsilon > 0$ we have

$$\langle \tilde{f}(\tilde{x}), (1+\varepsilon)\tilde{x} - G\tilde{f}(\tilde{x}) \rangle \quad \geq \quad \langle \tilde{f}(\tilde{x}), h * [(1+\varepsilon)\tilde{x} - G\tilde{f}(\tilde{x})] \rangle \tag{10}$$

□

IV. RESULTS EXPRESSED AS IQCS

In [13] a unified approach to robustness analysis via IQCs (integral quadratic constraints) was introduced. Such techniques may be used to analyze the robustness of MPC [6]. Here we express the various inequalities of the theorems in standard form. In particular, given $\phi : L_2^N \to L_2^N$, we adopt the notation of [9] and say

$$\phi \in \text{IQC}(\Pi) \tag{11}$$

when

$$\int_{-\infty}^{\infty} \begin{bmatrix} x(j\omega) \\ \phi(x)(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} x(j\omega) \\ \phi(x)(j\omega) \end{bmatrix} d\omega \geq 0 \tag{12}$$

for all $x \in L_2^N$. Similarly if $\phi : l_2^N \to l_2^N$ we say

$$\phi \in \text{IQC}(\Pi) \tag{13}$$

when

$$\int_{-\pi}^{\pi} \begin{bmatrix} x(e^{j\omega}) \\ \phi(x)(e^{j\omega}) \end{bmatrix}^* \Pi(e^{j\omega}) \begin{bmatrix} x(e^{j\omega}) \\ \phi(x)(e^{j\omega}) \end{bmatrix} d\omega \geq 0 \tag{14}$$

for all $x \in l_2^N$.

Let H be the continuous (or discrete) Fourier transform of h. Then following [2] we may say:

Corollary 2 (Results expressed as IQCs):

1) Inequality (1) may be expressed as $\phi \in \mathrm{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} 0 & (1-H^*)I \\ (1-H)I & 0 \end{bmatrix} \tag{15}$$

2) Inequality (2) may be expressed as $\phi \in \mathrm{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} -\alpha\beta(2-H-H^*)I & \alpha(1-H)I+\beta(1-H^*)I \\ \alpha(1-H^*)I+\beta(1-H)I & (-2+H+H^*)I \end{bmatrix} \tag{16}$$

3) Inequality (6) may be expressed as $f \in \mathrm{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} 0 & (1+\varepsilon)(1-H^*)I \\ (1+\varepsilon)(1-H)I & (-2+H+H^*)I \end{bmatrix} \tag{17}$$

4) Inequality (10) may be expressed as $\tilde{f} \in \mathrm{IQC}(\Pi)$ with

$$\Pi = \begin{bmatrix} 0 & (1+\varepsilon)(1-H^*)I \\ (1+\varepsilon)(1-H)I & (-2+H+H^*)G \end{bmatrix} \tag{18}$$

□

V. Conclusion

We have shown that Zames-Falb multipliers may be applied to the nonlinearity that arises in constrained linear MPC, provided 0 is feasible and the nonlinearity is static. The result may be incorporated into a robustness analysis of input constrained MPC based on IQCs [6].

Appendix: Proof of the Lemmas

Proof of Lemma 1:

1) Since $b \succeq 0$, zero is feasible and it follows trivially that $f(0) = 0$.
2) The Karoush Kuhn Tucker (KKT) conditions [5] for f give

$$\begin{aligned} f(x)+M^T\zeta(x)+L^T\lambda(x)-x &= 0 \\ Mf(x) &= 0 \\ Lf(x)+s(x) &= b \\ s(x)^T\lambda(x) &= 0 \end{aligned} \tag{19}$$

with $s(x) \succeq 0$ and $\lambda(x) \succeq 0$. Substituting for x (and y) from the first condition gives:

$$
\begin{aligned}
&[f(x)-f(y)]^T(x-y) \\
&= [f(x)-f(y)]^T[f(x)+M^T\zeta(x)+L^T\lambda(x)-f(y)-M^T\zeta(y)-L^T\lambda(y)] \\
&= ||f(x)-f(y)||^2+[f(x)-f(y)]^TM^T[\zeta(x)-\zeta(y)]+[(f(x)-f(y)]^TL^T[\lambda(x)-\lambda(y)] \\
&= ||f(x)-f(y)||^2+0+[s(y)-s(x)]^T[\lambda(x)-\lambda(y)] \\
&= ||f(x)-f(y)||^2+s(y)^T\lambda(x)+s(x)^T\lambda(y) \\
&\geq 0
\end{aligned} \tag{20}
$$

3) Given x, suppose f_x is feasible with $||f_x||^2 > ||x||^2$ and hence $||f_x||^2 > f_x^T x$. Let $\tilde{f}_x = (1-\varepsilon)f_x$ with $\varepsilon > 0$. By convexity $\tilde{f}_x$ is also feasible. Put $J_x(f) = \frac{1}{2}f^Tf - f^Tx$. Then

$$
\begin{aligned}
J_x(\tilde{f}_x) &= \frac{1}{2}(1-\varepsilon)^2 f_x^T f_x - (1-\varepsilon)f_x^T x \\
&= J_x(f_x) - \varepsilon(f_x^T f_x - f_x^T x) + \frac{1}{2}\varepsilon^2 f_x^T f_x \\
&< J_x(f_x) \text{ for } \varepsilon \text{ sufficiently small}
\end{aligned} \tag{21}
$$

Hence f_x cannot be equal to $f(x)$ and so we must have $||f(x)||^2 \leq ||x||^2$.

