

Local ISS of large-scale interconnections and estimates for stability regions

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Abstract

We consider large-scale interconnections of locally input-to-state stable (LISS) systems. The class of LISS systems is quite large, in particular it contains input-to-state stable (ISS) and integral input-to-state stable (iISS) systems. We prove local small-gain conditions both for LISS trajectory and Lyapunov-formulations that guarantee LISS of the composite system. Furthermore, we discuss and provide estimates for the resulting stability region of the composite system. This in particular provides an advantage over the linearization approach, as we shall discuss. An example demonstrates these quantitative results, and also the usefulness of monotone aggregation functions (MAFs) in this context.

Key words: Local input-to-state stability, interconnected systems, large-scale system, small-gain condition, Lyapunov function

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1. Introduction

In this paper we study local stability properties of interconnected nonlinear systems. One of the most popular frameworks for such interconnections is input-to-state stability (ISS) introduced in [22]. This notion has been used successfully for the investigation of continuous and discrete time systems, systems with time delays, and hybrid systems. In particular the first small-gain stability condition for a feedback interconnection of two ISS systems which were given in terms of ordinary differential equations was derived in [12]. A corresponding construction of an ISS Lyapunov function for feedback interconnections has been given in [11]. These results were extended for the case of an interconnection of $n \geq 2$ systems in [3, 5] and [7], respectively. Small-gain theorems for hybrid systems can be

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found in [14], [16]. Interconnected systems with time delays have been studied in the ISS framework in [17]. A small-gain theorem for interconnections of a more general type of systems that do not satisfy the classical semigroup property has been developed in [13].

In some applications the ISS property can be rather restrictive. A less restrictive property is for example the integral input-to-state stability (iISS) property [23]. The set of iISS systems contains ISS systems as a proper subset. Small-gain theorems for interconnections of iISS systems can be found in [9], [10]. Another way to weaken the ISS property is to consider its local version, local input-to-state stability (LISS), but see also [8, 21, 24] for different local stability properties. It turns out that LISS leads to an even bigger class of nonlinear systems than iISS systems (cf. [1, Theorem 1]: iISS implies 0-GAS and [24, Lemma I.1]: 0-GAS implies LISS). In broad terms, a system is LISS if the ISS property holds locally with respect to inputs and initial states. Systems with such restrictions and a corresponding small-gain condition for feedback interconnections of two systems have been discussed in [12]. Large-scale interconnections of such systems have been considered in [4] for the first time.

Provided that the stability regions of allowable inputs and initial conditions are quantified and suitably large, LISS is a rather interesting property from an application perspective, as it allows to estimate transient and asymptotic behavior of solutions of nonlinear systems in a well-understood framework.

This paper is devoted to stability investigations of large-scale interconnected nonlinear systems. To this extent, we consider $n \geq 2$ subsystems given by

$$\dot{x}_i = f_i(x_1, \dots, x_n, u_i), \quad i = 1, \dots, n, \quad (1)$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{M_i}$, and $f_i : \mathbb{R}^{\sum_j N_j + M_i} \rightarrow \mathbb{R}^{N_i}$, $i = 1, \dots, n$, are assumed to be continuous and locally Lipschitz in x uniformly for u_i in compact sets, which guarantees existence (at least on small time intervals) and uniqueness of solution for each of the systems. Let x^T denote the transposition of a vector. Introducing $x^T = (x_1^T, \dots, x_n^T) \in \mathbb{R}^N$, $N = \sum_{i=1}^n N_i$, $M = \sum_{i=1}^n M_i$, $u^T = (u_1^T, \dots, u_n^T)$, $f(x, u)^T = (f_1(x, u_1)^T, \dots, f_n(x, u_n)^T)$ we consider this interconnection as one composite system of a larger dimension N ,

$$\dot{x} = f(x, u). \quad (2)$$

Our main results are small-gain theorems that provide sufficient conditions for the stability of such interconnections: Under the assumption that each system (1) is LISS (see below) and a small-gain condition, we show that the composite system (2) is also LISS. Notably, these results hold for almost arbitrary interconnection structures, including cascades and feedback loops as special cases. Furthermore, one of our results yields an explicit construction of a LISS Lyapunov function for the composite system.

The main questions that we consider in this paper are:

- Under what conditions is a composite system (2) comprised of $n \geq 2$ LISS systems (1) itself LISS from u to the overall state x ?

- How can one construct a LISS Lyapunov function for the composite system if LISS Lyapunov functions for the subsystems are known?
- How large is the stability region for the composite system?

The paper is organized as follows. The next section introduces the necessary notions and formally states the problem. Section 5 briefly motivates advantages of LISS versus linearization approaches. In Section 3 we recall corresponding global results for the stronger ISS property. Our local small gain condition is introduced in Section 4 where we also prove some auxiliary results related to this condition. In Section 5 we briefly discuss advantages of LISS compared to linearization approaches. Section 6 contains the main results of the paper. An illustrative example is considered in Section 7. Section 8 concludes the paper.

2. Notation and definitions, problem formulation

2.1. Notation

Let $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$ denote the positive orthant in \mathbb{R}^n . For $a, b \in \mathbb{R}_+^n$ let $a \ll b$ denote that $a_i < b_i$ for all $i = 1, \dots, n$ and $a \leq b$ denote $a_i \leq b_i$ for all $i = 1, \dots, n$. We write $a < b$ iff $a \leq b$ and $a \neq b$. With respect to this partial order, the minimum and maximum of two or more vectors is taken component-wise. For a vector $a \in \mathbb{R}^n$ by $|a|$ we denote the vector $(|a_1|, \dots, |a_n|)^T \in \mathbb{R}_+^n$. Observe that $|a| = \max\{a, -a\}$. The logical negation of the relation \geq is denoted by $a \not\geq b$ and it means that there is at least one $i \in \{1, \dots, n\}$ such that $a_i < b_i$. It is *not* the same as the relation $<$. For $a, b \in \mathbb{R}_+^n$ we write $[a, b] := \{s \in \mathbb{R}_+^n : a \leq s \leq b\}$, $(a, b) := \{s \in \mathbb{R}_+^n : a < s < b\}$, and similarly $[a, b)$, $(a, b]$ to denote order intervals in \mathbb{R}_+^n . By $\|x\|$ we denote the Euclidean norm of $x \in \mathbb{R}^n$ and by $\|u\|_{L_\infty(T)} = \text{ess. sup}_{t \in T} \|u(t)\|$ we denote the essential supremum norm of a measurable function u . Reference to the time interval T is usually omitted in the case $T = \mathbb{R}_+$. The set of all measurable and essentially bounded functions is denoted by L_∞ . By $B(x, r)$ we denote the open ball with respect to the Euclidean norm around x of radius r . A continuous operator $A : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ is called monotone, if $r \leq s$ implies $A(r) \leq A(s)$. For a vector $x \in \mathbb{R}_+^n$ we denote by $x|_I$ the vector in \mathbb{R}_+^n with elements

$$(x|_I)_i = \begin{cases} x_i & \text{if } i \in I \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Before we move on to the stability concepts, we first recall the definition of comparison functions. A function $\gamma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class \mathcal{K} if it is continuous, increasing and $\gamma(0) = 0$. It is of class \mathcal{K}_∞ if, in addition, it is unbounded.

