

Systems Theory

Dr. Steve Weller

steven.weller@newcastle.edu.au

School of Electrical Engineering and Computer Science

The University of Newcastle, Callaghan, NSW 2308

Australia



Controllability, stabilizability and reachability

- Important question that lies at the heart of control using state-space models:

*“Can we steer the state, via the control input,
to certain locations in the state space?”*

□ Controllability

- can initial state be driven back to origin?

□ Stabilizability

- can *all* states be taken back to origin?

□ Reachability

- can a certain state be reached *from* origin?



Controllability

Issue of *controllability* concerns whether a given initial state x_0 can be steered to the origin in finite time using the input $u(t)$

Definition 1:

- A state x_0 is said to be *controllable* if there exists a finite time interval $[0, T]$ and an input $\{u(t), t \in [0, T]\}$ such that $x(T) = 0$
- If all states are controllable, then the system is said to be *completely controllable*

Reachability

Converse to controllability is *reachability*:

Definition 2:

- A state $\bar{x} \neq 0$ is said to be *reachable* (from the origin) if, given $x(0) = 0$, there exist a finite time interval $[0, T]$ and an input $\{u(t), t \in [0, T]\}$ such that $x(T) = \bar{x}$.
- If all states are reachable, the system is said to be *completely reachable*

Controllability and reachability—not *quite* the same

For continuous-time, linear time-invariant systems:

complete controllability \Leftrightarrow complete reachability

Following example illustrates subtle difference in discrete-time...

❖ consider the following shift-operator state space model:

$$x[k + 1] = 0$$

- system is completely controllable since every state goes to origin in one time-step
- *but* no non-zero state is reachable, so system is *not* completely reachable

Controllability or reachability?

- In view of the subtle distinction between controllability and reachability in discrete-time, we will use the term *controllability* in the sequel to cover the stronger of the two concepts
 - ⇒ discrete-time proofs for the results are a little easier
- We will thus present results using the following discrete-time model, written in terms of the delta operator:

$$\begin{aligned}\delta x[k] &= \mathbf{A}_\delta x[k] + \mathbf{B}_\delta u[k] \\ y[k] &= \mathbf{C}_\delta x[k] + \mathbf{D}_\delta u[k]\end{aligned}$$

A test for controllability

We now present a simple algebraic test for controllability that can easily be applied to a given state-space model

Theorem 2: Consider the state-space model

stated for delta model,
but holds for shift and
continuous-time models too

$$\mathbf{x}[k] = \mathbf{A} \mathbf{x}[k] + \mathbf{B} \mathbf{u}[k]$$

$$\mathbf{y}[k] = \mathbf{C} \mathbf{x}[k] + \mathbf{D} \mathbf{u}[k]$$

(i) The set of all controllable states is the range space of the *controllability matrix* $\Gamma_c[\mathbf{A}, \mathbf{B}]$, where

$$\Gamma_c[\mathbf{A}; \mathbf{B}] = [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}]$$

(ii) The model is completely controllable if and only if $\Gamma_c[\mathbf{A}, \mathbf{B}]$ has full row rank

Example: A completely controllable system

Consider a state-space model with

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The controllability matrix is given by

$$\Gamma_c[\mathbf{A}; \mathbf{B}] = [\mathbf{B}; \mathbf{A}\mathbf{B}] = \begin{bmatrix} 1 & 4 \\ 1 & 2 \end{bmatrix}$$

- $\text{rank } \Gamma_c[\mathbf{A}, \mathbf{B}] = 2$

⇒ the system is completely controllable

□ **Observation:** complete controllability of a system is independent of **C** and **D**

Example: A non-completely controllable system

For

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the controllability matrix is given by:

$$\Gamma_c[\mathbf{A}; \mathbf{B}] = [\mathbf{B}; \mathbf{A}\mathbf{B}] = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

