Burst Erasure Correcting LDPC Codes

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Abstract

In this paper low-density parity-check (LDPC) codes are designed for burst erasure channels. Firstly, lower bounds for the maximum length erasure burst that can always be corrected with message-passing decoding are derived as a function of the parity-check matrix properties. We then show how parity-check matrices for burst erasure correcting LDPC codes can be constructed using superposition, where the burst erasure correcting performance of the resulting codes is derived as a property of the stopping set size of the base matrices and the choice of permutation matrices for the superposition. This result is then used to design both single burst erasure correcting LDPC codes which are also resilient to the presence of random erasures in the received bits and LDPC codes which can correct multiple erasure bursts in the same codeword.

I. Introduction

A low-density parity-check (LDPC) code is a block code defined by a sparse parity-check matrix, H, and decoded iteratively with message-passing decoding [1]. LDPC codes are well known to provide excellent decoding performances on memoryless channels (see e.g. [2]), and recent interest has focussed on their performance on channels with memory [3], [4], [5], [6], [7], [8], [9], [10]. Such channels encompass many real-world communications systems including fading environments, packet based communications such as internet transmissions, and magnetic storage devices where the burst errors caused by thermal asperity and media defects are the dominant error type.

For memoryless channels, the simple binary erasure channel (BEC) has provided a useful framework to understand the performance of LDPC codes (see e.g. [11], [12]), and many of the observations made using the BEC can be usefully applied to more general memoryless channels. In this paper we similarly employ bursty binary erasure channels as a natural starting point for considering LDPC codes for channels with memory.

While our motivation in considering burst erasure channels to model channels with memory is their simplicity, burst erasure channels do occur in some important applications. Any system where the receiver

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is able to distinguish deep fades, for example by employing training sequences, can treat the fading period as an erasure burst. As well, packet losses in Internet transmissions can be modeled as erasure bursts, and forward error correction is becoming more attractive for these channels, particularly in real time or multicast applications were automatic repeat request schemes are less practical.

In this work we will use the classical burst erasure channel as defined in [13]. This channel has two states; the burst space, in which the channel output carries no information about the inputs, and the guard space, in which channel outputs are erasure free or are randomly corrupted with some small background erasure probability p [13]. We will consider both the classical burst erasure channel, with one erasure burst per codeword, and the multi-burst channel where multiple erasure bursts can occur in the one codeword.

Several previous authors have considered the use of LDPC codes in single burst erasure channels. Yang and Ryan, [3], [4], have developed an efficient exhaustive search algorithm for finding the longest string of erasures guaranteed to be corrected by any given LDPC code, called its *maximum resolvable erasure burst length*, $L_{\max}$ and Song and Cruz in [5] presented a pseudo-random construction for parity-check matrices with large $L_{\max}$. Subsequent research has considered the burst erasure correcting capacity of algebraic LDPC codes [6] or proposed pseudo-random search methods to find parity-check matrices with good $L_{\max}$ [7], [8]. For codes which correct multiple erasure bursts, Hosoya et al. have proposed a pseudo-random column permutation construction method in [9].

In this paper we build upon these results by presenting structured LDPC codes which achieve close to optimal $L_{\max}$ for both single and multiple burst erasures. In Section II we introduce burst erasure correcting LDPC codes and present lower bounds for the maximum length erasure burst that can always be corrected using iterative message-passing decoding. In Section III we describe superposition for constructing burst erasure correcting LDPC codes and the burst erasure correcting performance of the resulting codes is derived as a property of the stopping set size of the base matrices and the choice of permutation matrices for the superposition. In Section IV we design LDPC codes to correct a single erasure burst and in Section V we extend our analysis to design LDPC codes which can correct multiple erasure bursts in the same codeword. Section VI concludes the paper.

II. Low density parity-check codes and burst erasure correction

A length-n, rate-k/n, code with minimum distance d can correct any $d-1$ erasures in a single codeword. Such a code is optimal, and called *maximum distance separable* (MDS), if it achieves the Singleton bound

$$d \leq n - k + 1.$$

While non-binary codes which meet the Singleton bound exist over a range of lengths and rates, such as the Reed Solomon codes, (see e.g. [14]), we are interested in binary codes. Our motivation for considering binary codes for these channels is twofold; firstly, we consider channels which also include random erasures, for which binary LDPC codes are known to perform very well, and secondly we wish to consider codes with low complexity implementation even for very long lengths, for which binary LDPC codes are an ideal choice.

Unfortunately, the only binary MDS codes are trivial [14, p. 72], i.e. the $[n, 1, n]$ repetition codes or the $[n, n-1, 2]$ single parity-check codes. However, for the burst erasure channel, the erasures occur together in a clump and so a code which can correct a burst of $n-k$ erasures need not have minimum distance $n-k+1$. We will say that a code is *burstMDS* if it can correct one or more bursts of erasures with combined length of $n-k$ bits. Our objective in this paper is to design binary LDPC codes which are as close as possible to burstMDS when decoded with iterative message passing decoding.

We will say that the *efficiency* of the code with maximum resolvable erasure burst length, $L_{\max}$, is

$$\eta = \frac{L_{\max}}{n-k}.$$

Thus, a single burst erasure correcting code which is burstMDS will have $L_{\max}$ equal to $n-k$ and so have the maximum efficiency of 1. For multiple burst erasure correcting codes we define the efficiency of a code which can correct N_b length L bursts as

$$\eta = \frac{N_b L}{n-k}.$$

A binary LDPC code is defined by a sparse binary *parity-check matrix,* H. Each bit in the codeword corresponds to a column of H and each parity-check equation satisfied by the codewords corresponds to a row of H. The (j, i)-th entry of H is ‘1’, if the i-th codeword bit is included in the j-th parity-check equation. If the (j, i)-th entry of H is ‘1’, we say that the j-th row is incident on the i-th column, or, equivalently, that bit i is connected to check j. Two columns of H are said to overlap if they both contain a '1' entry in the same row.

A binary LDPC code is also described by a bipartite graph, $\mathcal{T}$, called a *Tanner graph*. Each bit in the codeword corresponds to a bit vertex of $\mathcal{T}$ and each parity-check equation satisfied by the codewords corresponds to a check vertex of $\mathcal{T}$. An edge joins the i-th bit vertex and j-th check vertex of $\mathcal{T}$, if the i-th codeword bit is included in the j-th parity-check equation. A *cycle* in a Tanner graph is a sequence of connected vertices which start and end at the same vertex in the graph and contain no other vertices more than once. The length of the cycle is the number edges it contains and the size and distribution of cycles, particularly small cycles, plays an important role in the decoding performance of the code.

Iterative message-passing decoding of LDPC codes on erasure channels is particularly straightforward since a transmitted bit is either received correctly or completely erased. It is assumed that the receiver is able to detect an erasure and so deletions are not considered. If only one of the bits in any given code parity-check equation is erased, the erased bit can be determined by choosing the value which satisfies that parity-check equation. Message passing iterative decoding of an LDPC code is a process of finding parity-check equations which check on only one erased bit. In a decoding iteration all such parity-check equations are found and the erased bits corrected. After these bits have been corrected any new parity-check equations checking on only one erased bit are then corrected in the subsequent iteration. The process is repeated until all the erasures are corrected or all the remaining uncorrected parity-check equations check on two or more erased bits. The latter will occur if the erased bits include a set of code bits, $\mathcal{S}$, called a stopping set.

A *stopping set*, $\mathcal{S}$, is a set of code bits with the property that every parity-check equation which checks on a bit in $\mathcal{S}$ checks on at least two bits in $\mathcal{S}$ [12]. If all of the bits in a stopping set are erased none of them can be corrected and so the stopping set distribution of an LDPC code determines the erasure patterns for which the message-passing decoding algorithm will fail [12]. The size of a stopping set is the number of bits it includes, and the minimum stopping set size of a parity-check matrix, H, is given by $S_{\min}$, and determines the minimum number of erased bits which can cause a decoding failure. On burst erasure channels, the location of the stopping set bits within the codeword, rather than just the size of the stopping sets, will be the important factor in determining the code's performance when using iterative message-passing decoding.

The spacing between the codeword bits in a stopping set is determined by the spacing of the non-zero entries in the parity-check matrix H. Indeed, a lower bound on the maximum resolvable burst erasure length of an LDPC code can be given by its *minimum zero span*, s, which is the minimum number of zeros between any two non-zero entries in any row of the parity-check matrix. Song and Cruz observed in [5] that for a parity-check matrix with a minimum zero span of s, a single burst erasure of length up to $s+1$ bits can always be recovered in one iteration of the message-passing decoding algorithm. We repeat this result here with a proof.

***Lemma** 1:* An LDPC code with parity-check matrix, H, having all columns with weight greater than zero and a minimum zero span of s, has $L_{\max} \geq s+1$ and can always correct a burst erasure of length up to $s+1$ bits in one iteration of message-passing decoding.