We will frequently use the class \mathcal{K}_∞ notation for functions that are defined only on bounded intervals $[0, r]$. In this case the function will obviously be bounded.

A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is said to be of class \mathcal{KL} if, for each fixed t , the function $\beta(\cdot, t)$ is of class \mathcal{K} and, for each fixed s , the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity.

2.2. Local input-to-state stability (LISS)

The concept of input-to-state stability (ISS) has been first introduced in [22]. Its local version has first appeared in [24].

Definition 2.1. *System (2) is locally input-to-state stable (LISS), if there exist $\rho^0 > 0$, $\rho^u > 0$, $\gamma \in \mathcal{K}_\infty$, and $\beta \in \mathcal{KL}$, such that for all $\|\xi\| \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$*

$$\|x(t, \xi, u)\| \leq \beta(\|\xi\|, t) + \gamma(\|u\|_{L_\infty}) \quad \forall t \geq 0. \quad (\text{LISS})$$

Here γ is called LISS gain.

If $\rho^0 = \rho^u = \infty$, then system (2) is called *input-to-state stable* (ISS). It is known that ISS defined this way is equivalent to the existence of an ISS Lyapunov function. Here we give the definition of a LISS Lyapunov function:

Definition 2.2. *A smooth function $V : \mathcal{D} \rightarrow \mathbb{R}_+$, with $\mathcal{D} \subset \mathbb{R}^N$ open, is a LISS Lyapunov function of (2) if there exist $\rho^0 > 0$, $\rho^u > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}_\infty$, and a positive definite function α such that $B(0, \rho^0) \subset \mathcal{D}$ and*

$$\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|), \quad \forall x \in \mathbb{R}^N, \quad (3)$$

$$V(x) \geq \gamma(\|u\|) \implies \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)), \quad (4)$$

for all $\|x\| \leq \rho^0$, $\|u\| \leq \rho^u$. The function γ is called LISS Lyapunov gain. If $\rho^0 = \rho^u = \infty$ then V is called an ISS Lyapunov function.

A related and strictly weaker stability concept (just think of the skaler system $\dot{x} = 0$) is that of local stability:

Definition 2.3. *System (2) is locally stable (LS), if there exist $\rho^0 > 0$, $\rho^u > 0$, $\sigma, \gamma \in \mathcal{K}_\infty$, such that for all $\|\xi\| \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$*

$$\|x(\cdot, \xi, u)\|_{L_\infty} \leq \sigma(\|\xi\|) + \gamma(\|u\|_{L_\infty}). \quad (\text{LS})$$

Also related is the concept of asymptotic gains.

Definition 2.4. *System (2) has the local asymptotic gain property (LAG), if there exist $\rho^0 > 0$, $\rho^u > 0$, $\gamma \in \mathcal{K}_\infty$, such that for all $\|\xi\| \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$*

$$\limsup_{t \rightarrow \infty} \|x(t, \xi, u)\| \leq \gamma(\|u\|_{L_\infty}). \quad (\text{LAG})$$

Note that inequality (LAG) is equivalent to

$$\limsup_{t \rightarrow \infty} \|x(t, \xi, u)\| \leq \gamma(\text{ess. } \limsup_{t \rightarrow \infty} \|u\|). \quad (\text{LAG}')$$

2.3. Monotone aggregation functions (MAFs)

The concept of monotone aggregation functions has been introduced in [19] and has subsequently been used in [6, 7, 20]. It is useful to cover different formulations for the aggregation of multiple inputs in a unified way. Examples of MAFs include all monotone norms on \mathbb{R}_+^n (which includes all p -norms).

Definition 2.5 (Monotone aggregation functions). *A function $\mu : \mathbb{R}_+^n \rightarrow \mathbb{R}_+$ is called a monotone aggregation function (MAF_n) if μ is continuous and*

- (M1) *nonnegative: $\mu(s) \geq 0$ for all $s \in \mathbb{R}_+^n$;*
- (M2) *strictly increasing: if $x \ll y$ then $\mu(x) < \mu(y)$.*

By $\mu \in \text{MAF}_n^m$ we denote vector monotone aggregation functions, i.e., $\mu_i \in \text{MAF}_n$ for $i = 1, \dots, m$, and if $m = n$ we simply write MAF^n instead of MAF_n^n .

A direct consequence of (M2) and continuity is that also the weaker monotonicity property $x \leq y \implies \mu(x) \leq \mu(y)$ holds for MAFs.

Further assumptions made for global results in [7, 20] include the properties

- (M3) *unboundedness: if $\|x\| \rightarrow \infty$ then $\mu(x) \rightarrow \infty$;*
- (M4) *sub-additivity: $\mu(x + y) \leq \mu(x) + \mu(y)$.*

Note that while we will require (M4) for a strict subset of our main results, none of the results derived in this paper assumes (M3). Standard examples satisfying (M1)–(M4) are summation and maximization, which we write as

$$\Sigma : (x_1, \dots, x_n)^T \mapsto \sum_{i=1}^n x_i \quad (5)$$

and

$$\oplus : (x_1, \dots, x_n)^T \mapsto \max_{i=1, \dots, n} x_i. \quad (6)$$

The induced vector MAFs will be denoted by the same symbols.

2.4. LISS for multiple inputs and gain matrices

The stability notions can be extended to the case of several inputs to one subsystem as in (1), where x_j , for $j \neq i$, is regarded as an independent input to the i th subsystem. This is possible for both, the trajectory formulation as well as the Lyapunov formulation. Here we settle some unifying notation.

