Barrier Function Based Model Predictive Control

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Abstract

A new formulation of nonlinear model predictive control (MPC) is developed by including a weighted barrier function in the control objective. While the barrier ensures that inequality constraints are strictly satisfied it also provides a smooth transition between points in the interior and those on the boundary of the constraint set. In addition, the resulting optimisation problem, to be solved at each control step, is effectively unconstrained and thus amenable to elegant optimisation techniques. The barrier must satisfy certain conditions in order that the state converges to the origin and we show how to construct such a barrier. Conventional MPC may be seen as a limiting case of the new class as the barrier weighting itself approaches zero. We pay particular attention to the novel approach of fixing the weighting parameter to some positive value – possibly large – and observe that this provides a degree of controller caution near constraint boundaries. We construct an ellipsoidal invariant set by exploiting the geometry of self-concordant functions and show nominal closed-loop stability for this class of controllers under full state feedback.

Key words: Nonlinear model predictive control, barrier functions, self-concordant functions.

1 Introduction

Model predictive control (MPC) is a receding horizon control strategy where the current control action is determined, on-line, by numerically solving an optimal control problem at each time interval. Constraints can be treated in a straightforward manner by explicitly adding them to the optimisation problem. Comprehensive surveys for MPC include those by Clarke et al. (1987), Garcia et al. (1989), Muske and Rawlings (1993), Michalska and Mayne (1993), Qin and Badgwell (1997), Chen and Allgöwer (1998), Mayne et al. (2000), and Maciejowski (2002). In particular, Mayne et al. (2000) offer an authoritative account of state feedback nonlinear MPC from a theoretical perspective.

In this paper, we adopt a similar philosophy to that used in interior-point methods and include a weighted barrier function directly into the MPC cost to ensure that constraints are strictly satisfied. The resultant cost function associates increasingly severe penalties for points approaching a constraint boundary which changes controller dynamics in such regions; a more cautious control action is observed when operating near or approaching a constraint boundary. Meanwhile, points far away from the constraint boundary receive negligible penalty from the barrier. This approach allows the control engineer to determine a smooth transition between interior and boundary points with a guarantee that inequality constraints are strictly satisfied (provided the constraint set has an interior).

This novel formulation of MPC is appealing because of its simplicity and generality; while the barrier approach is intuitive it also enjoys the elegant and powerful theory of self-concordant functions associated with interior-point methods (Nesterov and Nemirovskii, 1994). Typically, interior-point algorithms force the barrier weighting towards zero as the algorithm progresses; indeed the optimum is only achieved in the limit as the barrier weighting itself tends to zero. The current approach differs from this since we allow the barrier weighting to converge to some positive value – possibly much greater than zero. This requires a careful choice of barrier function to allow the state to converge to the origin in steady-state. The so-called *recentred barrier function* is used for this purpose and the resulting controller class is called *recentred barrier function MPC* (abbreviated as r-MPC). This class of controllers coincides with conventional MPC in

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the limit as the barrier weighting tends to zero and may thus be seen as a generalisation of conventional MPC. Some preliminary results can be found in (Wills and Heath, 2002a; Heath and Wills, 2002; Wills, 2003).

Other authors have considered barrier methods and interior-point methods within the context of MPC. Polak et al. (1990) presented an algorithm based on barrier functions; in particular, they use a method of centres (see e.g. Lieu and Huard, 1966) for a general optimal control problem with control and state inequality constraints. Their paper and references therein present applications of classical interior-point methods to constrained optimal control problems. Wright (1993) appears to be the first to have explicitly discussed new polynomial-time interior-point methods for the Quadratic Programs (QPs) and Sequential Quadratic Programs (SQPs) associated with MPC. These ideas are developed further in (Wright, 1997), where infeasible-start interior-point methods are used. Lim et al. (1996) also study constrained control from an interior-point framework.

Gopal and Biegler (1998) offer an authoritative account of large-scale optimisation and control. Their analysis and observations are primarily from the perspective of SQP (see also Albuquerque et al., 1997). Gopal and Biegler (1998) also point out that certain formulations of MPC result in structure that can be exploited and consider this structure in the context of SQP. Rao et al. (1998) show how to exploit the structure of linear MPC and utilise a Riccati-recursion to solve the system of linear equations present at each iteration of the interior-point algorithm (see Wills and Heath, 2003) for discussion. The proposal in this paper differs from all of these approaches since we do not require that the barrier weighting tends to zero.