4) Given x, y suppose f_x and f_y are feasible with $(f_x-f_y)^T(x-y) < ||f_x-f_y||^2$. Let $\tilde{f}_x = f_x+\varepsilon(f_y-f_x)$ and $\tilde{f}_y = f_y+\varepsilon(f_x-f_y)$ for some $\varepsilon > 0$. By convexity $\tilde{f}_x$ and $\tilde{f}_y$ are also feasible. Furthermore

$$
\begin{aligned}
&J_x(\tilde{f}_x)+J_y(\tilde{f}_y) \\
&= J_x(f_x)+\varepsilon f_x^T(f_y-f_x)+\frac{1}{2}\varepsilon^2||f_x-f_y||^2-\varepsilon(f_y-f_x)^Tx \\
&\quad +J_y(f_y)+\varepsilon f_y^T(f_x-f_y)+\frac{1}{2}\varepsilon^2||f_y-f_x||^2-\varepsilon(f_x-f_y)^Ty \\
&= J_x(f_x)+J_y(f_y)-\varepsilon(1-\varepsilon)||f_x-f_y)||^2+\varepsilon(f_x-f_y)^T(x-y) \\
&< J_x(f_x)+J_y(f_y) \text{ for } \varepsilon \text{ sufficiently small}
\end{aligned} \tag{22}
$$

Hence f_x and f_y cannot be concurrently equal to $f(x)$ and $f(y)$, so we must have $[f(x)-f(y)]^T(x-y) \geq ||f(x)-f(y)||^2$, and hence $[f(x)-f(y)]^T[f(x)-f(y)-(x-y)] \leq 0$.

□

Proof of Lemma 2: It is well-known (e.g. [1]) that f is both piecewise affine [16] and continuous [4]. Within each affine region R_i, the KKT conditions give (c.f. [16], [5])

$$\begin{bmatrix} I & L_i^T & M^T \\ L_i & 0 & 0 \\ M & 0 & 0 \end{bmatrix} \begin{bmatrix} f_i(x) \\ \lambda_i(x) \\ \zeta(x) \end{bmatrix} = \begin{bmatrix} x \\ b_i \\ 0 \end{bmatrix} \tag{23}$$

where $f = f_i$ within region R_i, where L_i and b_i correspond to the constraints which are active in the region (i.e. L_i is made up of rows of L, and b_i is made up of the corresponding entries in b, and we have the equality $f_i(x) = b_i$ within R_i), and where $\lambda_i(x)$ are the corresponding Lagrange multipliers. Thus the piecewise affine control law is

$$f_i(x) = F_i x + a_i \tag{24}$$

with

$$\begin{aligned} F_i &= I - \begin{bmatrix} L_i \\ M \end{bmatrix}^T \left(\begin{bmatrix} L_i \\ M \end{bmatrix} \begin{bmatrix} L_i \\ M \end{bmatrix}^T \right)^{-1} \begin{bmatrix} L_i \\ M \end{bmatrix} \\ a_i &= \begin{bmatrix} L_i \\ M \end{bmatrix}^T \left(\begin{bmatrix} L_i \\ M \end{bmatrix} \begin{bmatrix} L_i \\ M \end{bmatrix}^T \right)^{-1} \begin{bmatrix} b_i \\ 0 \end{bmatrix} \end{aligned} \tag{25}$$

In particular $F_i = F_i^T \geq 0$ and within each region R_i we can form the potential function

$$P_i(x) = \frac{1}{2} x^T F_i x + x^T a_i \tag{26}$$

such that

$$f_i(x) = \nabla P_i(x) \tag{27}$$

This is sufficient to show the integral is independent of path within each region [15].

So it suffices to show that if x_A and x_B lie on the border of adjoining regions R_i and R_j, then the integral from A to B is the same via region R_i or region R_j—see Fig 1.