We call the i th subsystem LISS, provided there exist $\rho_i^0 > 0$, $\rho_i^j > 0$, $\rho_i^u > 0$ and functions $\gamma_{ij}, \gamma_{iu} \in (\mathcal{K}_\infty \cup \{0\})$, $\beta_i \in \mathcal{KL}$, and a monotone aggregation function $\mu_i \in \text{MAF}_{n+1}$, such that for all $\xi_i \in \mathbb{R}^{N_i}$ such that $\|\xi_i\| \leq \rho_i^0$, for all $x_j \in L_\infty(\mathbb{R}_+; \mathbb{R}^{N_j})$ such that $\|x_j\|_{L_\infty} \leq \rho_i^j$ (where $j \neq i$), and for all $u_i \in L_\infty(\mathbb{R}_+; \mathbb{R}^{M_i})$ such that $\|u_i\|_{L_\infty} \leq \rho_i^u$, the following estimate holds for all $t \geq 0$:

$$\|x_i(t; \xi_i, x_j : j \neq i, u_i)\| \leq \beta_i(\|\xi_i\|, t) + \mu_i(\gamma_{i1}(\|x_1\|_{L_\infty([0,t])}), \dots, \gamma_{in}(\|x_n\|_{L_\infty([0,t])}), \gamma_{iu}(\|u_i\|_{L_\infty([0,t])})). \quad (7)$$

Similarly, for the Lyapunov formulation of LISS we have in the case of several inputs the following extension of Definition 2.2: A smooth function $V_i : \mathcal{D}_i \rightarrow \mathbb{R}_+$, $\mathcal{D}_i \subset \mathbb{R}^{N_i}$ open, such that for some $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$,

$$\psi_{i1}(\|x_i\|) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|), \quad \forall x_i \in \mathcal{D}_i, \quad (8)$$

is a LISS Lyapunov function for subsystem (1) if there exist $\rho_i^0 > 0$, $\rho_i^j > 0$, $\rho_i^u > 0$, functions $\gamma_{ij}, \gamma_{iu} \in \mathcal{K}_\infty \cup \{0\}$, a positive definite function α_i , and a monotone aggregation function $\mu_i \in \text{MAF}_{n+1}$, such that $B(0, \rho_i^0) \subset \mathcal{D}_i$ and for all $x_i \in \mathbb{R}^{N_i}$ with $\|x_i\| < \rho_i^0$ and inputs satisfying $\|x_j\| < \rho_i^j$ for $j \neq i$ and $u_i \in \mathbb{R}^{M_i}$, $\|u_i\| < \rho_i^u$, the following implication holds:

$$\begin{aligned} V_i(x) &\geq \mu_i(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{in}(V_n(x_n)), \gamma_{iu}(\|u_i\|)) \\ &\implies \nabla V_i(x_i) \cdot f_i(x, u_i) \leq -\alpha_i(V_i(x_i)). \end{aligned} \quad (9)$$

If all n subsystems in (1) are LISS, we can collect the gains in a matrix

$$\Gamma = (\gamma_{ij})_{i,j=1}^n, \quad \text{with } \gamma_{ij} \in \mathcal{K}_\infty \cup \{0\} \quad (10)$$

where we use the convention that $\gamma_{ii} \equiv 0$ for $i = 1, \dots, n$. Similarly, we collect the external gains γ_{iu} in a column vector $\Gamma^e(s) = (\gamma_{1u}(s_1), \dots, \gamma_{nu}(s_n))^T$.

The matrix Γ is called *gain matrix* of the interconnection (1). Note that $\gamma_{ij} \equiv 0$ means that there is no input from system j to system i , i.e., f_i does not depend on x_j .

The gain matrix Γ together with $\mu = (\mu_1, \dots, \mu_n)^T$ defines monotone operators, denoted by the symbols $\Gamma_\mu : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ and $\bar{\Gamma}_\mu : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+^n$, given by

$$\Gamma_\mu(s)_i := \mu_i(\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n), 0) \quad (11)$$

for $s \in \mathbb{R}_+^n$ and

$$\bar{\Gamma}_\mu(s)_i := \mu_i(\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n), \gamma_{iu}(s_{n+1})) \quad (12)$$

for $s \in \mathbb{R}_+^{n+1}$.

Throughout the paper we make the following assumption to rule out pathological cases that might otherwise occur when we use this notation:

Assumption 2.6 (Compatibility assumption). *Given $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$ and $\mu \in \text{MAF}^n$, we will from now on assume that Γ and μ are compatible in the following sense: For each $i = 1, \dots, n$, let I_i denote the set of indices corresponding to the nonzero entries in the i th row of Γ . Then it is understood that also the restriction of μ_i to the indices I_i satisfies (M2), i.e., if $x|_{I_i} \ll y|_{I_i}$ then $\mu_i(x|_{I_i}) < \mu_i(y|_{I_i})$.*

All standard examples of MAFs, i.e., monotone norms including Σ and \oplus satisfy this assumption.

2.5. A vector formulation for trajectory LISS estimates

For the trajectory formulation (7), we introduce a shorthand vector notation building upon gain operators:

We abbreviate $\|x_i(t)\|$ by $s_i(t)$, $\|x_i\|_{L_\infty([0,t])}$ by $s_{i,[0,t]}$ and form corresponding vectors $s(t) = (s_1(t), \dots, s_n(t))^T$ and $s_{[0,t]} = (s_{1,[0,t]}, \dots, s_{n,[0,t]})^T$. Analogously, by $e_{i,[0,t]}$ we refer to $\|u_i\|_{L_\infty([0,t])}$ and by $e_{[0,t]}$ to the corresponding vector.

For $v \in \mathbb{R}_+^n$ and $t \in \mathbb{R}_+$ let us write

$$B(v, t) = (\beta_1(v_1, t), \dots, \beta_n(v_n, t))^T.$$

To be able to give estimates for the stability regions, we define the vector notation $\rho^i = (\rho_1^i, \dots, \rho_n^i)^T$, for $i = 0, \dots, n$, and $\rho^e = (\rho_1^e, \dots, \rho_n^e)^T$. We also define $\rho^x := \min_{i=1, \dots, n} \rho^i$. Using this newly defined notation, the LISS estimates (7) for $i = 1, \dots, n$ can be written in vectorized form as follows:

Subsystems (1) are LISS for $i = 1, \dots, n$ if there exist vectors $\rho^0, \rho^x, \rho^e \in \mathbb{R}_+^n$, $\rho^0, \rho^x, \rho^e \gg 0$, such that for all $s(0) \ll \rho^0$, $t \geq 0$, and the corresponding solutions and inputs to (1) satisfying $s_{[0,t]} \ll \rho^x$ and $e_{[0,t]} \ll \rho^e$, the following estimate holds:

$$s(t) \leq B(s(0), t) + \mu([\Gamma(s_{[0,t]}), \Gamma^e(e_{[0,t]})]). \quad (13)$$

If μ satisfies M4, then estimate (13) implies

$$s(t) \leq B(s(0), t) + \Gamma_\mu(s_{[0,t]}) + \Gamma^e(e_{[0,t]}), \quad (14)$$

We observe that $s(0) \leq s_{[0,t]}$ for all $t \geq 0$ and hence without loss of generality we may assume that $\rho^0 \leq \rho^x$.