Proof: Since the column weights are non-zero, every one of the $s+1$ erased bits is checked by at least one parity-check equation, and, by the definition of the minimum zero span, no parity-check equation includes more than one of the $s+1$ erased bits. Thus each erased bit is included in a parity-check equation with all other bits known, and so can be corrected without knowledge of the other erased bits, thus requiring only one iteration of the decoder. ■

For better burst correcting LDPC codes, this bound on $L_{\max}$ can be doubled, for the same minimum zero span, by avoiding 4-cycles. A 4-cycle occurs in H if two columns of H contain non-zero entries in the same two rows. Avoiding 4-cycles can improve the performance of iterative decoding and, consequently, low-density parity-check matrices which are free of 4-cycles are very well studied, with many useful constructions available (see e.g. [15], [16], [17], [18]).

***Lemma** 2:* A 4-cycle free parity-check matrix with minimum column-weight $\gamma > 1$ and minimum zero span s has $L_{\max} \geq s+2\gamma-1$ if $s \geq \gamma-1$ and $L_{\max} \geq \gamma+2s$ otherwise. Furthermore, bursts of length up to $L_{\max}$ can be corrected in at most two iterations of message-passing decoding.

Proof: First we take the case that $s \geq \gamma-1$ and suppose that an erasure burst occurred in the set of bit locations $\mathcal{I} = \{i, \ldots, i+s+2\gamma-2\}$. The bits in the set of locations $\mathcal{I}_1 = \{i+\gamma-1, \ldots, i+s+\gamma-1\}$ can always be corrected in the first iteration; for each bit b_j in $\mathcal{I}_1$ there are at most $\gamma-1$ bits in $\mathcal{I}$ which are distance s or greater from b_j and hence which can be included in a check with b_j. However, there are γ checks on every codeword bit and so there are γ bits, distance s or greater from b_j, included in a check with b_j. Since 4-cycles are not allowed, no two equations on b_j can include the same second erased bit, and so at least one of the γ checks on b_j contains a bit which is not one of the $\gamma-1$ erased bits. Thus for each bit b_j in $\mathcal{I}_1$ this check can be used to correct b_j in the first iteration. Next, suppose that the set of bits $\mathcal{I}_2 = \{i, \ldots, i+\gamma-2\}$ are not corrected in the first iteration. Again each of these bits must be contained in γ checks, and again there are at most $\gamma - 1$ bits (the bits in locations $i+s+\gamma, \ldots, i+s+2\gamma-2$) of distance more than s away which were not corrected in the first iteration. Thus, without allowing a 4-cycle, there must similarly be one check on each of the bits $i, \ldots, i+\gamma-2$ which can be used to correct them in the second iteration. By symmetry, the same argument applies to the remaining erased bits in the set $\mathcal{I}_3 = \{i+s+\gamma, \ldots, i+s+2\gamma-2\}$. Secondly, we take the case that $s < \gamma - 1$ and suppose that an erasure burst occurred in bit locations $i, \ldots, i+2s+\gamma-1$. Again, by a similar argument, the bits in locations $i+s, \ldots, i+\gamma+s-1$ can always be corrected in the first iteration as not all of the γ checks on these bits can include a second erased bit without requiring an minimum zero span of less than s or allowing a 4-cycle. As above it is easy to see that if the remaining bits (bits $i, \ldots, i+s-1$ and $i+s+\gamma, \ldots, i+2s+\gamma-1$) are not corrected in the first iteration they can always be corrected in the second. ■

These results the $L_{\max}$ values which can be achieved by using codes with a particular minimum zero span, s, but not how to design H to achieve the required s. Existing methods use pseudo-random approaches to find parity-check matrices, either searching over s or directly searching over $L_{\max}$. The aim of this paper is to define deterministic constructions for structured parity-check matrices with close to optimal $L_{\max}$. In Section III we describe our superposition construction method for burst erasure correcting LDPC codes and derive some general results. We then design specific LDPC codes to correct a single erasure burst in Section IV and codes to correct multiple erasure bursts in Section V.

III. BURST ERASURE CORRECTING CODES USING SUPERPOSITION

We propose to design LDPC codes for burst erasure correction by using superposition (see e.g. [19]). Starting with an $M \times N$ base matrix H_{base}, the entries in H_{base} are replaced with $v \times v$ binary matrices, called the superposition matrices, to create an $m \times n$ LDPC code parity-check matrix, H, where $m = Mv$ and $n = Nv$.[1] Each zero entry in H_{base} is replaced with the $v \times v$ all zeros matrix, $\emptyset$, and each non-zero entry in H_{base} by a $v \times v$ circulant or permutation matrix (which may involve using a different

[1]We use N, M, and $K = N - rank_2(H_{\text{base}})$ for the dimensions of the base matrices and n, m, and $k = n - rank_2(H)$ for the dimensions of the resulting parity-check matrices.

superposition matrix for each non-zero entry). The burst erasure correcting performance of the resulting code is a function of both the minimum stopping set size of the base matrix and the properties of the superposition matrices. Encoding codes formed from systematic base matrices will be straightforward regardless of which permutation matrices are used to form H. For the non-systematic base matrices, however, the use of circulant permutations will ensure quasi-cyclic codes and thus ease of encoding using the method from [20].

We have previously used superposition to construct LDPC codes for packet loss channels in [21]. For the packet loss channel the erasure bursts can only occur within the packet boundaries and so it is sufficient to use arbitrary permutation matrices for the superposition to produce codes which can correct any $S_{\min} - 1$ lost packets [21]. For burst erasure channels however, codes which can correct bursts occurring across the columns of two or more adjacent superposition matrices are required.

Firstly, replacing every non-zero entry in H_{base} with the same $v \times v$ permutation matrix is, in practice, equivalent to independently encoding v codewords using H_{base} and interleaving them so that bits from the same codeword are always v bits apart. Interleaving, or interlacing, codewords in this way is commonly used in bursty channels and the resulting codes will be burstMDS if the base matrix is MDS (see e.g. [22]). However, the resulting LDPC codes are not very effective in the presence of random guard band erasures since the Tanner graph of H consists of disjoint subgraphs and since the repeated permutation matrices ensure stopping sets of size 2 in H. Furthermore, as the only binary MDS codes are the $[n, 1, n]$ repetition codes or the $[n, n-1, 2]$ single parity-check codes, burstMDS codes cannot be formed in this way for many code parameters of interest. In this paper we will design the superposition matrices so as to achieve binary LDPC codes which are almost burstMDS even when the base matrices are not MDS. Lemmas 3 and 4 show how the choice of superposition matrix can improve the code's burst erasure correction capability, and the remainder of this section describes the three base matrices which will be used in this paper.

***Lemma** 3:* If H_{base} has minimum stopping set size $S_{\min}$ and superposition with arbitrary $v \times v$ permutation matrices is used to form H, then H can correct any $\lceil S_{\min}/2 \rceil - 1$ erasure bursts of length v providing that the guard band is erasure free.

Proof: Since arbitrary permutation matrices are chosen for each non-zero entry in H_{base} the resulting parity-check matrix, H, can have adjacent non-zero entries in the same row. However, since permutation matrices have row weight one, every row of H has at most two non zero entries in every set of $2v$ columns, and so each burst can erase at most two bits in each parity-check equation. Since H_{base} has a minimum stopping set size of $S_{\min}$, and since each bit of the stopping set is in a different superposition matrix in H, at least $\lceil S_{\min}/2 \rceil$ erasure bursts are required to erase all of the bits in a single stopping set. ■

At best, i.e. when the base matrix is MDS, Lemma 3 guarantees an efficiency of only 0.5. Thus, while choosing the same superposition matrix for each entry produces poor codes for random erasures, Lemma 3 shows that choosing arbitrary superposition matrices is not not ideal either - producing poor codes for correcting burst erasures. In this paper a compromise between these two approaches is suggested, by using cyclic shifts of a $v \times v$ permutation matrix for the superposition matrices. In this way LDPC codes are produced which can correct $S_{\min} - 1$ erasure bursts of length slightly less than v bits. For any given permutation matrix A, we define $A^{(i)}$ as the matrix formed by a cyclic shift of A left by i columns. The value of i is the called the *order* of the shift.

Lemma 4: Given a base matrix, H_{base}, with minimum stopping set size $S_{\min}$, use superposition with $v \times v$ permutation matrices to form H, such that, for any permutation matrix A in H the closest non-zero matrix on its right is either A or $A^{(1)}$. Then, if r is the maximum shift order across any row of H, any $S_{\min} - 1$ erasure bursts of length $v - r$ can be corrected using H, provided that the guard band is erasure free.