In the spirit of Michalska and Mayne (1993) and Chen and Allgöwer (1998) we determine a terminal constraint set which is invariant under some local controller. Specifically, we extend their results to discrete-time systems with both state and control constraints and define an ellipsoidal terminal constraint set using the level set of an appropriately chosen Lyapunov function. Furthermore, we choose the particular level set in a novel manner by exploiting the properties of self-concordant functions and the Dikin's ellipsoid. Using this terminal constraint set and arguments from (Mayne et al., 2000) we show nominal closed-loop asymptotic stability of state feedback r-MPC using gradient recentred self-concordant barriers. The result is quite general since Nesterov and Nemirovskii (1994) showed that every closed convex domain admits a self-concordant barrier and it is possible to recentre any self-concordant barrier using its gradient. We illustrate the approach using a nonlinear simulation example.

The paper is organised as follows. Relevant notation and definitions for recentred barrier functions and r-MPC are given in Section 2. An ellipsoidal invariant set is constructed in Section 3. The method for constructing this invariant set is useful in its own right and is also pertinent to the stability analysis given in Section 4. A nonlinear example is given in Section 5.

2 Problem statement

We adopt notation, where appropriate, from (Mayne et al., 2000). Furthermore, by $\|x\|$ and $\|x\|_M$ we mean $\sqrt{x^T x}$ and $\sqrt{x^T M x}$, where M is a symmetric positive semi-definite matrix.

2.1 System

Consider the following time-invariant, discrete-time system with integer k representing the current discrete time event,

$$x(k+1) = f(x(k), u(k)). \tag{1}$$

In the above, $u(k) \in \mathbb{R}^m$ is the system input and $x(k) \in \mathbb{R}^n$ is the system state. The mapping $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be Fréchet differentiable on $\mathbb{R}^n \times \mathbb{R}^m$ and to satisfy $f(0,0) = 0$. Given some positive integer N let $\mathbf{u}$ denote a sequence of control moves given by $\mathbf{u} = \{u(0), u(1), \ldots, u(N-1)\}$ and let $x^{\mathbf{u}}(\cdot\,; x)$ denote the state sequence (of length N) starting at state x and evolving under the influence of $\mathbf{u}$ according to (1), i.e. $x^{\mathbf{u}}(\cdot\,; x) = \{x, f(x, u(0)), \ldots, f(x^{\mathbf{u}}(N-1; x), u(N-1))\}$. The i^{th} element of $x^{\mathbf{u}}(\cdot\,; x)$ is denoted by $x^{\mathbf{u}}(i; x)$ and $x^{\mathbf{u}}(0; x) = x$.

2.2 Constraints

The state is restricted to lie in the closed and convex set $\mathbb{X}$ while the input is restricted to the compact and convex set $\mathbb{U}$. Furthermore, the final state $x^{\mathbf{u}}(N; x)$ may be further restricted to lie in a *terminal* constraint set X_f which is assumed to be a closed and convex subset of $\mathbb{X}$. Further, each set is assumed to have a *non-empty* interior denoted by $\mathbb{X}^o$, X_f^o and $\mathbb{U}^o$ respectively. Define the feasible set $\mathscr{U}_N(x)$ as

$$\begin{aligned} \mathscr{U}_N(x) \;=\; \Big\{ \mathbf{u} : \; & u(i) \in \mathbb{U},\; i = 0, \ldots, N-1, \\ & x^{\mathbf{u}}(i; x) \in \mathbb{X},\; i = 1, \ldots, N-1, \\ & x^{\mathbf{u}}(N; x) \in X_f \Big\}. \end{aligned}$$

Denote the interior of $\mathscr{U}_N(x)$ by $\mathscr{U}_N^o(x)$. Note that since $\mathbb{U}$ is compact, $\mathbb{X}$ is closed, X_f is closed and f is continuous then it follows from Proposition 3.7.8 in (Sutherland, 1975) that $\mathscr{U}_N(x)$ is also compact.

2.3 Objective Function

In what follows we consider one possible construction of the time-invariant MPC objective function by adding a weighted barrier to a quadratic penalty for state and input deviations. By construction, the minimum of each quadratic term occurs at the origin. In order to preserve this property for the overall objective, we require that the barrier term also attains its minimum at the origin. Hence we define a gradient recentred barrier function which has this property.