Also note that in general $\|x\|_{L_\infty([0,T])} \neq \|s_{[0,T]}\|$, e.g., for $x(t) = (\cos t, \sin t)^T$ with $s_1(t) = |\cos t|$ and $s_2(t) = |\sin t|$. Here we find $\|x\|_{L_\infty([0,2\pi])} = 1$, while $\|s_{[0,2\pi]}\| = \sqrt{2}$. For the Euclidean norm we have the following estimate:

Lemma 2.7. *For the above defined notation in general it holds that*

$$\|x\|_{L_\infty([0,T])} \leq \|s_{[0,T]}\| \leq \sqrt{n} \|x\|_{L_\infty([0,T])}.$$

3. Known global results

Before we answer the main questions of the paper let us recall the small-gain condition related to the global ISS property.

The global small-gain condition assuring the ISS property for an interconnection of $n \geq 2$ ISS systems has first been derived in [3]. An alternative proof has been given in [5]. We quote the following result from these papers.

Theorem 3.1. *Consider system (2) and suppose that each subsystem (1) is ISS, i.e., condition (7) holds for all $\xi_i \in \mathbb{R}_+^n$, $u_i \in L_\infty$, $i = 1, \dots, n$. Let Γ be given by (10) and let the monotone aggregation be Σ . If there exists an $\alpha \in \mathcal{K}_\infty$, such that*

$$(\Gamma_\Sigma \circ D)(s) \not\geq s, \quad \forall s \in \mathbb{R}_+^n \setminus \{0\}, \quad (15)$$

with $D = \text{diag}_n(\text{id} + \alpha)$ then the system (2) is ISS from u to x .

A version of this result for general μ satisfying M4 follows along the lines of the same proofs using the result [20, Theorem 6.1].

Furthermore it is known that under the same small-gain condition an ISS Lyapunov function for (2) can be explicitly constructed as a combination of the ISS Lyapunov functions of subsystems, see [7, Corollary 5.5]:

Theorem 3.2. *Consider the interconnected systems (1), where each of the subsystems Σ_i is assumed to have an ISS Lyapunov function V_i and the corresponding gain matrix is given by (10). Assume that each $\mu_i \in \text{MAF}_{n+1}$ satisfies (M3) and is additive in the last argument, i.e.,*

$$\mu_i(s, r) = \mu_i(s, 0) + r, \quad \text{for all } s \in \mathbb{R}_+^n, r \in \mathbb{R}_+. \quad (16)$$

If Γ_μ is irreducible and if there exists an $\alpha \in \mathcal{K}_\infty$ such that for $D = \text{diag}(\text{id} + \alpha)$ the gain operator Γ_μ satisfies the condition

$$D \circ \Gamma_\mu(s) \not\leq s \quad (17)$$

then the interconnected system is ISS and there exists a vector valued function $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+^n$ satisfying $(\Gamma \circ D)(\sigma(r)) \ll \sigma(r)$ for all $r > 0$, such that each component function σ_i is of class \mathcal{K}_∞ and piecewise linear on $(0, \infty)$. Moreover,

$$V(x) = \max_i \sigma_i^{-1}(V_i(x_i))$$

is a nonsmooth ISS Lyapunov function for the system (2).

Note that $V(x)$ in this case is not smooth but only Lipschitz continuous. In [7] it has been pointed out that a Lipschitz continuous ISS Lyapunov function is sufficient to deduce input-to-state stability. The argument is the same for LISS Lyapunov functions.

A local version of the function σ can be constructed explicitly as we will show below.

4. Local small gain condition

Motivated by the global small-gain conditions we introduce its local counterpart as follows. We say that Γ_μ satisfies the *local small-gain condition* on $[0, w^*]$, provided that

$$\Gamma_\mu(w^*) \ll w^* \text{ and } \Gamma_\mu(s) \not\leq s, \quad \forall s \in [0, w^*], s \neq 0. \quad (\text{LSGC})$$

In this paper we give local results similar to Theorem 3.1 and 3.2. The following lemmas will be used to obtain the main result. We start with a simple criterion that guarantees (LSGC).

Lemma 4.1. *Let Γ be a gain matrix as in (10), $\mu \in \text{MAF}^n$, and let $w^* \in \mathbb{R}_+^n$ satisfy $\Gamma_\mu(w^*) \ll w^*$. Consider the trajectory $\{w(k)\}$ of the discrete monotone system $w(k+1) = \Gamma_\mu(w(k))$, $k = 0, 1, 2, \dots$ with $w(0) = w^*$. Then $w(k) \rightarrow 0$ for $k \rightarrow \infty$ if and only if Γ_μ satisfies the small-gain condition (LSGC) on $[0, w^*]$.*

Remark 4.2. *This lemma gives an idea of how to check the small-gain condition (LSGC). First one looks for $w^* \in \mathbb{R}^n$ with $\Gamma_\mu(w^*) \ll w^*$. Then instead of checking $\Gamma_\mu(s) \not\leq s$ for all $s \in [0, w^*] \setminus \{0\}$ one needs to check whether the sequence $w(k)$ converges to the origin — and the latter is quite an easy task. For the first task, i.e., finding a suitable w^* , there exist numerical algorithms which can be efficiently implemented. See, e.g., [19, Chapter 4] and [18].*

Proof. To prove sufficiency let $w(k) \rightarrow 0$ for $k \rightarrow \infty$ and suppose there exists a point $v \in [0, w^*]$ with

$$\Gamma_\mu(v) \geq v \quad (18)$$

and $v \neq 0$. Since Γ_μ is monotone, so is Γ_μ^k , i.e., the k -times application of Γ_μ . Hence (18) implies $\Gamma_\mu^k(v) \geq v \geq 0$, so $\Gamma_\mu^k(v)$ does not tend to zero as k approaches infinity. But $v \leq w^*$ implies $\Gamma_\mu^k(v) \leq \Gamma_\mu^k(w^*) = w(k)$, which is assumed to tend to zero. This contradiction implies that $\Gamma_\mu(v) \not\geq v$ for all $v \in [0, w^*]$ and sufficiency is proved.

Now assume that Γ_μ satisfies (LSGC) on $[0, w^*]$ and consider the sequence $\{w(k)\}$, $k = 0, 1, \dots$ defined by $w(k+1) = \Gamma_\mu(w(k))$. By the monotonicity of Γ_μ and $\Gamma_\mu(w^*) \ll w^*$ this sequence is bounded in \mathbb{R}^n and hence it contains a convergent subsequence that converges to some $v \in \mathbb{R}^n$. Then by the continuity of Γ_μ we have $\Gamma_\mu(v) = v$ contradicting (LSGC). This proves the necessity. \square

For the case that Γ has no zero rows and Γ_μ satisfies (LSGC), the result [20, Proposition 5.2] shows that a linear interpolation of the sequence $\{\Gamma_\mu^k(w^*)\}_{k \geq 0}$ gives a path, called an Ω -path, satisfying

$$\sigma : [0, 1] \rightarrow [0, w^*], \quad \Gamma_\mu(\sigma(r)) \ll \sigma(r), \quad \text{for } r \in (0, 1], \quad (19)$$

and that each component function σ_i is strictly increasing. Furthermore, σ is piecewise linear and satisfies $\sigma(0) = 0, \sigma(1) = w^*$. An extension is the following result.