Proof: Consider two superposition matrices, P_1 and P_2, which replace adjacent entries in the same rows of H_{base}, with P_1 to the left of P_2. If P_1 and P_2 are identical, then each pair of non-zero entries in each row of H across P_1 and P_2 are distance v columns apart. Each cyclic shift left by one column in P_2 increases the minimum zero span of the row of P_2 which was incident on the leftmost column, but reduces this distance by one for the remaining rows in P_2. Since every superposition matrix must be shifted by either the same number of columns, or by one further column, to the matrix on its left, H will have a minimum zero span of $s = v - 2$. Thus any erasure burst of length less than $v - 1$ bits can only erase one entry in any one parity-check equation. However, a single burst can still erase two of the columns of a stopping set in H, if that stopping set corresponds to a stopping set in H_{base} which has two adjacent, but non-overlapping, columns. I.e. the two columns in H are incident in superposition matrices which are in different rows of H but are part of a bigger stopping set. In this case, a shift order of i in the superposition matrix in the column on the right, and j in the superposition matrix in the column on the left, will bring the entries of the stopping set $i-j$ columns closer. The maximum possible difference in shifts, by r places, will occur if the circulants in the left column are not shifted at all while the circulants in the right column are shifted by the maximum r shifts. In this case, the columns of the corresponding stopping set in H can be brought up to r columns closer. Thus, bursts of length $v-r$ or less cannot erase two columns of the stopping set, and so restricting the burst length to $v-r$ bits ensures that at least $S_{\min}$ bursts are required to erase a stopping set. ■

Where the columns of H_{base} are re-ordered so that adjacent columns either overlap or are not involved in a stopping set together, the case of adjacent but non-overlapping stopping set columns is avoided and the bound can be improved to any $S_{\min}-1$ bursts of length up to $v-1$ bits can be corrected. In Sections IV and V we will show that, in certain cases, an even better result of $S_{\min}$ bursts corrected can be achieved by using carefully chosen shifted permutation matrices.

A. Base matrices

In Lemmas 3 and 4 we saw that the burst erasure correction capability of the code is a function of the minimum stopping set size of the base matrix. It is important to note that the existence of a binary t-erasure correcting code is not sufficient; we require as well that the base code can be represented by a parity-check matrix with $S_{\min}=t+1$ so that the t bursts can be corrected using iterative message passing decoding. While any base matrix with a given $S_{\min}$ can be used to correct $S_{\min}-1$ bursts using Lemma 4, we will use three base matrices with $S_{\min}$ close to $N-K$ so as to produce burst correcting codes with efficiency as high as possible.

Firstly, an optimal base matrix will have $S_{\min}=N-K+1$, i.e. it will be MDS. As mentioned earlier, such codes are limited, however, the $[N, N-1, 2]$ single parity-check codes provide the MDS base matrices:

$$H_{\text{base}} = \left[\begin{array}{ccccc} 1 & 1 & 1 & \dots & 1 \end{array}\right]. \tag{1}$$

It is easy to see that the codes in (1) have $S_{\min}=2$. Indeed, all binary base matrices have $S_{\min}$ at least 2 since any stopping set requires two columns. We will also use the matrices [21]:

$$H_{\text{base}} = \left[\begin{array}{ccccccc} 1 & 0 & 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & 1 & 1 & 1 & \dots & 1 \end{array}\right], \tag{2}$$

as base matrices for our constructions. These matrices are not MDS since $S_{\min} = N - K = 2$, however the extra parity-check equation in these base matrices will be used to construct double burst correcting codes in Section V, where the superposition matrices will be designed to overcome the fact that $S_{\min}$ is only 2.

Lastly, for codes with larger $S_{\min}$ we will use the base matrices:

$$H_{\text{base}} = \begin{bmatrix} 0 & 0 & 0 & & 1 & 1 \\ 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 1 & & 1 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & & 0 & 0 \\ 1 & 1 & 0 & & 0 & 0 \end{bmatrix}. \tag{3}$$

Lemma 5 proves that these length-N parity-check matrices have a minimum stopping set size of N. Of course codes using (3) as the base matrix will produce codes which have very low rate. However, in the following sections we will show how concatenations of multiple copies of (3) can be used to construct single and multiple burst erasure correcting codes by carefully choosing the superposition matrices.

***Lemma** 5:* The LDPC codes described by parity-check matrices in the form of (3) have $S_{\min} = N$.

Proof: Start with any column of H_{base}, which we label the i-th column. Since H_{base} has weight 2 rows, to form a stopping set, $\mathcal{S}$, requires that each row of H_{base} incident on the i-th column has the second column on which it is incident is also included in $\mathcal{S}$. Thus $\mathcal{S}$ must include the $(i-2)$-th, i-th and $(i+2)$-th columns of H. The same is then true for the extra rows incident on the $(i-2)$-th and $(i+2)$-th columns (which are not incident on the i-th column) and so on until the $(i-l)$-th and $(i+l)$-th columns of H_{base}, for all l even, are included in $\mathcal{S}$. The two end columns however include adjacent checks and so are incident on one check in the set $i \pm l$, l even, and one in the set $i \pm l$, l odd. The process of completing the stopping set is thus repeated to include the $(i-l)$-th and $(i+l)$-th columns, for all l odd. Therefore, a stopping set can be formed only by including all of the columns in H_{base}, and thus $S_{\min} = N$. ■

IV. Single burst erasure correcting LDPC codes

In this Section two different types of burst erasure correcting codes will be considered. The first, row-circulant codes, correspond to using the single parity-check codes, (1), for the base matrices and applying superposition with circulant matrices. The second type of codes are more general quasi-cyclic codes which are constructed using the base matrices of (3) and applying superposition with permutation matrices.

A. Row-circulant LDPC codes

Row-circulant LDPC codes offer low complexity encoding advantages and for this reason have previously been designed for memoryless channels by choosing circulants which improve the code's distance properties (see e.g. [23], [24]). Here we will choose the circulant matrices to instead maximize the burst erasure correcting performance of the codes.

A $v \times v$ circulant, A, is defined by the polynomial

$$a(x) = a_1 + a_2 x + a_3 x^2 + \cdots a_v x^{v-1},$$

where a_i is the entry in the i-th row of the first column of A and can have a value of '0' or '1'. Lemmas 6 and 7 give the properties of the circulants we will use in Constructions 1 and 2. By some abuse of terminology we will talk about the burst erasure correction performance of a binary matrix, A, using the assumption that A is the parity-check matrix of an LDPC code, even though a code with such a parity-check matrix may have zero rate.

***Lemma** 6:* An LDPC code with parity-check matrix A, where A is a column-weight 2 circulant with $a(x) = 1 + x^b$ and $b < v/2$, has $L_{\max} \geq \max\{2b, v - b\}$.

Proof: Consider the leftmost column l in any given stopping set, $\mathcal{S}$, in A (see e.g. Fig. 2). Since this is the leftmost column of $\mathcal{S}$ the rows incident on this column must have their second entry to the right in order to complete $\mathcal{S}$. Since A is circulant one of these rows is incident on a second column distance b away and the other is incident on a second column distance $v - b$ away. Thus for both rows incident on column l to include two columns in the stopping set requires that $L_{\max} \geq \max\{v - b, b\} = v - b$. If $v - b = b$ the stopping set is completed and $L_{\max} = b$. However, if $v - b \neq b$, which is the case for A defined above since $b < v/2$, the second rows incident on each of these new columns must have both of their incident columns included in the stopping set. Consider the column which is distance b away from column l. The second check on this column is a cyclic shift of the first by b places making its second column distance a further b columns to the right. Note that if the second entry was not b columns to the right it would be $b - v$ columns to the left violating our requirement that column l was the left most column in the stopping set. Thus, for this second entry to be included in the stopping set gives $L_{\max} \geq 2b$ and the proof follows. ∎

The first part of Lemma 6 can alternatively be proven by noting that setting $b < v/2$ avoids 4-cycles and substituting $\gamma = 2$ and $s = b - 1$ into Lemma 2. If $b > v/2$ the analogous result is that $L_{\max} \geq \max\{2(v-b), b\}$, however these matrices are not used in this paper.

Extending this result to column-weight 3, $v \times v$ circulants, the three possible zero spans are labeled by s_1, s_2 and s_3, where $s_1 \leq s_2 \leq s_3$ and $s_1 + s_2 + s_3 + 3 = v$.

***Lemma** 7:* An LDPC code with parity-check matrix A, where A is a column-weight 3 circulant with maximum zero span of s_3, can always correct a burst erasure of length up to $L_{\max} \geq s_3 + 1$ bits.

Proof: Suppose that a burst of length $L = s_3 + 1$ occurs across any subset of the v bits. Take the parity-check equation which includes the L-th erased bit and no other erased bits. By the definition of the circulant such a parity-check equation must exist. The other two bits included in this parity-check equation are the last bit before the burst and a bit distance $s' + 1$ past the burst, where s' is either s_1 or s_2. Now, this parity-check equation, and the parity-check equations corresponding to the s' rows above it in the circulant, contain only one erased bit. Thus these $s' + 1$ erased bits can be corrected in the first decoding iteration. Since the $(L - s')$-th to L-th bits are now known, the next s' equations above the previous s' in the circulant only contain one erased bit, and thus the $(L - 2s' - 1)$-th to $(L - s' - 1)$-th erased bits can be corrected. This process can be repeated until all of the erased bits in a burst of length up to $s_3 + 1$ bits are corrected, using at most $\left\lceil \frac{s_3+1}{s'+1} \right\rceil$ iterations. ■

In Construction 1, burst erasure correcting parity-check matrices without repeated columns are achieved while maximizing the resulting $L_{\max}$.

***Construction** 1:* Construct a length Nv, rate $\approx (N-1)/N$, 4-cycle free parity-check matrix, H, using superposition into the length-N base matrix from (1), replacing the i-th entry of H_{base} with the $v \times v$ circulant A_i, where

$$a_i(x) = 1 + x^{b_i} \quad \text{and} \quad b_i = \left\lceil \frac{v}{2} \right\rceil - i. \qquad \square$$

Construction 1 provides 4-cycle free, column-weight 2 parity-check matrices with close to optimal $L_{\max}$, but which can also correct a small number of random erasures.