The underlying barrier function is assumed to be self-concordant (see Section 2.3 of Nesterov and Nemirovskii, 1994). Loosely speaking, a ϑ-self-concordant barrier $B : \mathbb{R}^n \to \mathbb{R}$ defined on an open non-empty convex subset $Q \subset \mathbb{R}^n$ has the following properties (see also Renegar, 2001):

(1) The sublevel sets $\{z \in Q : B(z) \le t\}$ are closed in $\mathbb{R}^n$ for every $t \in \mathbb{R}$.
(2) Given $z \in Q$, $r \in [0,1)$ and any y such that $(z-y)^T \nabla^2 B(z)(z-y) < r^2$, then for all $h \in \mathbb{R}^n$ the Hessian of B may be bounded by

$$\begin{aligned}(1-r)^2 h^T \nabla^2 B(z) h &\le h^T \nabla^2 B(y) h \\ &\le (1-r)^{-2} h^T \nabla^2 B(z) h,\end{aligned}$$

(3) For any $z \in Q$ and $r \in [0,1)$ it follows that $\left\{h \in \mathbb{R}^n : h^T \nabla^2 B(z) h < r^2\right\} \subset Q$.
(4) For any $z \in Q$ it follows that

$$\sup\left\{\nabla B^T(z)[\nabla^2 B(z)]^{-1} \nabla B(z)\right\} \le \vartheta.$$

We define a gradient recentred self-concordant barrier function as follows.

Definition 2.1 *Let Q be an open non-empty convex set which contains the origin. Let $B : Q \to \mathbb{R}$ be a self-concordant barrier function over Q. Define a function $\bar{B} : Q \to \mathbb{R}$ as*

$$\bar{B}(z) \;=\; B(z) - B(0) - [\nabla B(0)]^T z.$$

Then $\bar{B}$ is called a gradient recentred self-concordant barrier function over Q (about the origin). □

Since self-concordant barrier functions are convex, it follows by construction that the gradient recentred barrier is also convex and achieves its global minimum at the origin. This is illustrated in Figure 1 which shows the logarithmic barrier function for the constraints $-1 \le u \le 2$ given by

$$B(u) = -\ln(2-u) - \ln(1+u). \tag{2}$$

Its corresponding gradient recentred logarithmic barrier function

$$\bar{B}(u) = \ln(2) - \ln(2-u) - \ln(1+u) + \tfrac{1}{2}u, \tag{3}$$

is also shown in Figure 1.

A remarkable result from Nesterov and Nemirovskii (1994) (Section 2.5) is the existence of a self-concordant barrier function for an arbitrary closed convex set $G \subset \mathbb{R}^n$. As a consequence, we can construct the gradient recentred self-concordant barrier function about any interior point in G. Unfortunately, their so-called universal barrier function does not – in general – have easily computable gradients and Hessians. However, in most important practical cases there exist self-concordant barriers that are amenable to Newton's method (in particular see Chapter 5 of Nesterov and Nemirovskii, 1994).

Let B_f, B_x and B_u be gradient recentred self-concordant barrier functions for X_f, $\mathbb{X}$ and $\mathbb{U}$ respectively. Let $V_N(x, \mathbf{u})$ denote the time-invariant MPC objective function with prediction horizon N defined as

$$\begin{aligned}V_N(x,\mathbf{u}) = F(x^{\mathbf{u}}(N;x)) &+ \ell_0(x,u(0)) \\ &+ \sum_{i=1}^{N-1} \ell(x^{\mathbf{u}}(i;x), u(i)),\end{aligned}$$

where

$$\begin{aligned}F(x) &= \|x\|_P^2 + \mu B_f(x), \\ \ell_0(x,u) &= \|x\|_Q^2 + \|u\|_R^2 + \mu B_u(u) \quad \text{and} \\ \ell(x,u) &= \|x\|_Q^2 + \|u\|_R^2 + \mu B_x(x) + \mu B_u(u).\end{aligned}$$

The matrices P, Q and R are assumed to be symmetric and positive definite and the scalar μ that weights the barrier terms is assumed to be positive. Since B_f, B_x and B_u are recentred barriers then by construction $V_N(0,0) = 0$ which is also the global minimum.

Let $X_N = \{x \in \mathbb{R}^n : \mathscr{U}_N^o(x) \ne \emptyset\}$ be the set of states which can be steered to the origin by strictly admissible control sequences. The control strategy may be described as follows: at each time interval k, given the state $x(k) \in X_N$, compute the following and apply the first control move to the system.

$$(\mathscr{P}_N): \qquad \mathbf{u}^*(x(k)) = \arg\min_{\mathbf{u}} V_N(x(k), \mathbf{u}).$$

Note that $\mathbf{u}^*(x(k))$ is strictly feasible according to the following argument. Given $x \in X_N$ let $\{\mathbf{u}_k \in \mathscr{U}_N^o(x)\}$ be a sequence of input trajectories converging to $\bar{\mathbf{u}}$ on the boundary of $\mathscr{U}_N(x)$. It follows immediately from the definition of a self-concordant barrier that $\lim_{k\to\infty} V_N(x, \mathbf{u}_k) = \infty$. Furthermore, since $\mathscr{U}_N^o(x) \ne \emptyset$

and $\mathscr{U}_N(x)$ is compact for all $x \in X_N$ then by Corollary 8 from Section 3.3. of (Fiacco and McCormick, 1968) it follows that the optimal input sequence is strictly feasible.