Proposition 4.3. *Let $\Gamma_\mu : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ be given by (11). Assume that Γ has no zero row and satisfies (LSGC). Then there exists a path $\sigma : [0, 1] \rightarrow [0, w^*]$ and a function $\varphi \in \mathcal{K}_\infty$ such that*

$$\bar{\Gamma}_\mu(\sigma(r), \varphi(r)) \ll \sigma(r), \quad \text{for all } r \in (0, 1]. \quad (20)$$

Proof. Let σ be given by [20, Proposition 5.2], i.e., as a linear interpolation of the sequence points

$$\{\Gamma_\mu^k(w^*)\}_{k \geq 0}, \quad (21)$$

so that σ satisfies (19). For each $\rho \in (0, 1]$ let $\phi_\rho = \sup\{s > 0 : \bar{\Gamma}_\mu(\sigma(r), s) \ll \sigma(r)\} \in [0, \infty]$. By monotonicity and continuity of $\bar{\Gamma}_\mu$ we have $\phi_\rho > 0$ for all $r \in (0, 1]$, so we may take φ to be any \mathcal{K}_∞ function satisfying $\varphi(r) < \phi_\rho$ for all $r \in (0, 1]$. \square

The following result is a local version of the main ingredient which has been used in [5] to prove the global ISS small-gain theorem.

Proposition 4.4. *Let $w^* \in \mathbb{R}_+^n$, $w^* \gg 0$. Let Γ_μ given by (11) satisfy (LSGC) on $[0, w^*]$. Assume Γ has no zero row. Assume μ satisfies M4. Then for each $w' \in (\Gamma_\mu(w^*), w^*)$ there exists a function $\varphi \in \mathcal{K}_\infty$, such that for all $w \in \mathbb{R}_+^n$, $0 \leq w \leq w'$, and all $v \in \mathbb{R}_+^n$ we have*

$$(\text{id} - \Gamma_\mu)(w) \leq v \implies w \leq \text{diag}(\varphi)(v). \quad (22)$$

Proof. This proof essentially goes along the lines of the proof of [5, Lemma 13] (which requires use of (M4)), once it is established that there exists an operator $D = \text{diag}(\text{id} + \kappa)$, $\kappa \in \mathcal{K}_\infty$, such that $D \circ \Gamma_\mu(s) \not\leq s$ for all $s \neq 0$, $s \in [0, w']$. The existence of this operator D can be guaranteed using essentially the same technical construction as in [20, Proposition 5.8]. Combining all these ingredients, one obtains the estimate $w \leq \min\{w', \text{diag}(\text{id} + \kappa^{-1})^n(v)\}$. From here we can conclude $w_i < (\text{id} + \kappa^{-1})^n(v_i)$ if v_i is small. So we may take $\varphi(r) = (\text{id} + \kappa^{-1})^n(r)$. \square

Remark 4.5. *Note that “large” φ can lead to strong restrictions on initial conditions and inputs, see Remark 6.4. However in some cases this kind of conservativeness is inevitable.*

Consider Γ given as the matrix

$$\begin{bmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{bmatrix}$$

where $\gamma_{12}, \gamma_{21} \in \mathcal{K}_\infty$ satisfy $\gamma_{12} \circ \gamma_{21}(r) < r$ for $0 < r \leq r^$. Let μ be either \oplus or Σ . It follows that the origin is locally attractive with respect to the dynamics $s(k+1) = \Gamma_\mu(s(k))$ with the point $(r^*, r^*)^T$ and due to monotonicity also the order interval $[0, (r^*, r^*)^T]$ belonging to the region of attraction. By [20, Proposition 4.1] for all $s \neq 0$ in the region of attraction $\Gamma_\mu(s) \not\leq s$ and by [20, Theorem 3.3] there exists a point $w^* \gg 0$ such that*

$$\Gamma_\mu(w^*) \ll w^* \text{ and } \Gamma_\mu(s) \not\leq s \text{ for all } s \in [0, w^*], s \neq 0.$$

For $w \leq w^$ we have $\Gamma_\mu(w) \leq w^*$ and hence both $w - \Gamma_\mu(w) \leq w^*$ as well as $\Gamma_\mu(w) - w \leq w^*$. It follows that $0 \leq |w - \Gamma_\mu(w)| \leq w^*$. So without loss of generality, we may assume $v \leq w^*$.*

Figure 1 depicts the current situation for two different points w_i within the order interval $[0, w^]$. Due to the geometric properties of Γ_μ , although $w_2 \ll w_1$ holds, the inequality is reversed for the distances $\|w_i - \Gamma_\mu(w_i)\|$, depicted by the black line segments.*

5. Advantages of LISS vs. linearization of systems

LISS is a local stability concept. A much more established local stability concept arises by linearizing the system of interest around some equilibrium point. Linearization translates the question of stability into a question about the

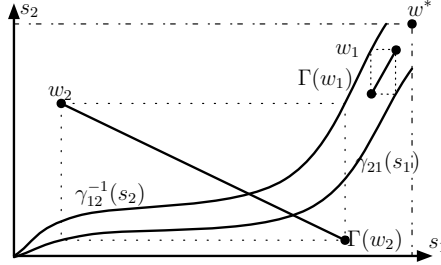


Figure 1: Geometric interpretation of the estimates in Proposition 4.4.

locations of eigenvalues of the system matrix of the linearized system. Stability results for large-scale interconnections of linear systems have been obtain at least since the 1970s.

This section exemplifies by means of an example, what the possible benefit of considering LISS systems over linearized ones can be: In addition to tools to assess interconnection stability, one also obtains knowledge about basins of attraction and the allowable magnitude of disturbances. To this end consider a simple first order nonlinear system given by

$$\dot{x} = -x + x^2 + u. \quad (23)$$

Its linearization is given by

$$\dot{x} = -x + u, \quad (24)$$

and by looking at (24) we readily observe that the origin must be asymptotically stable in the sense of Lyapunov with respect to (23) (or (24) for that matter), if the input is taken to be $u \equiv 0$. Unfortunately in this case, the linearization does not tell us anything about the basin of attraction, especially if u is allowed to live in some small compact set.

Take $V(x) = |x|$ as a Lyapunov function candidate. For system (24) we have $\dot{V} = \nabla V(x) \cdot (-x + u) \leq -|x| + u < 0$ if $V(x) = |x| > |u|$. So system (24) is ISS with gain³ $\gamma = \text{id}$.