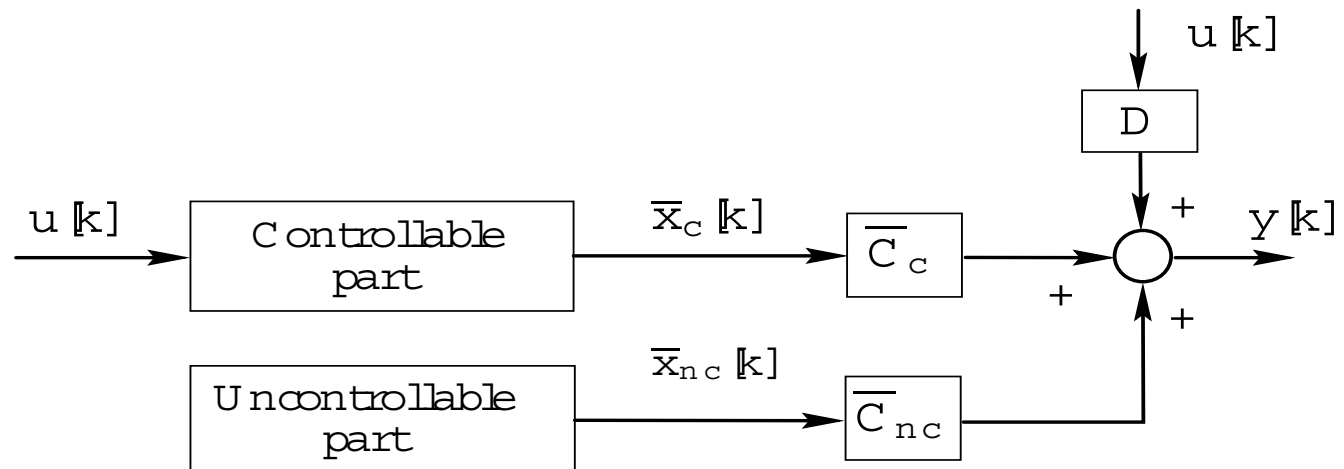
- $\text{rank } \Gamma_c[\mathbf{A}, \mathbf{B}] = 1$ since row 2 = -row 1
- ⇒ $\Gamma_c[\mathbf{A}, \mathbf{B}]$ is *not* full row rank
- ⇒ system is *not* completely controllable

Controllability—a word of caution

- Controllability is a black and white issue: a model either is completely controllable or it's not
- Knowing that a system is controllable (or not) is a valuable piece of information, but...
- knowing that a system is controllable really tells us nothing about the *degree* of controllability, i.e., about the difficulty that might be involved in achieving a certain objective
 - for example: how much energy is required to drive system state to origin?
 - this issue lies at the heart of the fundamental design trade-offs in control

Controllable–uncontrollable decomposition

If a system is not completely controllable, it can be decomposed into a controllable and a completely uncontrollable subsystem



However these states behave, it's independent of input $u[k]$

Partitioning the state-space model

Key to the controllable–uncontrollable decomposition is the transformation of \mathbf{A} , \mathbf{B} , and \mathbf{C} into suitably partitioned form:

$$\begin{aligned} \begin{bmatrix} \bar{\mathbf{x}}_c[k] \\ \bar{\mathbf{x}}_{nc}[k] \end{bmatrix} &= \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{nc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c[k] \\ \bar{\mathbf{x}}_{nc}[k] \end{bmatrix} + \begin{bmatrix} \bar{\mathbf{B}}_c \\ 0 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} \bar{\mathbf{C}}_c & \bar{\mathbf{C}}_{nc} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_c[k] \\ \bar{\mathbf{x}}_{nc}[k] \end{bmatrix} + D u[k] \end{aligned}$$

Controllable decomposition

Lemma 1: Consider a system having $\text{rank}\{\Gamma_c[\mathbf{A}, \mathbf{B}]\} = k < n$. Then there exists a similarity transformation T such that $\bar{x} = T^{-1}x$,

$$\bar{\mathbf{A}} = T^{-1} \mathbf{A} T ; \quad \bar{\mathbf{B}} = T^{-1} \mathbf{B}$$

and $\bar{\mathbf{A}}, \bar{\mathbf{B}}$ have the form

$$\bar{\mathbf{A}} = \begin{bmatrix} \bar{\mathbf{A}}_c & \bar{\mathbf{A}}_{12} \\ 0 & \bar{\mathbf{A}}_{nc} \end{bmatrix} ; \quad \bar{\mathbf{B}} = \begin{bmatrix} \bar{\mathbf{B}}_c \\ 0 \end{bmatrix}$$

where $\bar{\mathbf{A}}_c$ has dimension k and $(\bar{\mathbf{A}}_c, \bar{\mathbf{B}}_c)$ is completely controllable.