Remark 2.1 *Computational issues are beyond the scope of this paper. However, it is worth mentioning that if the system model is linear then ($\mathscr{P}_N$) may be solved using efficient algorithms; three such algorithms are developed in (Wills and Heath, 2002b; Wills, 2003). In fact, the larger the barrier weighting value is the more efficient the algorithms become.*

3 Invariant set

Invariant sets play an important role in the analysis of constrained MPC (see e.g. Gilbert and Tan, 1991; Michalska and Mayne, 1993; Chen and Allgöwer, 1998; Blanchini, 1999; Mayne et al., 2000). In particular, (Blanchini, 1999) surveys invariant sets for control, reiterating the two prominent invariant set structures as being ellipsoidal and polytopic. Michalska and Mayne (1993) provide an example of ellipsoidal invariant sets (see also Chen and Allgöwer, 1998)) while (Gilbert and Tan, 1991) developed the polytopic Maximal Output Admissible Set. Blanchini (1999) concludes that ellipsoidal sets are more conservative than polytopic sets and the latter are usually more complex to construct.

In this section we construct an ellipsoidal invariant set by choosing a particular level of a Lyapunov function in a similar manner to (Michalska and Mayne, 1993) and (Chen and Allgöwer, 1998) but extended to discrete-time systems with both state and control constraints. We choose the particular level by exploiting properties of self-concordant functions and their associated Dikin's ellipsoids. The resulting terminal set will be used in Section 4 where nominal closed-loop stability is discussed.

Consider the Jacobian linearisation of system (1) about the origin,

$$x(k+1) = Ax(k) + Bu(k), \quad (4)$$

where $A = f_x(0,0)$ and $B = f_u(0,0)$

Assumption 3.1 *The linear system described in* (4) *is assumed to be stabilisable.* □

Under the above assumption there exists a matrix K such that system (4) is asymptotically stable under the control law

$$u(k) = Kx(k).$$

Let A_K be defined as $A_K = A + BK$ and let $X_K = \{x \in \mathbb{X} : Kx \in \mathbb{U}\}$ with interior denoted by X_K^o. Since the properties of self-concordant barriers are preserved under affine transformation (see Section 2.3 in Nesterov and Nemirovskii, 1994), it follows immediately that we may define a gradient recentred self-concordant barrier function B_K for X_K as

$$B_K(x) = B_x(x) + B_u(Kx).$$

We are interested in finding some positive definite matrix P and a positive value α such that the set $X_P(\alpha) = \{x \in \mathbb{R}^n : \|x\|_P \leq \alpha\}$ is positively invariant under the locally stabilising control law $u(k) = Kx(k)$. In a similar manner to (Michalska and Mayne, 1993) and (Chen and Allgöwer, 1998) we find a value α^* such that $X_P(\alpha^*) \subseteq X_K$ and then show that there exists some $\alpha \in (0, \alpha^*]$ such that $f(x, Kx) \in X_P(\alpha)$ for all $x \in X_P(\alpha)$.

Since A_K is discrete Hurwitz, then for any positive definite matrices H_1 and H_2 there exists a positive definite matrix P which solves the discrete Lyapunov matrix equation

$$A_K^T P A_K - P = -H_1 - H_2, \quad (5)$$

Suitable choices for H_1 and H_2 will be given in Section 4 where nominal stability is discussed.

With P defined we wish to obtain $\alpha > 0$ such that $X_P(\alpha) \subseteq X_K$. In the literature this is usually achieved by solving a maximisation problem for α (see e.g. Michalska and Mayne, 1993).