***Lemma** 8:* The parity-check matrices of Construction 1 have $L_{\max} \geq 2\left\lceil \frac{v}{2} \right\rceil - 2N$.

Proof: From Lemma 6, $L_{\max} \geq 2b_i$ within the i-th circulant, thus a burst of length less than $2b_N$ can not erase a stopping set within a circulant. Next, consider a burst across the A_i-th and A_{i+1}-th circulants. Since $b_i = \left\lceil \frac{v}{2} \right\rceil - i$ the b_i-th row of H has a zero span of 0 between the A_i-th and A_{i+1}-th circulants in H. However, the rows of H above the b_i-th row are cyclic shifts of the first, all with a zero span of $v - 1$

across A_i and A_{i+1} and, similarly, the rows of H below the b_i-th row are cyclic shifts of the (b_i+1)-th row, all with a zero span of $v-2$ across A_i and A_{i+1}. Therefore, if the last column in the A_i-th circulant is not involved in the stopping set, Lemma 1 dictates that any burst of length $v-1$ across the A_i and A_{i+1} circulants can be corrected. Consequently, any stopping set across A_i and A_{i+1} with a width less than $2b_N$ must include the v-th column of the A_i-th circulant. Considering the stopping sets across A_i and A_{i+1} which include the v-th column of A_i we see that the second entry in this column, which is in the last row of H, has a zero span of $v-2$ to the right or b_i-1 to the left, and so a stopping set with width less than $v-2$ must include the $(v-b_i)$-th column of A_i. However, the $(v-b_i)$-th column of A_i has, in turn, its second entry in a row with zero spans of $v-2$ on the right and b_i-1 on the left. Whichever of these columns is chosen, the columns of the stopping set will be distance at least $2b_i$ columns apart and the proof follows. ■

For example, setting $N=5$ and $v=300$ produces the parity-check matrix

$$H=[A_1,A_2,A_3,A_4,A_5],$$

of the length-1500, rate-$4/5$, LDPC code shown in Fig. 4, where the circulants, A_1 to A_5, are

$$a_1=1+x^{149},\quad a_2=1+x^{148},\quad a_3=1+x^{147},\quad a_4=1+x^{146},\quad a_5=1+x^{145}.$$

Applying Lemma 8, this code has $l_{\max}=291$ and so has an efficiency of 0.97.

Since, in general, performance improvements can be obtained by considering LDPC parity-check matrices with weight-3, rather than weight-2, columns [2] our second construction will use weight-3 circulants. Using substitution into base matrices of the form shown in (1), by replacing the i-th entry of H_{base} with the circulant $a_i(x)=1+x^{b_i}+x^{c_i}$, where $b_i=i+1$ and $c_i=b_i+N+i$, guarantees parity-check matrices with $L_{\max}=v-3$ by applying Lemma 7. However, these parity-check matrices will not be 4-cycle free. To obtain column-weight 3 parity-check matrices without 4-cycles requires that each circulant has unique zero spans, not just unique minimum zero span. In Construction 2 column-weight 3, 4-cycle free burst erasure correcting codes are designed.

***Construction** 2:* Construct a length Nv, rate $\approx (N-1)/N$, 4-cycle free parity-check matrix, H, using superposition into the length-N base matrix from (1), replacing the i-th entry with the $v\times v$, circulant

A_i, where $v > 8N$ and

$$a_i(x) = 1 + x^{b_i} + x^{c_i}, \quad b_i = 2i, \quad \text{and} \quad c_i = \left\lceil \frac{3v}{8} \right\rceil + i. \tag{4}$$

□

For example, setting $N = 2$ and $v = 250$ produces the parity-check matrix

$$H = [A_1, A_2],$$

of the length-500, rate-1/2, LDPC code shown in Fig. 3, with circulants

$$a_1 = 1 + x^2 + x^{94}, \quad a_2 = 1 + x^4 + x^{95},$$

having $L_{\max} = 220$ and an efficiency of 0.88.

***Lemma** 9:* The parity-check matrices from Construction 2 are 4-cycle free.

Proof: A 4-cycle will be formed if two rows of H are incident in more than one column together. In a $v \times v$ circulant $a = 1 + x^b + x^c$ every pair of rows which are b, $c - b$ or $v - c$ rows apart will be incident in a column together. Thus, for the codes in Construction 2 to be 4-cycle free requires that the sets $[b_1, \ldots, b_N]$, $[c_1 - b_1, \ldots, c_N - b_N]$ and $[v - c_1, \ldots, v - c_N]$ do not contain any common entries. From (4):

$$\begin{aligned} b_1, \ldots, b_N &= 2, 4, 6, \ldots, 2N \\ c_1 - b_1, \ldots, c_N - b_N &= \left\lceil \frac{3v}{8} \right\rceil - 1, \left\lceil \frac{3v}{8} \right\rceil - 2, \ldots, \left\lceil \frac{3v}{8} \right\rceil - N \\ v - c_1, \ldots, v - c_N &= v - \left\lceil \frac{3v}{8} \right\rceil - 1, v - \left\lceil \frac{3v}{8} \right\rceil - 2, \ldots, v - \left\lceil \frac{3v}{8} \right\rceil - N. \end{aligned}$$

Since $v > 8N$, $\lceil \frac{3v}{8} \rceil - N > 2N$ and $v - \lceil \frac{3v}{8} \rceil - N > \lceil \frac{3v}{8} \rceil - 1$ the proof follows. ■

B. Quasi-cyclic LDPC codes

In general, the minimum stopping set size of row-circulant codes is upper bounded by 2γ, where γ is the column weight of the circulants. Due to this limitation, row-circulant LDPC codes are outperformed by randomly constructed LDPC codes when sufficiently large code lengths are considered. To construct burst erasure correcting LDPC codes without this limitation we consider base matrices in the form of (3).

Although these base matrices are not MDS, we show in the following that, by judiciously selecting the superposition matrices, codes with close to burstMDS performance can still be obtained.

Firstly, we show in Lemmas 10 and 11 that for some choices of superposition matrices, up to $S_{\min}$ corrected burst erasures can be obtained. This is achieved by careful positioning of the superposition matrices into the entries of the smallest stopping sets in H_{base}. These results will be used to construct single burst erasure correcting in Constructions 3 and 4, and again in Section V when multiple burst erasure correcting codes are constructed.

***Lemma** 10:* Consider a stopping set $\mathcal{S}$, in the base matrix H_{base}, which contains weight-2 columns such that each parity-check equation in H_{base} is connected to either zero or two stopping set bits in $\mathcal{S}$. Suppose that the parity-check matrix H is formed from H_{base} using superposition with $v \times v$ permutation matrices. $H(\mathcal{S})$ is defined as the columns of H corresponding to the columns of H_{base} which are involved in $\mathcal{S}$. Suppose that to form H every entry of $\mathcal{S}$ is replaced by the same permutation matrix A, except for the entries in the bottom most row involved in $\mathcal{S}$, where the entry in the l-th column is replaced by $A^{(l)}$. If r is the maximum difference in shift order between two entries in any row of $\mathcal{S}$, then if all but r consecutive bits in $H(\mathcal{S})$ are erased, and no other codeword bits are erased, all of the erased bits can be corrected.

Proof: Suppose that we wish to locate a stopping set $\mathcal{S}'$ within $H(\mathcal{S})$. We start $\mathcal{S}'$ with the i-th column of one of the permutation matrices within $H(\mathcal{S})$. Since $\mathcal{S}'$ will include a column from every superposition matrix which replaced an entry of $\mathcal{S}$, and since each column of $\mathcal{S}$ was replaced with at least one copy of A, the i-th column of each of the A matrices must be included in $\mathcal{S}'$ (see e.g. the highlighted entries in Fig. 1). The final permutation matrix in the j-th row of $\mathcal{S}$, P_r, was shifted left by r columns relative to the first, P_a, and so the $(i-r)$-th column in P_r will also need to be included in $\mathcal{S}'$ so that the check on the i-th bit in P_a checks on a second stopping set bit. Then the $(i-r)$-th column of the other permutation matrix in the same column as P_r, which must be A, is also erased. As above, the $(i-r)$-th column of every permutation matrix must now be included in $\mathcal{S}'$ to complete the stopping set on the checks of $\mathcal{S}'$. Again, the $(i-r \mod v)$-th column in the last permutation matrix will subsequently need to be included in $\mathcal{S}'$ so that the check on the $(i-r)$-th bit in the first permutation matrix in the j-th row checks on a second bit in $\mathcal{S}'$ and so on until every r-th column in $H(\mathcal{S})$ must be included to form a stopping set. Thus if any one of these columns is received the erased bits do not form a stopping set and the proof follows. ■

Lemma 10 can be extended to the case where more than one row of the stopping set includes shifted permutation matrices. The spacing between the columns of the resulting stopping set $\mathcal{S}'$ will depend on the difference in column shifts between the entries of $\mathcal{S}$ on the same row. We will use the result for stopping sets of size three in our constructions and so present it here.