Details of actually computing
matrix T not considered here

Controllable subspace

Output has a component $\mathbf{C}_{nc} \bar{x}_{nc} [k]$ that does not depend on the manipulated input $u[k]$, so...

⇒ caution must be exercised when controlling a system which is not completely controllable

- same holds when *model* used for control design is not completely controllable

Definition 3: The *controllable subspace* of a state-space model is composed of all states generated through every possible linear combination of the states in \bar{x}_c

stability of
controllable subspace



stability of all
eigenvalues of \bar{A}_c

Uncontrollable models in control design

- Uncontrollable models are often a very convenient way of describing *disturbances* when modeling for control design

Example: constant disturbance can be modeled by the following state-space model:

$$\underline{\dot{x}}_d = 0$$

- uncontrollable *and* non-stabilizable

⇒ very common to employ uncontrollable models in control-system design

Stabilizability

Definition 4: The *uncontrollable subspace* of a state-space model is composed of all states generated through every possible linear combination of the states in \bar{x}_{nc}

stability of
uncontrollable subspace \Leftrightarrow stability of all
eigenvalues of \bar{A}_{nc}

A state-space model is said to be *stabilizable* if its uncontrollable subspace is stable.

In other words: system is stabilizable only if those states that cannot be controlled decay to origin “by themselves”

Canonical forms

If a system is completely controllable, there exist similarity transformations that convert it into special “standard forms”, or *canonical forms*:

- ❑ *controllability canonical form*

- ❑ *controller canonical form*

- These canonical forms present **A** and **B** matrices in highly structured ways
- ☹ Physical interpretation of states is lost
- ☺ Can be written directly from knowledge of system poles

Observability, detectability and reconstructibility

$$x[k] = A x[k] + B u[k]$$

$$y[k] = C x[k] + D u[k]$$

□ Observability

- What do observations of output tell us about *initial* state of system?

□ Detectability

- Do observations of output tell us everything “important” about the internal state of system? (Yes, if non-observable states decay to origin)

□ Reconstructibility

- Can we establish *current* state of system from past output response?
- Same as observability for continuous-time systems, but subtly different for discrete-time systems

Observability

Observability is concerned with what can be said about the initial state when given measurements of the plant output

Definition 5:

- A state $x_0 \neq 0$ is said to be *unobservable* if, given $x(0) = x_0$, and $u[k] = 0$ for $k \geq 0$, then $y[k] = 0$ for $k \geq 0$
 - state is doing something “interesting”, (or at least is non-zero!), yet output is zero
- The system is said to be *completely observable* if there exists no non-zero initial state that it is unobservable

Reconstructability

Reconstructability is concerned with what can be said about $x(T)$, on the basis of the past values of the output, i.e., $y[k]$ for $0 \leq k \leq T$

For *continuous-time* LTI systems:

complete reconstructability \Leftrightarrow complete observability

Observability vs. reconstructibility

Consider discrete-time system:

$$x[k+1] = 0$$

$$x[0] = x_0$$

$$y[k] = 0$$

- ❖ we know for certain that $x[T] = 0$ for all $T \geq 1 \Rightarrow$ system is reconstructable
- ❖ but $y[k] = 0 \forall k$, irrespective of the value of $x_0 \Rightarrow$ completely unobservable

In view of the subtle difference between observability and reconstructibility, we will use the term observability in the sequel to cover the stronger of the two concepts

A test for observability

A test for observability of a system is established in the following theorem.

Theorem 3: Consider the state model

holds for discrete (shift) and continuous-time models too

$$\mathbf{x}[k] = \mathbf{A} \mathbf{x}[k] + \mathbf{B} u[k]$$

$$y[k] = \mathbf{C} \mathbf{x}[k] + \mathbf{D} u[k]$$

- (i) The set of all unobservable states is equal to the null space of the *observability matrix* $\Gamma_0[\mathbf{A}, \mathbf{C}]$, where

$$\Gamma_0[\mathbf{A}; \mathbf{C}] = \begin{bmatrix} \mathbf{C} & \mathbf{C} \mathbf{A} & \mathbf{C} \mathbf{A}^2 & \dots & \mathbf{C} \mathbf{A}^{n-1} \end{bmatrix}$$

- (ii) The system is completely observable if and only if $\Gamma_0[\mathbf{A}, \mathbf{C}]$, has full column rank n

Example: A completely observable system

Consider the following state space model:

$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Then

$$\Gamma_0[\mathbf{A}; \mathbf{C}] = \begin{bmatrix} \mathbf{C} \\ \mathbf{C}\mathbf{A} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$$

Hence, $\text{rank } \Gamma_0[\mathbf{A}, \mathbf{C}] = 2$, and the system is completely observable.