Lemma 3.1 below offers an alternative approach based on exploiting properties of self-concordant functions and the Dikin's ellipsoid; the latter is defined as follows (see Section 2.1 in Nesterov and Nemirovskii, 1994). Let g be a convex twice continuously differentiable function on the open domain $Q \subset \mathbb{R}^n$. The Dikin's ellipsoid of g with radius r about the point $\bar{z} \in Q$ is given by

$$W_{g,r}(\bar{z}) = \{z \in \mathbb{R}^n : (\bar{z} - z)^T \nabla^2 g(\bar{z})(\bar{z} - z) \leq r^2\}.$$

Lemma 3.1 *Given* $\beta \in (0,1)$ *then* $X_P(\alpha^*) \subset X_K$ *if*

$$\alpha^* = \sqrt{\frac{\beta^2}{\lambda_{\max}(P^{-1}\nabla^2 B_K(0))}}. \quad (6)$$

Proof. Theorem 2.1.1 from (Nesterov and Nemirovskii, 1994) shows that the open unit Dikin's ellipsoid of a strongly 1-self-concordant barrier function is contained in the domain of the same function. Furthermore, gradient recentred self-concordant barrier functions are by definition strongly 1-self-concordant functions. Hence the Dikin's ellipsoid of B_K about the origin with radius $\beta < 1$ (denoted by $W_{B_K,\beta}(0)$) is contained within X_K i.e. $W_{B_K,\beta}(0) \subset X_K$. Since P is positive definite and $\beta \in (0,1)$ it follows immediately that there exists $\alpha > 0$ such that $X_P(\alpha) \subseteq W_{B_K,\beta}(0) \subset X_K$. The maximum value of α for which this holds is given by (6). □

If there exists some $\alpha \in (0, \alpha^*]$ such that $f(x, Kx) \in X_P(\alpha)$ for all $x \in X_P(\alpha)$ then $X_P(\alpha)$ is positively invariant under the linear stabilising control law $u = Kx$. If we define $V(x) = \|x\|_P^2$ and define $\phi(x) = V(f(x, Kx)) - V(x)$ then this is equivalent to showing that $\phi(x) \leq 0$ for all $x \in X_P(\alpha)$. The following Lemma shows that such an α exists for the stronger requirement that $\phi(x) \leq -\|x\|_{H_2}^2$ for all $x \in X_P(\alpha)$.

Lemma 3.2 *There exists $\alpha \in (0, \alpha^*]$ such that*

$$\phi(x) \leq -\|x\|_{H_2}^2.$$

Proof. Let $r(x) = f(x, Kx) - A_K x$ then $V(f(x, Kx)) = \|A_K x\|_P^2 + \|r(x)\|_P^2 + 2x^T A_K^T P r(x)$ and from (5) it follows that

$$\phi(x) = \|r(x)\|_P^2 + 2x^T A_K^T P r(x) - \|x\|_{H_1+H_2}^2.$$

Let $c = \frac{2\|A_K P\|}{\sqrt{\lambda_{\min}(P)\lambda_{\min}(H_1)}}$ then

$$\begin{aligned} 2x^T A_K^T P r(x) &\leq 2\|A_K P\| \, \|x\| \, \|r(x)\|, \\ &\leq c\|x\|_{H_1} \|r(x)\|_P. \end{aligned}$$

Hence $\phi(x) \leq c\|x\|_{H_1} \|r(x)\|_P + \|r(x)\|_P^2$. Since f is differentiable it follows that for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that

$$\|x\|_P \leq \delta \quad \Rightarrow \quad \|r(x)\|_P \leq \epsilon \|x\|_{H_1},$$

It follows immediately that there exists $\alpha \in (0, \alpha^*]$ such that

$$\|x\|_P \leq \alpha \quad \Rightarrow \quad \|r(x)\|_P^2 + 2x^T A_K^T P r(x) \leq \|x\|_{H_1}^2.$$

□

Finding a suitable α is not necessarily trivial (this is true for nonlinear MPC in general). One approach is to solve the following optimisation problem starting at $\alpha = \alpha^*$ and reducing α until the solution is not positive (Michalska and Mayne, 1993).

$$(\mathscr{P}_\alpha): \qquad \begin{aligned} \max_x \quad & \|r(x)\|_P^2 + 2x^T A_K^T P r(x) - \|x\|_{H_1}^2, \\ \text{s.t.} \quad & \|x\|_P \leq \alpha. \end{aligned}$$

Remark 3.1 *With a view to practical considerations, it is highly likely that the origin will shift at each control step due to new target calculations and integral action (see e.g. Qin and Badgwell, 2000). This requires that α be recalculated at every control step.*

Remark 3.2 *If the system is linear then the matrix H_1 in (5) can be set to zero and $\alpha = \alpha^*$ satisfies Lemma 3.2 since $r(x) = 0$ in this case. Hence $(\mathscr{P}_\alpha)$ need not be solved.*

4 Stability

Mayne et al. (2000) stipulate sufficient conditions for nominal stability of state feedback MPC as follows.