***Lemma** 11:* Consider a stopping set $\mathcal{S}$, in the base matrix H_{base}, which contains three weight-2 columns such that each parity-check equation in H_{base} is connected to either zero or two stopping set bits in $\mathcal{S}$. Suppose that the parity-check matrix H is formed from H_{base} using superposition with $v \times v$ permutation matrices. Suppose that both entries, P_1 and P_2, in the first row of H_{base} which is incident on the columns of $\mathcal{S}$, are replaced with the same permutation matrix, A, and the second entries in these two columns, entries P_3 and P_4, are replaced by $A^{(a)}$ and $A^{(b)}$ respectively. Further, the two entries of the third column of $\mathcal{S}$, entries P_5 and P_6, where P_5 is the entry on the same row of H_{base} as P_3, are replaced by $A^{(c)}$ and $A^{(d)}$ respectively. Then, if all but $r = |a - c + d - b| > 0$ consecutive bits in $H(\mathcal{S})$ are erased, and no other codeword bits are erased, all of the erased bits can be corrected.

Proof: We wish to find a stopping set $\mathcal{S}'$ within the columns of $H(\mathcal{S})$. To begin, the i-th column of P_1 is included in $\mathcal{S}'$. Since P_2 is the on the same row and contains the same permutation matrix as P_1 the i-th column of P_2 is also in $\mathcal{S}'$. This requires the $(i - (c - a))$-th column in P_5 to be included in $\mathcal{S}'$ so that the corresponding check in P_3 includes two stopping set bits. Then the $(i - (c - a) - (b - d))$-th column in P_4 must now be included in $\mathcal{S}'$ so that the corresponding check in P_6 includes two stopping set bits. A stopping set will occur if $(-(c - a) - (b - d)) = 0$, otherwise the process begins again with the $(i - (c - a) - (b - d))$-th column of P_1 and the proof follows in the same manner as for Lemma 10.

■

In a column-weight 2 code, a 6-cycle translates directly into a stopping set of size 3 and so the requirement on the allowed circulants for the quasi-cyclic codes in Lemma 11 is related to the restriction on the allowed circulants for quasi-cyclic codes to avoid 6-cycles in [25].

Using Lemma 11 burst erasure correcting codes can now be constructed for a wide range of code lengths and rates:

***Construction** 3:* Construct the $3 \times 3p$ base matrix H_{base} from the concatenation of p copies of the 3×3 base matrices from (3). Then construct a length $3pv$, rate $\approx (p - 1)/p$, parity-check matrix, H, using superposition. The last nonzero entry in the columns of H_{base} corresponding to the i-th copy of (3), $i \in \{1, \dots, p\}$, is replaced with $I_v^{(i+1)}$. Every other non-zero entry of H_{base} is replaced by I_v and every zero entry by $\emptyset$. □

***Lemma** 12:* The codes from Construction 3 have $L_{\max} = 3v - p - 1$.

Proof: The base matrices from (3) have minimum stopping set size $S_{\min} = 3$. Although concatenating multiple copies of these matrices reduces the minimum stopping set size of H_{base} to 2, any adjacent set of 3 columns does not contain a stopping set of size less than 3. From Lemma 11 a burst across any one of these size 3 stopping sets can be corrected if $p + 1$ consecutive bits are received correctly and the proof follows. ■

For example, the rate-4/5 burst erasure correction codes with $p = 5$:

$$H = \begin{bmatrix} \emptyset & I_v & I_v & \emptyset & I_v & I_v & \emptyset & I_v & I_v & \emptyset & I_v & I_v & \emptyset & I_v & I_v \\ I_v & \emptyset & I_v^{(1)} & I_v & \emptyset & I_v^{(2)} & I_v & \emptyset & I_v^{(3)} & I_v & \emptyset & I_v^{(4)} & I_v & \emptyset & I_v^{(5)} \\ I_v^{(1)} & I_v^{(1)} & \emptyset & I_v^{(2)} & I_v^{(2)} & \emptyset & I_v^{(3)} & I_v^{(3)} & \emptyset & I_v^{(4)} & I_v^{(4)} & \emptyset & I_v^{(5)} & I_v^{(5)} & \emptyset \end{bmatrix} \quad (5)$$

have $L_{\max} = 3v - 6$. The entry $I_v^{(i)}$ in (5) represents superposition with a permutation matrix that is the cyclic shift of the $v \times v$ identity matrix by i columns left, and the entry $\emptyset$ represents superposition with a $v \times v$ all zeros matrix. Setting $v = 100$ gives the parity-check matrix of the length-1500, rate-$4/5$, LDPC code, with $L_{\max} = 294$ and efficiency 0.98, with performance shown in Fig. 4.

Construction 3 can be generalized to all of the base matrices in the form of (3), where the codes can correct bursts of length close to vN bits if the sum of shifts in the entries allocated to any stopping set, which occurs within N adjacent columns of H_{base}, do not add to zero mod N.

***Construction** 4:* Construct the $N \times pN$ base matrix H_{base} from the concatenation of p copies of the $N \times N$ base matrices from (3). Then construct a length Npv, rate $\approx (p-1)/p$, parity-check matrix, H, using superposition where every zero entry in the H_{base} is replaced by $\emptyset$, first non-zero entry of each column of H_{base} is replaced by the $v \times v$ identity matrix, I_v, and the second non-zero entry of each column of H_{base} is replaced by $I_v^{(l_i)}$, with the orders l_i chosen so that the sum of the shift orders in any adjacent set of N columns is non-zero mod N. □

For example, the rate-$1/2$ burst erasure correction codes with $p = 2$ and $N = 5$:

$$H = \begin{bmatrix} \emptyset & \emptyset & \emptyset & I_v & I_v & \emptyset & \emptyset & \emptyset & I_v & I_v \\ \emptyset & \emptyset & I_v & \emptyset & I_v & \emptyset & \emptyset & I_v & \emptyset & I_v^{(6)} \\ \emptyset & I_v & \emptyset & I_v & \emptyset & \emptyset & I_v & \emptyset & I_v^{(5)} & \emptyset \\ I_v & \emptyset & I_v & \emptyset & \emptyset & I_v & \emptyset & I_v^{(4)} & \emptyset & \emptyset \\ I_v & I_v^{(1)} & \emptyset & \emptyset & \emptyset & I_v^{(2)} & I_v^{(3)} & \emptyset & \emptyset & \emptyset \end{bmatrix} \quad (6)$$

have $L_{\max} = 5v - 2$. Setting $v = 50$ gives the parity-check matrix of the length-500, rate-$1/2$ LDPC code, with $L_{\max} = 248$ and efficiency 0.992, with performance shown in Fig. 3.

Table I shows the parameters of codes from Constructions 1-4 compared to recently published results for burst erasure correcting LDPC codes, and Figs. 3 and 4 show their erasure correction performance on the burst erasure channel with one randomly located erasure burst. Fig. 3 shows length-500, rate-$1/2$ codes and Fig. 4 shows length-1500, rate-$4/5$ codes. The base matrices of the codes from Construction 3 and 4 are from (5) with $v = 100$ and (6) with $v = 50$ respectively. Also shown is the performance of interleaved codes from the MDS base matrices in (1) with $N = 2$, $v = 250$ and $N = 5$, $v = 300$ respectively.

The codes from Construction 1 perform similarly to the traditional interleaved MDS codes, performing well in the channel with no guard band erasures but poorly once random erasures are admitted into the guard band. The codes from Construction 2 perform very well, significantly outperforming the MDS codes over channels with random erasures in the guard band but also providing a good level of protection from erasure bursts. The erasure correcting codes of Constructions 3 and 4 provide an excellent trade off between the two, providing burst erasure protection almost as good as that of the MDS codes, but with significantly better erasure correction performances as the channel becomes more random.

V. LDPC CODES FOR CORRECTING MULTIPLE ERASURE BURSTS

The classic bursty channel provides a reasonable model of burst erasure channels under the assumption of widely spaced bursts, however, it does not include the possibility of two or more bursts occurring in quick succession. Here we extend our design of structured LDPC codes to codes for channels with multiple randomly placed bursts per codeword. The channel we consider is an extension to the classical burst erasure channel where multiple erasure bursts are randomly located in the same codeword and the guard band is erasure free.

Hosoya et al in [9] have extended the method of [4] to define a pseudo-random construction for low-density parity-check matrices which can correct multiple erasure bursts by searching over all of the zero spans in a parity-check matrix and pseudo-randomly permuting columns involved in a short zero span.

Using this method the average zero span of the resulting parity-check matrix is optimized. Here we treat the problem differently, concentrating on designing parity-check matrices which can correct a fixed number, N_b, of bursts of length L bits or less with as high an efficiency as possible regardless of the zero span of H.