Example: A non-completely observable system

Consider

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Here

$$\text{rank } \Gamma_0[A; C] = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Hence, $\text{rank } \Gamma_0[A, C] = 1 < 2$, and the system is *not* completely observable.

The controllable—observable duality

It's no coincidence that:

complete controllability $\Leftrightarrow \Gamma_c[\mathbf{A}, \mathbf{B}]$ has full row rank

complete observability $\Leftrightarrow \Gamma_o[\mathbf{A}, \mathbf{C}]$ has full column rank

as the following Theorem shows:

Theorem 4 Consider a state-space model described by the $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$. Then

$(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$
completely controllable

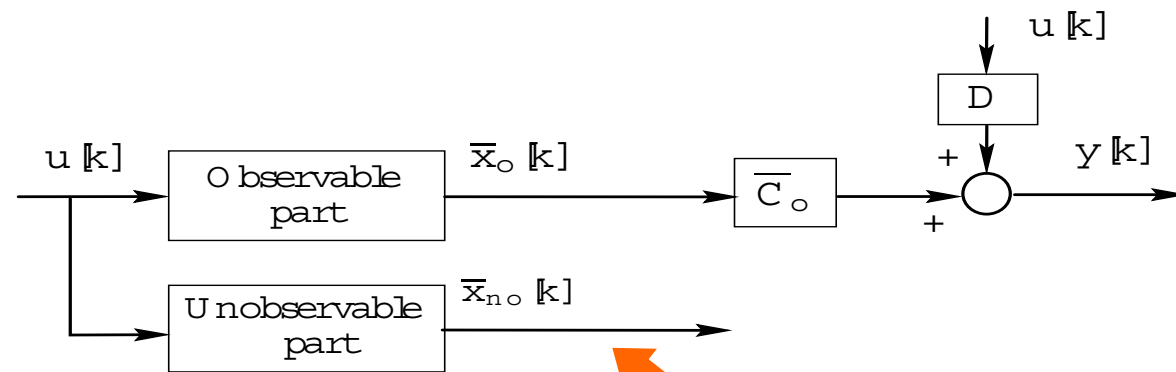
\Leftrightarrow

$(\mathbf{A}^T, \mathbf{C}^T, \mathbf{B}^T, \mathbf{D}^T)$
completely observable

the so-called
dual system

Observable–unobservable decomposition

If a system is not completely observable, it can be decomposed into an observable and a completely unobservable subsystem



This subset of states *in no way* influences output

Partitioning of A, B, and C

Decomposition of state-space into observable and non-observable parts relies on suitable partitioning of (similarity transformed) state-space matrices:

$$\begin{aligned} \begin{bmatrix} \bar{x}_o[k] \\ \bar{x}_{no}[k] \end{bmatrix} &= \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{no} \end{bmatrix} \begin{bmatrix} \bar{x}_o[k] \\ \bar{x}_{no}[k] \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{no} \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_o[k] \\ \bar{x}_{no}[k] \end{bmatrix} + D u[k] \end{aligned}$$

Observable decomposition

Lemma 4: If $\text{rank}\{\Gamma_o[\mathbf{A}, \mathbf{C}]\} = k < n$, there exists a similarity transformation T such that with $\bar{x} = T^{-1}x$, $\bar{\mathbf{A}} = T^{-1}\mathbf{A}T$, $\bar{\mathbf{C}} = \mathbf{C}T$, then $\bar{\mathbf{C}}$ and $\bar{\mathbf{A}}$ take the form

$$\bar{\mathbf{A}} = \begin{array}{cc} \bar{\mathbf{A}}_o & 0 \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{A}}_{no} \end{array} \quad \bar{\mathbf{C}} = \begin{array}{cc} \bar{\mathbf{C}}_o & 0 \end{array}$$

where $\bar{\mathbf{A}}_o$ has dimension k and the pair $(\bar{\mathbf{C}}_o, \bar{\mathbf{A}}_o)$ is completely observable

Observable subspace

Definition 6: *The observable subspace* of a state-space model is composed of all states generated through every possible linear combination of the states in \bar{x}_0

stability of
controllable subspace



stability of all
eigenvalues of \bar{A}_0

Detectability

Definition 7: The *unobservable subspace* of a state-space model is composed of all states generated through every possible linear combination of the states in \bar{x}_{n0} .

stability of unobservable subspace \Leftrightarrow stability of all eigenvalues of $\bar{\mathbf{A}}_{n0}$

A state-space model is said to be *detectable* if its unobservable subspace is stable.