A1: $X_f \subset \mathbb{X}$, X_f is closed with $0 \in X_f$.
A2: $x \in X_N$, i.e. there exists a feasible input.
A3: $Kx \in \mathbb{U}$ for all $x \in X_f$.
A4: $f(x, Kx) \in X_f$ for all $x \in X_f$.
A5: $F(f(x, Kx)) - F(x) + \ell(x, Kx) \leq 0$ for all $x \in X_f$.

In the following discussion we show how to construct r-MPC such that these sufficient conditions for nominal stability hold. In particular, under the assumption of feasibility (A2) we construct a terminal constraint set X_f which satisfies A1, A3 and A4 and then show that A5 follows.

Assuming the linearised system (4) is stabilisable about the origin then there exists a matrix K such that $A_K = A + BK$ is discrete Hurwitz. Let the local control law be given by

$$u = Kx. \tag{7}$$

Choose $\beta \in (0, 1)$ and let the matrix H_2 in (5) be given by

$$H_2 = Q + K^T R K + \frac{\mu}{2(1-\beta)^2} \nabla^2 B_K(0). \tag{8}$$

Let H_1 be some positive definite symmetric matrix and solve (5) for P. Choose $\alpha \in (0, \alpha^*]$ such that Lemma 3.2 holds. The terminal constraint set X_f may be defined as

$$X_f \;=\; X_P(\alpha). \tag{9}$$

Note that A1, A3 and A4 are satisfied with K and X_f given in (7) and (9). Let B_f be a gradient recentred self-concordant barrier function for X_f given by

$$B_f(x) \;=\; \ln\left(\alpha^2\right) - \ln\left(\alpha^2 - \|x\|_P^2\right). \tag{10}$$

Lemma 4.1 *Let X_f be given by (9), then*

$$F(f(x, Kx)) - F(x) + \ell(x, Kx) \leq 0, \quad \forall x \in X_f^o.$$

Proof. Let $x^+ = f(x, Kx)$. Recall that $B_K(x) = B_x(x) + B_u(Kx)$. We may write

$$\begin{aligned} F(x^+) - F(x) + \ell(x, Kx) = {} & \|x^+\|_P^2 - \|x\|_P^2 \\ & + \mu B_f(x^+) - \mu B_f(x) \\ & + \|x\|_Q^2 + \|Kx\|_R^2 + \mu B_K(x) \end{aligned}$$

From Theorem 2.1.1 in (Nesterov and Nemirovskii, 1994) and Taylor's theorem it follows immediately that for all $x \in X_f^o$ we have

$$B_K(x) \;\leq\; \frac{1}{2(1-\beta)^2} x^T \nabla^2 B_K(0) x.$$

We also have from Lemma 3.2 that

$$\begin{aligned} B_f(x^+) - B_f(x) &= \ln\left(\frac{\alpha^2 - \|x\|_P^2}{\alpha^2 - \|x^+\|_P^2}\right), \\ &\le \ln\left(\frac{\alpha^2 - \|x\|_P^2}{\alpha^2 - \|x\|_P^2 + \|x\|_{H_2}^2}\right), \\ &\le 0. \end{aligned}$$

□

Employing the same argument used in Section 3.3 of (Mayne et al., 2000), it follows immediately that for all $x \in X_N$

$$V_N(x^+, \mathbf{u}^*(x^+)) - V_N(x, \mathbf{u}^*(x)) \le \ell_0(x, u^*(0; x)),$$

where $x^+ = f(x, u^*(0; x))$. Hence, under the assumption of feasibility (A2) and local stabilisability about the origin, we may say that r-MPC is nominally asymptotically stable for any value of barrier weighting $\mu > 0$. Furthermore, in the limit as $\mu \to 0$ we may choose $\beta = 1$ in Lemma 3.1 and obtain $X_f = X_P(\alpha)$ for some $\alpha \in (0, \alpha^*]$ that satisfies Lemma 3.2.

5 Simulation Example

To illustrate r-MPC, we consider a discretisation of the model used in Section 5 of (Chen and Allgöwer, 1998). The resulting non-linear system is given by

$$\begin{aligned} x_1(k+1) &= x_1(k) + \delta(x_2(k) + (\sigma + (1-\sigma)x_1(k))u(k)) \\ x_2(k+1) &= x_2(k) + \delta(x_1(k) + (\sigma - 4(1-\sigma)x_2(k))u(k)) \end{aligned}$$

Let $\sigma = 0.5$ and $\delta = 0.1$. The input is required to satisfy

$$u(k) \in \mathbb{U} = \{u \in R : -1 \le u \le 2\}.$$

A gradient recentred self-concordant barrier function for $\mathbb{U}$ about the origin is given by (3) (see Figure 1).