Given an $M \times N$ base matrix, H_{base}, with minimum stopping set size $S_{\min}$, parity-check matrices which can correct $S_{\min} - 1$ erasure bursts of length v can be constructed using superposition with I_v for every non-zero entry of H_{base}, which is equivalent to code interleaving. Alternatively, using the same base matrix, Lemma 4 showed that LDPC codes which produce connected Tanner graphs, and can correct $S_{\min} - N + 1$ erasure bursts of length almost v bits, can be constructed using superposition with cyclically shifted permutation matrices. This is made possible by arranging the columns of the $M \times N$ base matrix such that the columns with final non-zero entry in the M-th row are first, followed by the columns with final non-zero entry in the $(M-1)$-th row and so on. The last non-zero entry in the i-th column, $i > 1$, of H_{base} is replaced by $I_v^{(i-1)}$ and every other non-zero entry of H_{base} is replaced by I_v.

For example, using the length-7, rate-4/7 Hamming code parity-check matrix as the base matrix, and applying superposition with $v = 100$ shifted permutation matrices, gives parity-check matrices which can correct any two erasure bursts of length up to 94 bits with efficiency $2 * 94/300 = 0.627$. However, codes constructed in this way will only achieve a high efficiency (i.e. approach burstMDS performances) for those rates for which binary MDS base matrices exist, i.e. the codes can be burstMDS for $N_b > 1$ bursts only if the code rate is $1/(N_b + 1)$.

Thus 2-burst correcting codes are only burstMDS if H_{base} has only two rows, and, unfortunately, the only binary base matrix with two rows and $S_{\min} = 3$ is the rate-$1/3$ repetition code. To construct parity-check matrices which are close to burstMDS for $N_b > 1$ and rate $\geq 1/2$ requires that more than $S_{\min} - 1$ bursts can be corrected. We propose to achieve this using the $2 \times N$ base matrices from (2) and carefully choosing the superposition matrices. Lemmas 10 and 11 derived in the previous section are now applied to these multiple burst correcting codes.

Firstly, using Lemma 10, 2-burst correcting codes can now be constructed for a wide range of code lengths and rates:

Construction *5:* Construct H using superposition on the length-N base matrix from (2). The zero entries of H_{base} are replaced by $\emptyset$, the $(2, i)$-th entry, $i > 2$, of H_{base} is replaced by $I_v^{(i-2)}$ and all other non-zero entries of H_{base} are replaced by I_v. $\square$

***Lemma** 13:* The codes from Construction 5 can correct any two erasure bursts of length $v - N + 2$, provided that the guard band is erasure free.

Proof: Applying Lemma 10 gives that any stopping set in the columns of H corresponding to the weight-2 columns of H_{base} must include at least every $(N-2)$-th column of two of the permutation matrices. With only two erasure bursts of length $v - N + 2$ it is not possible to erase all of the columns in these stopping sets. Alternatively, the weight-1 columns of H_{base} are involved in stopping sets of size 3 in H_{base}, however, applying Lemma 4, gives that any two bursts of length $v - N + 2$ also cannot erase the corresponding stopping sets in H. ■

For example, rate-$3/4$, length-$8v$ LDPC codes, which can correct any two erasure bursts of length up to $v - 4$ bits each, with efficiency $2(v-4)/2v$, are constructed by:

$$H = \begin{bmatrix} I_v & \emptyset & I_v & I_v & I_v & I_v \\ \emptyset & I_v & I_v & I^{(1)} & I_v^{(2)} & I_v^{(3)} \end{bmatrix}. \tag{7}$$

Setting $v = 502$ and $N = 2$ gives the parity-check matrix of the length-2008, rate-$1/2$, LDPC code shown in Fig. 5, which can correct any two erasure bursts of length up to 500 bits, with efficiency 0.996. Setting $v = 504$ and $N = 6$ gives the parity-check matrix of the length-3024, rate-$2/3$, LDPC code shown in Fig. 7, which can correct any two erasure bursts of length up to 500 bits, with efficiency 0.992.

Where more than two bursts occur per codeword we also use base matrices with M equal to the number of bursts and with $S_{\min}$ less than $M+1$. LDPC codes to correct three erasure bursts can be constructed similarly to Construction 4, by using $p = 2$, but now to satisfy Lemma 11 the shifts must be chosen to ensure that the sum of the shifts for all size two or three stopping sets in H_{base} is greater than zero, rather than just those stopping sets with adjacent columns. However, this constraint alone does not guarantee a good burst erasure correction performance since three bursts can erase codeword bits which correspond to columns in H from as many as six different H_{base} columns. For example, using

$$H = \begin{bmatrix} \emptyset & \emptyset & I_v & I_v & I_v & I_v \\ I_v & I_v & \emptyset & \emptyset & I_v^{(4)} & I_v^{(5)} \\ I_v & I_v^{(1)} & I_v^{(2)} & I_v^{(3)} & \emptyset & \emptyset \end{bmatrix}, \tag{8}$$

and superposition with $v \times v$ circulants, three bursts of length up to $v - 5$ bits can be guaranteed to be corrected only if each burst is restricted to erase codeword bits which correspond to columns of H within the boundaries of a single column of superposition matrices. Without this restriction, three short bursts can erase a stopping set of size 4. For example, the stopping set corresponding to the $(2v+1)$-th, $4v$-th, $(4v+1)$-th, and $6v$-th columns of H can be erased by three bursts of length 2 bits each. While these stopping sets will always exist whichever way the columns of (8) are permuted, the situation that two of the columns of the stopping set are adjacent, and so erased by the same burst, can be avoided by instead choosing:

$$H = \begin{bmatrix} I_v & \emptyset & I_v & I_v & \emptyset & I_v \\ I_v & I_v & \emptyset & \emptyset & I_v & I_v^{(4)} \\ \emptyset & I_v & I_v^{(1)} & I_v^{(2)} & I_v^{(3)} & \emptyset \end{bmatrix}. \tag{9}$$

In this case, there is the problem that three adjacent bursts of length $2v/3$ or greater, across bits corresponding to the third and fourth superposition matrices of (9), will erase a stopping set in H from the two adjacent copies of the same column in H_{base}. Nevertheless, a single received bit between any two of the bursts will allow the correction of all of the erased bits. Any other set of three bursts with length up to $v - 6$ bits can always be corrected using the parity-check matrix in (9). Since the event of three consecutive bursts has a low probability of occurring the average decoder performance is not significantly effected by this single case.

The same principles can be extended to other base matrices with $S_{\min} = 3$.

***Construction** 6:* Construct H_{base} from the concatenation of 2 copies of an LDPC base matrix with $S_{\min} \geq 3$. Order the columns of H_{base} to avoid small stopping sets in adjacent columns. Form H using superposition where the final entry of each column of H_{base} is replaced by the permutation matrix I_v^l, with the shifts l chosen so that the sum of the shifts in any stopping set is non-zero. □

For example,

$$H = \begin{bmatrix} I_v & I_v & I_v & I_v & \emptyset & \emptyset & I_v & I_v \\ I_v & I_v & \emptyset & \emptyset & I_v & I_v & I_v^{(6)} & I_v^{(7)} \\ I_v & I_v^{(1)} & I_v^{(2)} & I_v^{(3)} & I_v^{(4)} & I_v^{(5)} & \emptyset & \emptyset \end{bmatrix}, \tag{10}$$

gives rate-$5/8$ 3-burst correcting LDPC codes. Setting $v = 506$ and $N = 6$ gives the parity-check matrix of the length-3036, rate-$1/2$, LDPC code with performance shown in Fig. 6, which can correct three erasure bursts of length up to 500 bits each. Setting $v = 506$ and $N = 9$ gives the parity-check matrix of the length-4554, rate-$2/3$, LDPC code with performance shown in Fig. 7, which can correct three erasure bursts of length up to 500 bits each.

Figs. 5 to 8 show the performance of LDPC codes on burst erasure channels with multiple length 500 bursts and varying burst erasure probabilities. Any given codeword can be effected by zero, one, or multiple bursts; with the probability of many bursts occurring in a single codeword increasing with increasing burst erasure probability. The length 2008 and 3024 burst correction codes are from Construction 5 and the length 3036 and 4554 burst correction codes are from Construction 6. Also shown is the performance of typical LDPC codes constructed pseudo-randomly to avoid 4-cycles, from [26], and codes formed by interleaving 500 codewords from the rate-1/2 and rate-2/3 repetition codes respectively. Lastly the performance of codes with the same parameters and designed using the method from [9] are also shown. For these codes, the average zero span in the parity-check matrix is labeled D_{avg} [9]. Although the codes from [9] perform equivalently to randomly constructed codes in these channels, they will out-perform random codes when smaller bursts are considered [9].

Figs. 5 to 8 clearly demonstrate the benefit of considering structured LDPC codes for burst erasure correction. That the structured LDPC codes outperform the random LDPC codes on a channel with memory should be unsurprising since, unlike for the random codes, the structure of the channel has been taken into account in their design. More significantly they can also perform much better than traditional binary burst erasure correction schemes. Although the interleaved codes are burstMDS for one burst, since single erasure correcting binary MDS base matrices exist, they do not achieve the performance improvements gained by the structured LDPC codes which can correct multiple bursts occurring in close proximity.

VI. Conclusion

For memoryless channels it is well established that long pseudo-random LDPC codes provide capacity approaching performances. However, for channels with memory in the form of burst erasures, we have shown that structured LDPC codes are certainly the best choice. Moreover, the ease of implementation of message-passing decoding on erasure channels, and the ease of encoding provided by the proposed codes, suggests that structured LDPC codes represent a promising candidate for applications which suffer from burst erasures and face low complexity or high throughput constraints.