Now consider the nonlinear system (23). It is not ISS, since $x^0 > 1$ and $u \equiv 0$ produces unbounded trajectories. But it may well be LISS. Indeed, taking the very same Lyapunov function candidate, we have

$$\nabla V(x) \cdot (-x + x^2 + u) \leq -|x| + x^2 + |u| < 0$$

if $|x| - x^2 > |u|$. The function $\kappa(s) = s - s^2$ is strictly increasing on the interval $[0, 1/2]$ with $\kappa(0) = 0$ and $\kappa(1/2) = 1/4$. Its inverse on $[0, 1/4]$ is given by

³For simplicity we only assume $\dot{V} < 0$, which is not in accordance with Definition 2.2. However, the gains γ that we compute in this subsection can be approximated arbitrary closely by gains $\tilde{\gamma} < \gamma$ in the sense of Definition 2.2, leading to estimates of the form $\dot{V} < -\alpha(V)$, α positive definite.

$\kappa^{-1}(r) = 1/2 - \sqrt{1/2 - r}$. So for $x \in \mathbb{R}$ such that $V(x) = |x| < 1/2 =: \rho^0$ we have the implication

$$V(x) > \kappa^{-1}(|u|) \implies \dot{V} < 0.$$

So as gain we may take $\gamma = \kappa^{-1}$ on $[0, 1/4]$ and extended it to some \mathcal{K}_∞ function. This restricts our admissible inputs $u \in \mathbb{R}$ to $|u| < \rho^u = 1/4$. Now we have a LISS estimate with LISS Lyapunov function V and LISS gain γ . Notably, we have precise information about the set where the LISS implication holds. We will see in the next section that such information for the subsystems can be aggregated for the composite system by the stability theorems.

6. Main results

In the first two of the following subsections we assume that each subsystem (1) is LISS and show that the local small-gain condition (LSGC) is sufficient to imply LISS of the composite system (2). Furthermore, we show how a LISS Lyapunov function can be constructed. Notably, in both cases estimates for the stability region are provided.

By linearizing the gain operator, the sufficient conditions for interconnection stability can be simplified at the expense of the estimates on the stability regions that one would otherwise obtain, see Section 6.3.

6.1. A local small-gain theorem

In this subsection we state a small-gain theorem for large-scale interconnected systems comprised of LISS subsystems in the spirit of [3], [5].

Theorem 6.1. *Let all subsystems (1), $i = 1, \dots, n$, satisfy (7). Suppose Γ_μ satisfies (LSGC) and Γ has no zero rows. Then there exist $\rho^0 > 0$, $\rho^u > 0$, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}_\infty$, such that system (2) satisfies (LISS).*

Remark 6.2. *Observe that the small-gain condition (LSGC) required in Theorem 6.1 appears substantially weaker than the corresponding condition (15) in global result of Theorem 3.1. Recall that the robustness term D is essential and cannot be omitted in (15), see Example 18 in [5]. It turns out that (LSGC) is equivalent to the same condition stated with $D \circ \Gamma_\mu$ in place of Γ_μ . This follows from the following observation. Since the set $\Omega = \{s \in \mathbb{R}_+^n : \Gamma_\mu(s) \ll s\}$ is open, the condition (LSGC) would also hold with w^* replaced by $w^* + \varepsilon$, where $\varepsilon \in \mathbb{R}_+^n$, $\varepsilon \gg 0$, is sufficiently small. By Lemma 4.1 we have some form of robustness, since $\Gamma_\mu^k(w^* + \varepsilon) \rightarrow 0$ as $k \rightarrow \infty$.*

For Proposition 4.4 instead of w^ a smaller point w' is used, which effectively amounts to having a local robustness operator D , and it makes the result more explicit. In fact, it is even possible to locally construct such a robustness operator by following the lines of the proof of [20, Proposition 5.8]. The structural assumptions made for this global result in [20] are not necessary for the local result.*

The proof divides into the following steps: First we establish that system (2) satisfies (LS) and (LAG), then we construct a \mathcal{KL} -function β for the (LISS) estimate.

Proof of Theorem 6.1. For brevity we use the notation of Section 2.5, i.e., we assume that (13),(14) hold. Throughout the proof let $w' \in (\Gamma_\mu(w^*), w^*)$ be fixed and φ be given by Proposition 4.4. Denote $F := \text{diag}(\varphi)$. Let $\varepsilon := \min\{\rho^x, \rho^0, w'\} \in \mathbb{R}_+^n$ and $\delta := F^{-1}(\frac{1}{2}\varepsilon) \in \mathbb{R}_+^n$.

Step 1 — Existence of solutions and local stability. Let $u \in L_\infty(\mathbb{R}_+; \mathbb{R}^M)$ such that $e_{[0,\infty]} = e_{[0,\infty]}(u)$ satisfies $e_{[0,\infty]} \ll \rho^e$ and $\Gamma^e(e_{[0,\infty]}) \ll \frac{1}{2}\delta$. Consider initial states $x(0) = \xi \in \mathbb{R}^N$ such that $s(0) = s(0; \xi, u) \in \mathbb{R}^n$ satisfies $B(s(0), 0) \ll \frac{1}{2}\delta$ and $s(0) \ll \varepsilon$.

Define $T^* := \inf\{t \geq 0 : s(t) = s(t; \xi, u) \not\leq \varepsilon\}$. Clearly $s_{[0,T^*]} \leq \varepsilon$ and hence $s_{[0,T^*]} \leq w'$. So we may apply Proposition 4.4 to the following inequality:

$$(\text{id} - \Gamma_\mu)(s_{[0,T^*]}) \leq B(s(0), 0) + \Gamma^e(e_{[0,\infty]}).$$

Hence

$$s_{[0,T^*]} \leq F(B(s(0), 0) + \Gamma^e(e_{[0,\infty]})) \ll F(\delta/2 + \delta/2) = \frac{1}{2}\varepsilon.$$

This implies that there is no finite minimal time T^* , such that the component-wise norm of the trajectory $x(\cdot, \xi, u)$ leaves the ε -neighborhood around the origin. Hence this trajectory stays in that open set for all times.

Now let $\rho^0 := \sup\{\|s\| : s \in \mathbb{R}_+^n, s \leq \varepsilon, B(s, 0) \ll \frac{1}{2}\delta\}$ and choose $0 < \rho^u < \sup\{\|e\| : e \in \mathbb{R}_+^n, e \leq \rho^e, \Gamma^e(e) \leq \frac{1}{2}\delta\}$. Then it follows that for $\|\xi\| < \rho^0$ and $\|u\|_{L_\infty} \leq \rho^u$ the solution $x(\cdot; \xi, u)$ exists for all times and is bounded in norm by $\frac{1}{2}\|\varepsilon\|$. In fact, we have (using Lemma 2.7)

$$\|x(\cdot; \xi, u)\|_{L_\infty} \leq \|F(B(s(0), 0) + \Gamma^e(e_{[0,\infty]}))\|$$

and by the weak triangle inequality [12] and the norm triangle inequality

$$\leq \|F(2B(s(0), 0))\| + \|F(2\Gamma^e(e_{[0,\infty]}))\|$$

and hence

$$\leq \sigma(\|\xi\|) + \gamma(\|u\|_{L_\infty})$$

for some $\sigma, \gamma \in \mathcal{K}_\infty$. This establishes (LS).