The weighting matrices for the cost function V_N are given by $Q = \mathrm{diag}\{0.5, 0.5\}$ and $R = 1$ and the prediction horizon $N = 15$. Using these matrices we also solve a discrete-time algebraic Riccati equation for the linearised system about the origin and obtain a locally stabilising state feedback gain

$$K = [2.0107 \;\; 2.0107].$$

Note that $\nabla^2 B_K(0)$ is given by

$$\nabla^2 B_K(0) = \tfrac{5}{4} K^T K.$$

Let $H_1 = 100(Q + K^T R K)$ and let H_2 be given by (8) with $\beta = 0.5$ and $\mu = 0.1$. Solve (5) for P and obtain $\alpha^* = 10.5839$ from (6). We found that $\alpha = 8.5$ satisfies Lemma 3.2, hence let $X_f = X_P(8.5)$. The terminal penalty F is given by $F(x) = \|x\|_P^2 + \mu B_f(x)$ with B_f given by (10).

Figure 2 shows state trajectories for various initial conditions under the control of r-MPC with $\mu = 0.1$. Figure 3 shows two input trajectories for different barrier weightings (starting from the same initial state); one large ($\mu = 0.1$) and one small ($\mu = 10^{-6}$). Figure 4 shows a close-up of the origin for the case where $\mu = 0.1$.

If the barrier function is not recentred then steady-state behaviour may be affected. For example, suppose we use the standard logarithmic barrier given by (2). Figure 5 shows a close-up of the origin when using this barrier. Note that the states do not converge to the origin, which illustrates the importance of recentring the barrier.

6 Conclusion

We have developed a new class of model predictive control based on the recentred barrier function. The simulations illustrate that control action becomes cautious near constraint boundaries. We conjecture that such behaviour may be desirable in physical systems where uncertainty exists near constraint boundaries. The degree of caution is directly related to the positive weighting parameter which characterises the controller class.

We have exploited the theory of self-concordant functions introduced by Nesterov and Nemirovskii (1994) to help determine an ellipsoidal invariant set. The associated geometry can also be used to demonstrate nominal stability via the incorporation of a terminal constraint set and an argument based on that of Mayne et al. (2000).

The controller is applicable to a wide class of nonlinear systems with general convex constraints. However, we stress that these results are subject to the usual caveats regarding feasibility and full state feedback. From a practical perspective, the controller has been successfully applied to an edible oil refining line using output feedback and a linear system model (Wills, 2003).

An appealing aspect of r-MPC is that the optimisation problem to be solved at each step is effectively unconstrained–in the sense that the solution always lies in the interior of the constraint set. Indeed this requirement motivated the introduction of the recentred barrier. Recently, Jadbabaie et al. (2001) have shown that it may be possible to demonstrate stability of unconstrained nonlinear model predictive controllers without the introduction of terminal constraints. A natural question would be whether the proposed class of controllers might be analysed in such a fashion. As the class of controller we have proposed includes more conventional

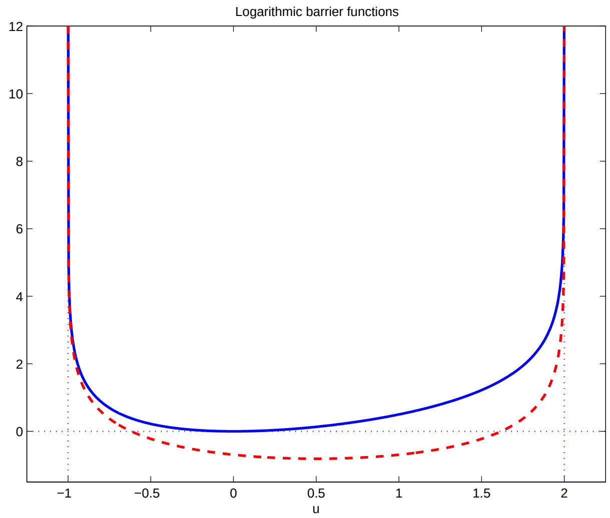

Fig. 1. Logarithmic barrier function (dashed) and gradient recentred logarithmic barrier function (solid) for the constraint $-1 \le u \le 2$. Note that the recentred barrier achieves its minimum at the origin with minimum value zero.