In other words: system is observable only if those states that cannot be observed decay to origin “by themselves”

- ❖ while non-stabilizable models are frequently used to model disturbances in control-system design, this is *not* true for non-detectable models.

Canonical decomposition

- ❖ Dual to controller and controllability canonical forms are *observer* and *observability canonical forms*
 - ◆ Precise forms aren't important here
- ❖ Can also combine controllable and observable decompositions into a *canonical decomposition* with subsystems which are:
 - ◆ Controllable and observable (A_{co}, B_1, C_1)
 - ◆ Controllable, not observable
 - ◆ Observable, not controllable
 - ◆ Not observable, not controllable

Only controllable *and* observable parts of system appear in transfer function!

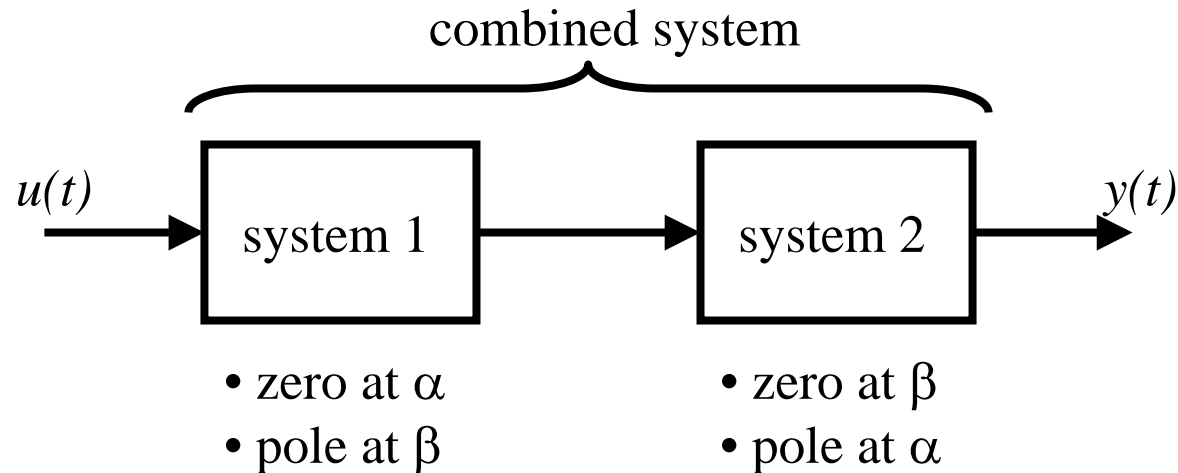
$$H(s) = C(sI - A)^{-1}B = C_1(sI - A_{co})^{-1}B_1$$

$$A = \begin{bmatrix} A_{co} & 0 & A_{13} & 0 \\ A_{21} & A_{22} & A_{23} & A_{24} \\ 0 & 0 & A_{33} & 0 \\ 0 & 0 & 0 & A_{44} \end{bmatrix} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = [C_1 \quad 0 \quad C_2 \quad 0]$$

Pole-zero cancellations

Systems which are either non-completely controllable and/or non-completely observable are associated with transfer functions having *pole-zero cancellations*



Then combined system:

- ❖ has a pole at β that is not observable from $y(t)$
- ❖ has a zero at α that is not controllable from $u(t)$

The big picture

- *Controllability*: can we use input to steer system state to origin in finite time?
 - *Stabilizability*: not controllable, but uncontrollable states well behaved
- *Observability*: can we infer system state from measurements of output?
 - *Detectability*: not observable, but unobservable states decay to origin
- Algebraic tests for controllability and observability
- Non-observable and/or non-controllable systems have transfer functions with pole-zero cancellations



*Drinking from
a firehose...*