Let I_i be the integral via region R_i, and R_j be the integral via region R_j. Thus

$$I_i = \int_A^B (F_i x + a_i)^T dx = \frac{1}{2} x_B^T F_i x_B + x_B^T a_i - \frac{1}{2} x_A^T F_i x_A - x_A^T a_i \tag{28}$$

and similarly

$$I_j = \int_A^B (F_j x + a_j)^T dx = \frac{1}{2} x_B^T F_j x_B + x_B^T a_j - \frac{1}{2} x_A^T F_j x_A - x_A^T a_j \tag{29}$$

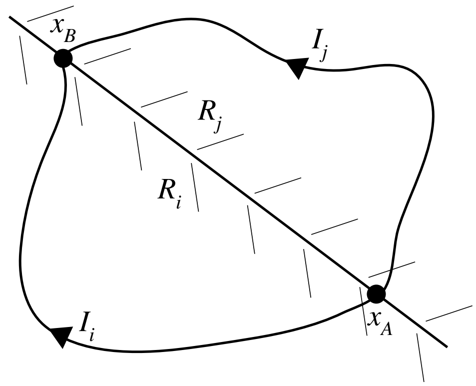

Fig. 1. This illustrates the final part of the proof of Lemma 4. Given x_A and x_B on the border of adjoining regions R_i and R_j, the integral from x_A to x_B is independent of choice of path.

Since x_A and x_B lie on the boundary of region R_i and R_2 we have the relations

$$\begin{aligned} F_i x_A + a_i &= F_j x_A + a_j \\ F_i x_B + a_i &= F_j x_B + a_j \end{aligned} \tag{30}$$

In particular

$$a_i - a_j = (F_j - F_i)x_A = (F_j - F_i)x_B \tag{31}$$

Thus

$$\begin{aligned} I_j - I_i &= \frac{1}{2}x_B^T F_j x_B + x_B^T a_j - \frac{1}{2}x_A^T F_j x_A - x_A^T a_j - \frac{1}{2}x_B^T F_i x_B - x_B^T a_i + \frac{1}{2}x_A^T F_i x_A + x_A^T a_i \\ &= \frac{1}{2}x_B^T (F_j - F_i)x_B + \frac{1}{2}x_A^T (F_i - F_j)x_A + (x_A - x_B)^T (a_i - a_j) \\ &= \frac{1}{2}(x_A - x_B)^T (a_i - a_j) \\ &= \frac{1}{2}(x_A - x_B)^T (F_j - F_i)x_A \\ &= 0 \end{aligned} \tag{32}$$

as required. □

References

[1] A. Bemporad, M. Morari, V. Dua, and E. N. Pistikopoulos. The explicit linear quadratic regulator for constrained systems. *Automatica*, 38:3–20, 2002.

[2] F. J. D'Amato, M. A. Rotea, A. V. Megretski, and U. T. Jönsson. New results for analysis of systems with repeated nonlinearities. *Automatica*, 37(5):739–747, 2001.

[3] C. A. Desoer and M. Vidyasagar. *Feedback systems: input-output properties*. Academic Press, New York, 1975.

[4] A. V. Fiacco. *Introduction to sensitivity and stability analysis in nonlinear programming*. Academic Press, Orlando, Florida, 1983.

[5] R. Fletcher. *Practical Methods of Optimization*. John Wiley & Sons Inc., New York, 1987.

[6] W. P. Heath, G. Li, A. G. Wills, and B. Lennox. The robustness of input constrained model predictive control to infinity-norm bound model uncertainty. ROCOND06, 5th IFAC Symposium on Robust Control Design, Toulouse, France, July 5-7, 2006.

[7] W. P. Heath and A. G. Wills. The inherent robustness of constrained linear model predictive control. 16th IFAC World Congress, Prague, July 3rd-8th, 2005.

[8] W. P. Heath, A. G. Wills, and J. A. G. Akkermans. A sufficient stability condition for optimizing controllers with saturating actuators. *Int. J. Robust Nonlinear Control*, 15:515–529, 2005.

[9] U. Jönsson. Lecture notes on integral quadratic constraints. Department of Mathematics, KTH, Stockholm. ISBN 1401-2294, 2000.

[10] V. V. Kulkarni and M. G. Safonov. All multipliers for repeated monotone nonlinearities. *IEEE Trans-AC*, 47(7):1209–1212, 2002.

[11] V. V. Kulkarni and M. G. Safonov. Incremental positivity nonpreservation by stability multipliers. *IEEE Trans-AC*, 47(1):173–177, 2002.

[12] R. Mancera and M. G. Safonov. All stability multipliers for repeated MIMO nonlinearities. *Systems and Control Letters*, 54(4):389–397, 2005.

[13] A. Megretski and A. Rantzer. System analysis via integral quadratic constraints. *IEEE Trans-AC*, 42(6):819–830, 1997.

[14] R. T. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, 1970.

[15] M. G. Safonov and V. V. Kulkarni. Zames-Falb multipliers for MIMO nonlinearities. *Int. J. Robust Nonlinear Control*, 10:1025–1038, 2000.

[16] E. Zafiriou. Robust model predictive control of processes with hard constraints. *Computers Chem. Engng.*, 14(4–5):359–371, May 1990.

[17] G. Zames and P.L. Falb. Stability conditions for systems with monotone and slope-restricted nonlinearities. *SIAM J. Control*, 6(1):89–108, 1968.