Step 2 — Establishing the local asymptotic gain property. An estimate of the form (LAG') can be established using essentially the same steps as in the proof of [5, Theorem 9].

Step 3 — Constructing the \mathcal{KL} -estimate. To this end let

$$\tilde{\beta}(r, t) := \sup_{\|u\|_{L_\infty} \leq \rho^u, \|\xi\| \leq r} (\|x(t, \xi, u)\| - \gamma(\|u\|_{L_\infty}))^+,$$

where a^+ denotes $\max\{a, 0\}$. By compactness of the set where the supremum is taken, the supremum is attained, finite, and for $t \rightarrow \infty$ the function $\tilde{\beta}$ tends to zero. It is clearly increasing in r and continuous and hence can be bounded above by a function β of class \mathcal{KL} . With this β and γ from Step 2 we obtain the LISS estimate for (2). This completes the proof. \square

Remark 6.3 (An alternative to the \mathcal{KL} -estimate). *Instead of constructing the \mathcal{KL} -estimate in the last step of the preceding proof we could have argued that the origin is locally asymptotically stable with respect to the composite, externally unforced system $\dot{x} = f(x, 0)$. Following the lines of [24, Lemma I.2] (a result showing that 0-GAS implies LISS) and thereby using [15, Theorem 14*] for a suitable converse Lyapunov result, local input-to-state stability could also be shown this way. See also [2, Theorem 1] for a similar result for hybrid systems.*

Remark 6.4 (Stability regions). *In the proof we have obtained*

$$\rho^0 := \sup \left\{ \|s\| : s \in \mathbb{R}_+^n, s \leq \varepsilon, B(s, 0) \ll \frac{1}{2} \delta \right\}$$

and, essentially,

$$\rho^u = \sup \left\{ \|e\| : e \in \mathbb{R}_+^n, e \leq \rho^e, \Gamma^e(e) \leq \frac{1}{2} \delta \right\},$$

where $\varepsilon = \min\{\rho^x, \rho^0, w'\} \in \mathbb{R}_+^n$ and $\delta = F^{-1}(\frac{1}{2}\varepsilon) \in \mathbb{R}_+^n$. Here w' had to be chosen in the open order interval $(\Gamma_\mu(w^*), w^*)$ and $F = \text{diag}(\varphi)$, with φ given by Proposition 4.4.

6.2. A small-gain theorem using LISS Lyapunov functions

In this section we assume that each subsystem i of (1) is LISS and admits a LISS Lyapunov function V_i with the corresponding gains γ_{ij} , γ_{iu} , MAFs μ_i and corresponding Γ_μ so that the implication (9) holds for all $\|x_i\| < \rho_i^0$ and $x \in \mathbb{R}^N$ with $\|x_j\| < \rho_j^j$ for $j \neq i$ and $u_i \in \mathbb{R}^{M_i}$ with $\|u_i\| < \rho_i^u$. We are looking for explicit expressions for the restrictions on the states x and inputs u such that the overall system is LISS.

First of all let us say that vectors $x = (x_1^T, \dots, x_n^T)^T \in \mathbb{R}^N$ with $\|x_i\| \geq \rho_i^0$ or $\|x_i\| \geq \rho_i^j$ for at least one i are out of interest, because the implication (9) is not available in this case. Similarly any x with $\|x_i\| > w_i^*$ is out of interest since then the small-gain condition does not apply. Hence a necessary restriction on states is already given in terms of ρ_i^0 , w^* and $\rho^x := (\min_j \rho_1^j, \dots, \min_j \rho_n^j)^T$. In some cases these restrictions are already enough, see Corollary 6.6, for the LISS property of the interconnection provided the local small-gain condition is satisfied. In general we have the following

Theorem 6.5. *Assume that each system i of the interconnection (1) is LISS and that it admits a LISS Lyapunov function V_i satisfying (9) for all $\|x_i\| < \rho_i^0$, $\|x_j\| < \rho_j^j$, and $\|u_i\| < \rho_i^u$. Let Γ and μ be given by (9) and assume Γ has*

no zero rows. Assume Γ_μ satisfies (LSGC). Then the composite system (2) is LISS. Moreover, a nonsmooth LISS Lyapunov function is given by

$$V(x) = \max_i \sigma_i^{-1}(V_i(x_i)) \quad (25)$$

where σ is given by Proposition 4.3. Moreover, with this Lyapunov function V implication (4) holds for all $x \in B_{\mathbb{R}^{N_1}}(0, \tilde{\rho}_1) \times \dots \times B_{\mathbb{R}^{N_n}}(0, \tilde{\rho}_n)$, with $\tilde{\rho}_i := \min\{\rho_i^x, \rho_i^0, w_i^*, \psi_{12}^{-1}(w_i^*)\}$ and $u \in B_{\mathbb{R}^{M_1}}(0, \rho_1^u) \times \dots \times B_{\mathbb{R}^{M_n}}(0, \rho_n^u)$.

Proof. By Proposition 4.3 the local small-gain condition implies the existence of strictly increasing functions $\sigma_i : [0, 1] \rightarrow [0, w_i^*]$, such that $\sigma = (\sigma_1, \dots, \sigma_n)^T$ satisfies (20). Note that $\sigma_i^{-1} : [0, w_i^*] \rightarrow [0, 1]$ is well defined and such that for any compact set $K \subset (0, \infty)$ there exist $c, C > 0$ such that $c < (\sigma_i^{-1})' < C$. We define $V(x)$ by (25). To assure that $\sigma_i^{-1}(V_i(x_i))$ is well defined we have required $\|x_i\| < \psi_{12}^{-1}(w_i^*)$ that implies $V_i(x_i) < w_i^*$ by (8).

The proof that $V(x)$ is a LISS Lyapunov function for the interconnection follows along the lines of the proof of Theorem 5.3 in [7]. \square

Corollary 6.6. *Consider interconnection (1) such that each system i has the same properties as in Theorem 6.5. If $V_i(x_i) < \|x_i\|$ holds for all i then the assertion of Theorem 6.5 holds for*

$$x \in B_{\mathbb{R}^{N_1}}(0, \min\{\rho_1^x, \rho_1^0, w_1^*\}) \times \dots \times B_{\mathbb{R}^{N_n}}(0, \min\{\rho_n^x, \rho_n^0, w_n^*\}).$$

and $u \in B_{\mathbb{R}^{M_1}}(0, \rho_1^u) \times \dots \times B_{\mathbb{R}^{M_n}}(0, \rho_n^u)$.