Future work will focus on multiple burst erasure correcting codes which are more robust to random erasures and the application of these codes to more general memoryless channels.

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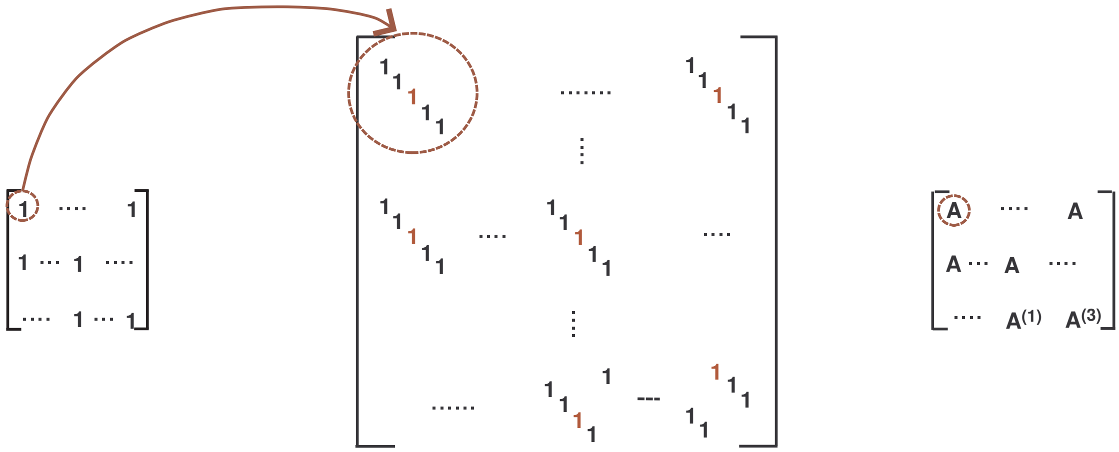

Fig. 1. Construction of an LDPC code via superposition. H_{base} is shown on the left and H in the center. Zero entries are not shown. On the right is the shorthand notation for H. In this case A is the 5×5 identity matrix, I_5. (The two entries in the first and middle rows of H_{base} are replaced by I_5 and the two entries in the final row of H_{base} are replaced by $I_5^{(1)}$ and $I_5^{(3)}$ respectively.)

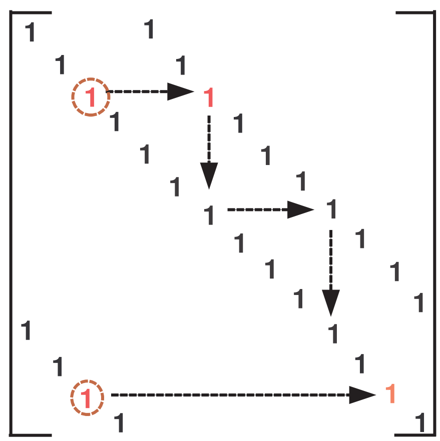

Fig. 2. Minimum stopping sets in a column-weight 2 circulant.

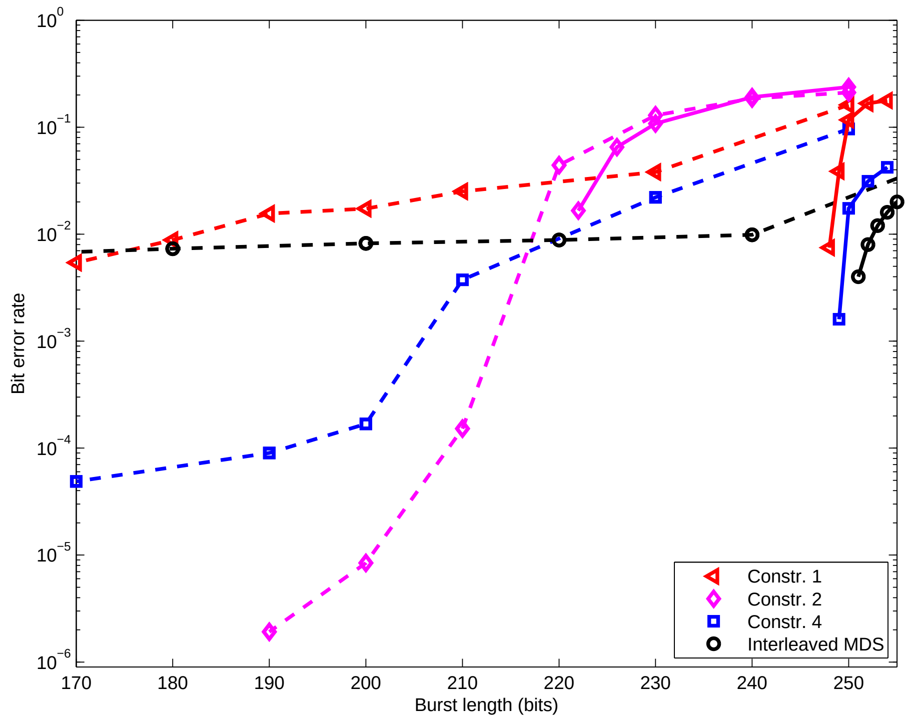

Fig. 3. The performance of length 500, rate-$1/2$ LDPC codes on an erasure channel with one randomly located burst erasure and guard band erasure probabilities of $p = 0$ (solid curves) and $p = 0.01$ (dashed curves).

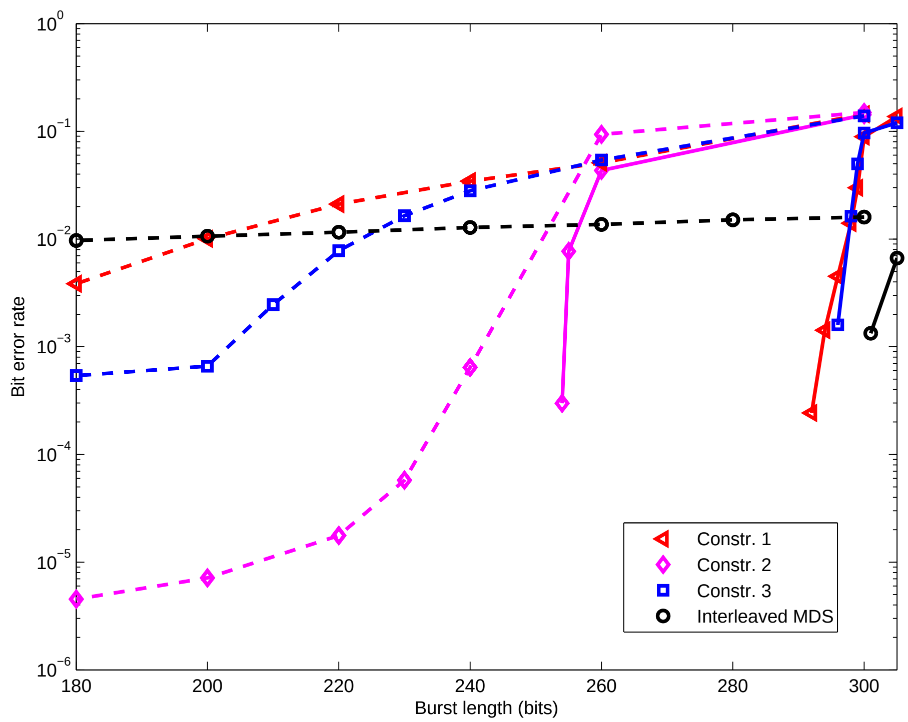

Fig. 4. The performance of length 1500, rate-$4/5$ LDPC codes on an erasure channel with one randomly located burst erasure and guard band erasure probabilities of $p = 0$ (solid curves) and $p = 0.01$ (dashed curves).

TABLE I

BURST CORRECTION PROPERTIES OF SELECTED LDPC CODES.