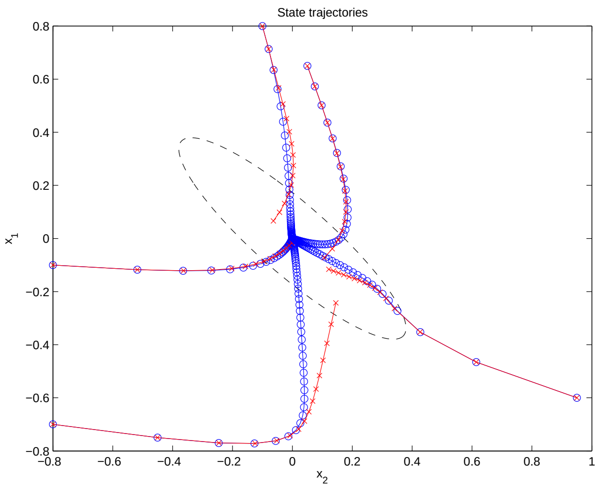

Fig. 2. State trajectories for different initial conditions and barrier weighting $\mu = 0.1$; actual(o), initial prediction(x). The ellipse describes the boundary of X_f.

model predictive control as a limiting case, this may suggest a new avenue for the analysis of constrained model predictive control.

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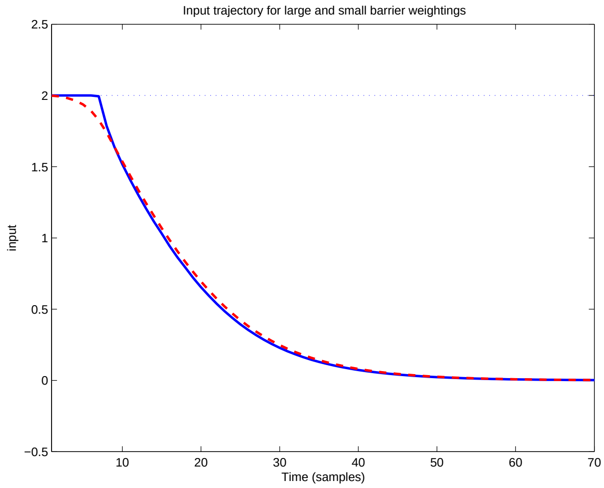

Fig. 3. Input trajectories for different barrier weightings: $\mu = 10^{-6}$ (solid) and $\mu = 0.1$ (dashed). Note that the large weighting results in a more cautious control action near constraint boundaries, while the trajectories are similar when away from the boundary.

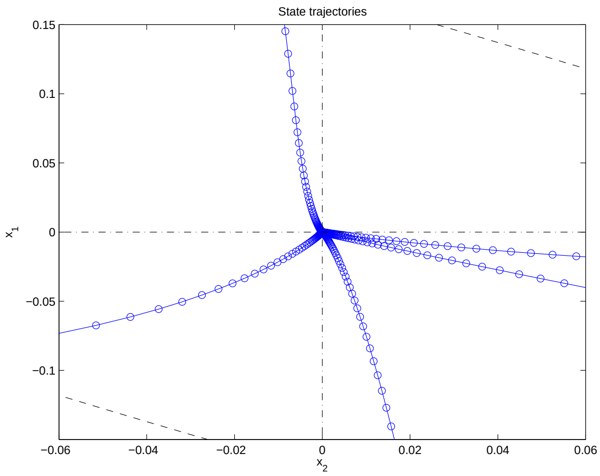

Fig. 4. Close-up of origin with recentred logarithmic barrier. Note the trajectories converge to the origin.

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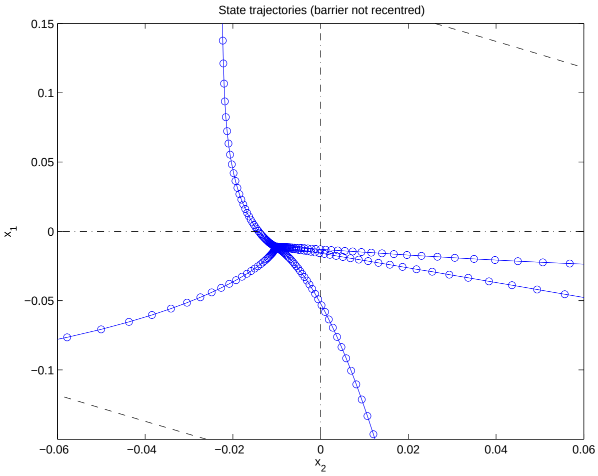

Fig. 5. Close-up of origin with standard logarithmic barrier. Note the trajectories *do not* converge to the origin since the barrier is not recentred in this case.

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