6.3. LISS small-gain theorems and linearization of gains

An important connection of the LISS approach to linear stability theory arises of course, when the subject of the stability condition happens to be linear, or, for that matter, can be linearized.

Assume given a gain operator $\Gamma_\mu : \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ we have its Jacobian matrix at zero, denoted by $J\Gamma_\mu(0) \in \mathbb{R}^{n \times n}$, at our disposal. Clearly the elements of $J\Gamma_\mu(0)$ have to be nonnegative, and the diagonal will have only zero entries. For any nonnegative matrix $G \in \mathbb{R}_+^{n \times n}$ let $\rho(G) = \max\{|\lambda|; \lambda \text{ is an eigenvalue of } G\}$ denote the spectral radius of G . Then the following statements are equivalent:

1. G satisfies $\rho(G) < 1$;
2. $G^k \rightarrow 0$ as $k \rightarrow \infty$;
3. the matrix $(I - G)^{-1}$ exists and is nonnegative;
4. there exists a positive vector $s \in \mathbb{R}_+^n$, $s \gg 0$, such that $Gs \ll s$.

For a proof see, e.g., [20, Lemma 1.1]. Notable is also the fact that

$$\rho(G) < 1 \iff \exists D = \text{diag}(1 + \varepsilon), \varepsilon > 0 : \rho(DG) < 1,$$

where D is a linear version of the robustness operator which appeared in (15) and (17).

So instead of using the nonlinear path in Proposition 4.3, a linear path σ is given by $\sigma(r) = sr \in \mathbb{R}_+^n$. From here the function φ can be constructed as before in Proposition 4.3. The composite Lyapunov function in Theorem 6.5 can then simply be taken as

$$V(x) = \max_i V_i(x_i)/s_i. \quad (26)$$

Similarly, for the trajectory result Theorem 6.1, the inverse $(I - J\Gamma_\mu(0))^{-1}$ can be used directly instead of estimate (22). To summarize, we obtain the following corollaries to Theorems 6.1 and 6.5, respectively.

Corollary 6.7. *Let all subsystems (1), $i = 1, \dots, n$, satisfy (7). Suppose Γ_μ is differentiable at $s = 0$ and assume Γ has no zero rows. If $\rho(J\Gamma_\mu(0)) < 1$ then system (2) is LISS.*

Corollary 6.8. *Assume that each system i of the interconnection (1) is LISS and that it admits a LISS Lyapunov function V_i satisfying (9) for all $\|x_i\| < \rho_i^0$, $\|x_j\| < \rho_i^j$, and $\|u_i\| < \rho_i^u$. Let Γ and μ be given by (9). Assume the Γ_μ is differentiable at zero and the Jacobian matrix $J\Gamma_\mu(0)$ satisfies $\rho(J\Gamma_\mu(0)) < 1$. Then the composite system (2) is LISS and a nonsmooth LISS Lyapunov function is given by (26).*

7. Example

The following example illustrates the use of MAFs and the application of the main result Theorem 6.5. Consider the following system of n coupled equations with subsystems given by

$$\dot{x}_i(t) = -(n+1)x_i + x_i^2 + \sum_{\substack{j=1 \\ j \neq i}}^n x_j + u_i, \quad x_i, u_i \in \mathbb{R}, \quad i = 1, \dots, n. \quad (27)$$

Consider the i th subsystem. Obviously, it is not 0-GAS, since for zero inputs and a large initial states $x_i(0)$ the trajectories are unbounded. However it can be shown that it is LISS with $\rho_i^0 = \frac{n}{2}$, $\rho_i^j = 1 - \varepsilon$ and $\rho_i^u = 1 - \varepsilon$ for arbitrary small $\varepsilon > 0$. To this end consider $V_i(x_i) := |x_i|$ as a LISS Lyapunov function candidate. We define $\gamma_{ij} := \text{id}$ for $i \neq j$, $\gamma_{iu} = \text{id}$ and $\mu_i : \mathbb{R}_+^{n+1} \rightarrow \mathbb{R}_+$ by

$$\mu_i(s_1, \dots, s_{n+1}) := \frac{1}{2} \left(n+1 + \varepsilon - \sqrt{(n+1 + \varepsilon)^2 - 4 \sum_{\substack{j=1 \\ j \neq i}}^{n+1} s_j} \right).$$

Note that μ_i satisfies M1 and M2 as well as the compatibility assumption for $s_i \leq 1 - \varepsilon$, $i = 1, \dots, n$ and $s_{n+1} \leq 1 - \varepsilon$. Now if

$$V_i(x_i) = |x_i| > \mu_i(\gamma_{i1}(|x_1|), \dots, \gamma_{in}(|x_n|), \gamma_{iu}(|u_i|))$$

it follows that

$$\dot{V}_i(x_i) < -(n+1)|x_i| + |x_i|^2 + \sum_{j=1}^n |x_j| + |u_i| < -\varepsilon(|x_i|).$$

This shows that each subsystem is LISS.

Let $x := (x_1, \dots, x_n)^T$ and $u := (u_1, \dots, u_n)^T$. Let $w^* := (1-\varepsilon, \dots, 1-\varepsilon)^T \in \mathbb{R}^n$. It is easy to check that $\Gamma_\mu(w^*) \ll w^*$ and that the local small-gain condition is satisfied for this w^* . Let $\sigma_i = (1-\varepsilon) \text{id}$ for $i = 1, \dots, n$ and $\sigma = (\sigma_1, \dots, \sigma_n)^T$ and $\varphi = (1-\varepsilon) \text{id}$. Choosing ε small enough it can be checked that for any $\tau \in (0, 1]$ it holds $\bar{\Gamma}_\mu(\sigma(\tau), \varphi(\tau)) \ll (\sigma(\tau))$. Hence by Corollary 6.6 we conclude that the interconnection is LISS with LISS Lyapunov function $V(x) := \max_i |x_i|$ and subject to the following restrictions on $x, u \in \mathbb{R}^n$: $|x_i| < 1-\varepsilon$ for $i = 1, \dots, n$ and $|u_i| < 1-\varepsilon$, i.e., $\rho^0 = \rho^u = 1-\varepsilon$ for some small positive number ε .

8. Conclusions

We have presented a new nonlinear tool for the stability assessment of nonlinear interconnected systems, which complements the existing linearization theory. Our approach allows to consider general interconnections of locally input to state stable systems, a class much larger than linear systems which also includes integral input-to-state stable systems. The results presented cover trajectory estimates as well as a Lyapunov version. Most notably, we also provide a constructive approach to aggregate Lyapunov functions of subsystems into a composite Lyapunov function. A nontrivial example illustrates how this method works. In contrast to the linear theory, the LISS approach readily provides estimates for the stability regions.

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