Code type	Length	Rate	$L_{\max}$	$L_{\max}/(n-k)$
[6, Example 1]	3066	0.504	1021	0.671
[6, Example 4]	3060	0.503	1155	0.759
eIRA (col weights 2,3,7,8) [7]	4000	0.5	1787	0.894
Construction 1 ($N=2, v=1500$)	3000	0.5	1496	0.997
Construction 2 ($N=2, v=1500$)	3000	0.5	1468	0.978
Construction 4 ($p=2, N=5, v=300$)	3000	0.5	1498	0.999
EG (col weight 64)	4095	0.82	376	0.51
eIRA (col weights 2,5) [27]	4095	0.82	507	0.688
PG (col weight 65)	4161	0.82	303	0.405
eIRA (col weights 2,3,7,8) [7]	4161	0.824	602	0.822
Construction 1 ($N=6, v=693$)	4158	0.833	682	0.927
Construction 2 ($N=6, v=693$)	4158	0.833	613	0.833
Construction 3 ($p=6, v=231$)	4158	0.833	686	0.932
[6, Example 1]	16352	0.877	1021	0.469
[6, Example 2]	16513	0.903	509	0.318
Construction 1 ($N=10, v=1650$)	16500	0.9	1648	0.999
Construction 3 ($p=10, v=550$)	16500	0.9	1639	0.993

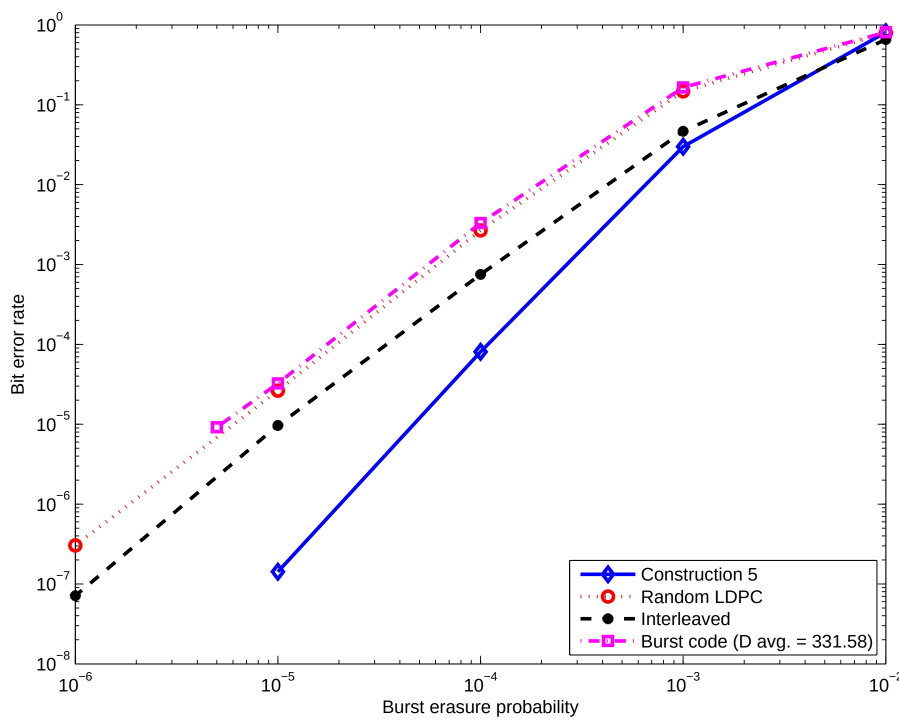

Fig. 5. The performance of length-2008, rate-$1/2$ LDPC codes on a burst erasure channel with multiple length 500 bursts and varying burst erasure probability.

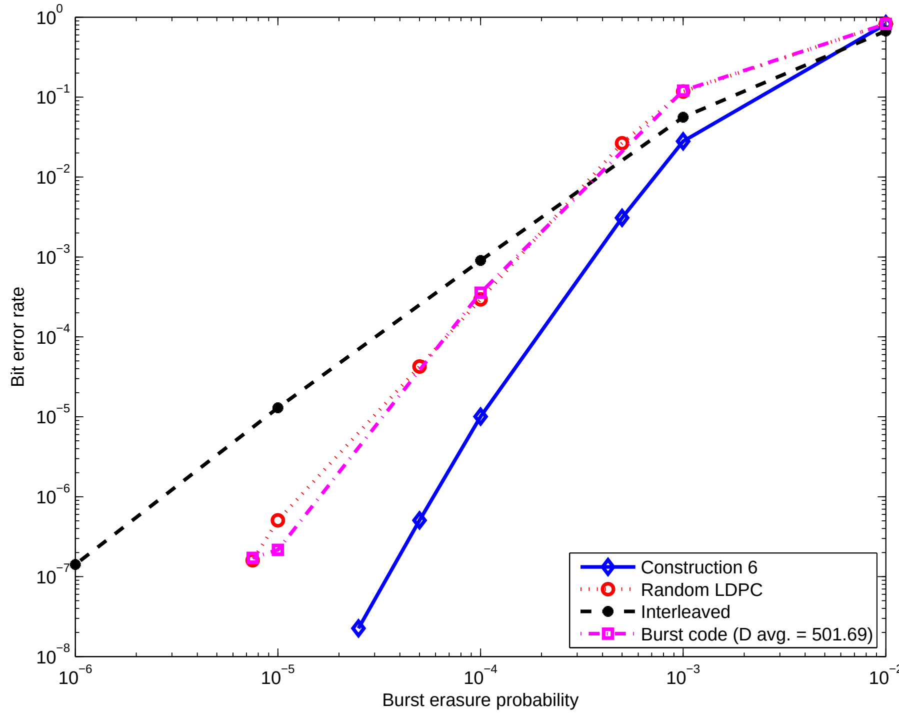

Fig. 6. The performance of length-3036, rate-$1/2$, LDPC codes on a burst erasure channel with multiple length 500 bursts and varying burst erasure probability.

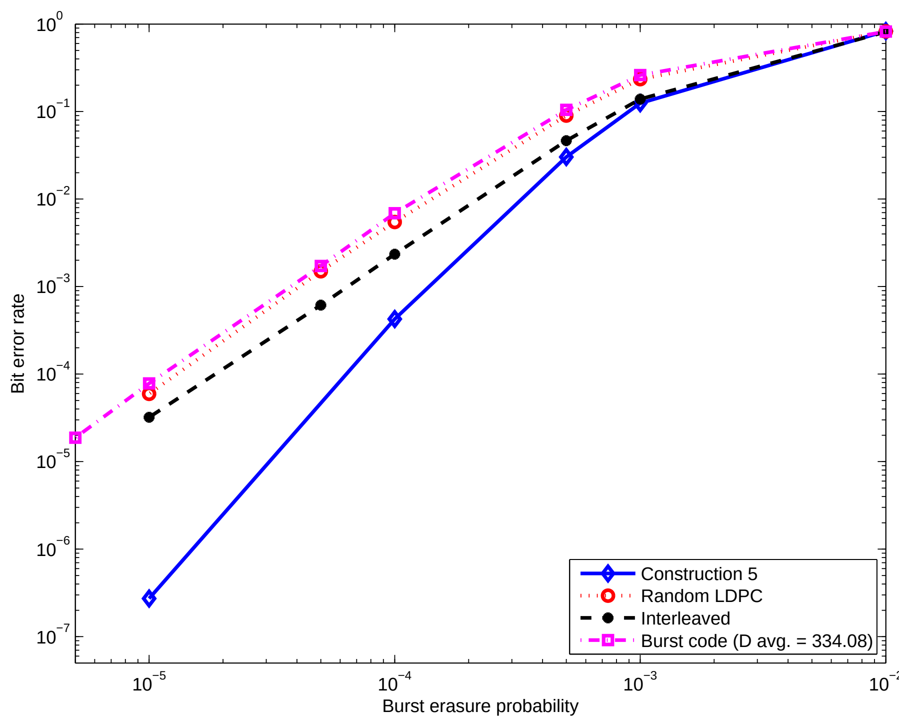

Fig. 7. The performance of length-3024, rate-$2/3$ LDPC codes on a burst erasure channel with multiple length 500 bursts and varying burst erasure probability.

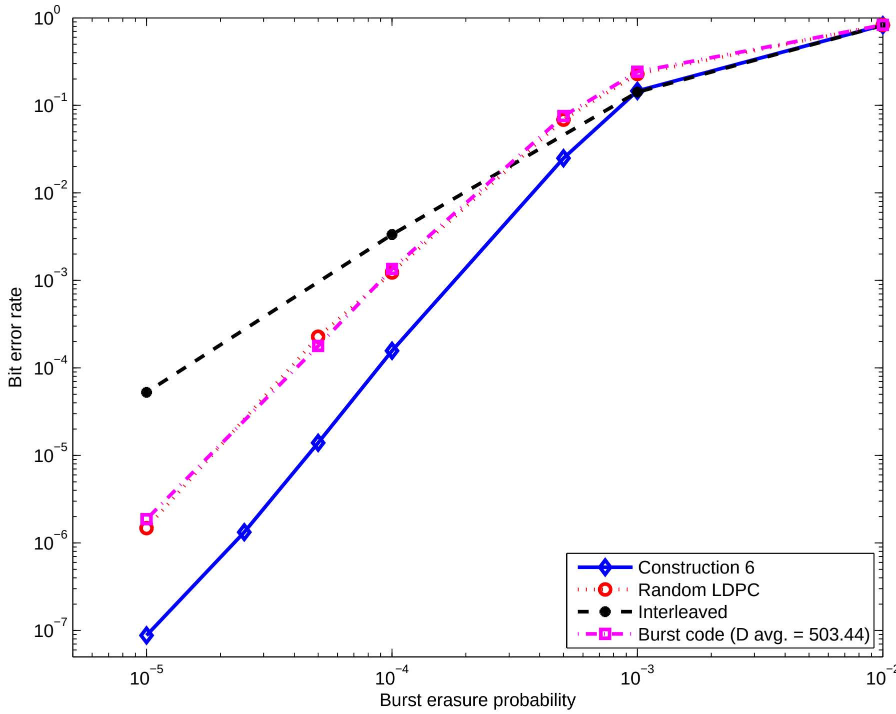

Fig. 8. The performance of length-4554, rate-$2/3$ LDPC codes on a burst erasure channel with length 500 bursts and varying burst erasure probability.